# The Linear Quadratic Dynamic Game for Discrete-Time Descriptor Systems

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**Abstract**

The linear quadratic zero-sum dynamic game for discrete time descriptor systems is considered. A method, which involves solving a linear quadratic zero-sum dynamic game for a reduced-order discrete time state space system, is developed to find the linear feedback saddle-point solutions of the problem. Checkable conditions, which are described in terms of two dual algebraic Riccati equations and a Hamiltonian matrix, are given such that the linear quadratic zero-sum dynamic game for the reduced-order discrete time state space system is available. Sufficient conditions for the existence of the solutions are obtained. In contrast with the dynamic game in state space systems, the dynamic game in descriptor systems admits uncountably many linear feedback saddle-point solutions. All these solutions have the same existence conditions and achieve the same value of the dynamic game.

## 1 Introduction

In this paper, we consider the linear quadratic zero-sum dynamic game for discrete time descriptor systems $Ex_{k+1} = Ax_k + B^1 u_k^1 + B^2 u_k^2$, where $E$ is in general a singular matrix and the system structure is in general noncausal(Ref. 1). For the descriptor system described above, if the system is causal, it can be transformed to a regular full-order or reduced-order state space system depending on whether the matrix $E$ is invertible, or not. It is well known that, under certain conditions, the linear quadratic zero-sum dynamic game for state space systems admits a unique linear feedback saddle-point solution (Ref. 2).

In papers (Refs. 3,4), two methods based on the "completion of square" technique and the dynamic programming technique have been provided to solve the linear quadratic zero-sum differential game for continuous time descriptor systems. However, it seems that those methods are not suitable for the same problem of discrete time descriptor systems. In fact, we have not found any appropriate (generalized) discrete time Riccati equation which allow us to apply the dynamic programming technique to solve the problem.

In this paper, we provide a different method, which involves solving a linear quadratic zero-sum dynamic game for a reduced-order discrete time state space system, to solve the linear quadratic zero-sum dynamic game for discrete time descriptor systems. Checkable conditions depending on the solvability of two dual algebraic Riccati equations and the invertible condition of a Hamiltonian matrix are given such that the linear quadratic zero-sum dynamic game for the reduced-order discrete time state space system is available. Sufficient conditions for the existence of the solutions are given in terms of the conditions of the linear quadratic zero-sum dynamic game for the reduced-order discrete time state space system. Similar to the counterpart results of continuous time descriptor systems, we show that the dynamic game in discrete time descriptor systems admits uncountably many linear feedback saddle-point solutions. All these solutions have the same existence conditions and achieve the same value of the dynamic game. Therefore, the nonunique feature of the linear feedback saddle-point solutions in this paper is different from the so-called informational nonuniqueness in state space systems, where players have access to closed-loop state information (with memory) and different saddle-point equilibria do not necessarily require the same existence conditions (Ref. 2).

## 2 Problem Formulation

Consider the linear discrete time descriptor system

$$Ex_{k+1} = Ax_k + B^1 u_k^1 + B^2 u_k^2, \qquad (1)$$

for $k \in \mathbf{K} := \{0,1,2,...,N-1\}$ where $x_k$ is the $n$-dimensional descriptor vector, $u_k^1$ is the $m$-dimensional control vector of Player 1, $u_k^2$ is the $l$-dimensional control vector of Player 2. The matrix $E$ is a square matrix of rank $r \leq n$. The pencil $(sE-A)$ is assumed to be regular (i.e., $|(sE-A)| \not\equiv 0$). It is assumed that the initial state $x_0$ is known to both players. The cost function is given by

$$J = \frac{1}{2}x_N^T E^T Q_N E x_N + \frac{1}{2}\sum_{k=0}^{N-1}\{x_k^T Q x_k + u_k^{1T} u_k^1 - u_k^{2T} u_k^2\}, \quad (2)$$

which is to be minimized by Player 1 and to be maximized by Player 2, where $Q_N \geq 0$ and $Q \geq 0$. The superscript $T$ denotes the transpose of the matrix. The information structure to be used in the paper is the closed-loop no-memory information on $x_k$, under which the strategy spaces for Player 1 and Player 2 at each stage $k \in \mathbf{K}$ are denoted by $\Gamma_k^1$ and $\Gamma_k^2$ respectively. $\Gamma_k^1$ and $\Gamma_k^2$ are composed of linear feedback strategies of $x_k$, denoted by $\gamma_k^1$ and $\gamma_k^2$, such that the resulting closed-loop system is causal(Refs. 5,6). Their open-loop realizations are $u_k^1$ and $u_k^2$ respectively. Let us introduce the notation

$$\gamma^i \in \Gamma^i := \{\gamma_k^i \in \Gamma_k^i,\ k \in \mathbf{K}\},\ i = 1,2. \quad (3)$$

Then, we have the notion of a saddle-point equilibrium.

**Definition 1.** For the dynamic game posed above, a pair of strategies $(\gamma^{1*}, \gamma^{2*}) \in \Gamma^1 \times \Gamma^2$ is in a saddle-point equilibrium if

$$J(\gamma^{1*}, \gamma^2) \leq J(\gamma^{1*}, \gamma^{2*}) \leq J(\gamma^1, \gamma^{2*}) \quad (4)$$

for all $(\gamma^1, \gamma^2) \in \Gamma^1 \times \Gamma^2$.

We now seek the linear feedback saddle-point solution to the problem formulated above. At this point it is important to recall that the dynamic programming technique is a useful tool to find the linear feedback saddle-point solution for the problem formulated in state space systems. However, the dynamic programming technique is not valid (at least, at present time) for the problem formulated in this paper. The reason is that we have not yet found any appropriate discrete time Riccati equation to describe the cost-to-go function. It has been indicated that an obvious modification to the usual Riccati equation of state space system may result in no solution in descriptor system (Refs. 5,6). Therefore, an alternative method is needed to solve the problem of this paper. In the next section, we construct the reduced-order linear quadratic dynamic game for state space systems whose solution plays an important role in the solution of the dynamic game for the descriptor system.

## 3 A Reduced-Order Dynamic Game in State Space Systems

Suppose that the positive semidefinite weighting matrix $Q$ is factored into $Q = C^T C$ and $y_k = Cx_k$ is the output of the system (1). Then, a basic assumption is made throughout the paper.

**Assumption 1.** The descriptor system (1) is causally controllable and reconstructible.

Moreover, there exist nonsingular matrices $M$ and $H$ such that

$$MEH = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},\ r = rank\ E. \quad (5)$$

Based on the transformation (5), we define

$$MAH = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ MB^i = \begin{bmatrix} B_1^i \\ B_2^i \end{bmatrix},\ i = 1,2, \quad (6a)$$

$$H^T Q H = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad (6b)$$

$$M^{-T} Q_N M^{-1} = \begin{bmatrix} Q_{11N} & Q_{12N} \\ Q_{12N}^T & Q_{22N} \end{bmatrix}. \quad (6c)$$

Obviously, Assumption 1 holds true if and only if the rows of the matrices $[A_{22}\ B_2^1\ B_2^2]$ and $[A_{22}^T\ C_2^T]$ are independent respectively(Refs. 5,6).

Let us further define some other matrices(Ref. 7).

$$T_1 = \begin{bmatrix} A_{11} & -S_{11} \\ Q_{11} & A_{11}^T \end{bmatrix},\ T_2 = \begin{bmatrix} A_{12} & -S_{12} \\ Q_{12} & A_{21}^T \end{bmatrix}, \quad (7a)$$

$$T_3 = \begin{bmatrix} A_{21} & -S_{12}^T \\ -Q_{12}^T & -A_{12}^T \end{bmatrix},\ T_4 = \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix}, \quad (7b)$$

where

$$S_{11} = B_1^1 B_1^{1T} - B_1^2 B_1^{2T}, \quad (8a)$$

$$S_{12} = B_1^1 B_2^{1T} - B_1^2 B_2^{2T}, \quad (8b)$$

$$S_{22} = B_2^1 B_2^{1T} - B_2^2 B_2^{2T}. \quad (8c)$$

**Assumption 2.** The matrix $T_4$ is invertible.

**Assumption 3.** The following two dual algebraic Riccati equations admit real symmetric solutions $X$ and $Y$ respectively,

$$XA_{22} + A_{22}^T X - XS_{22}X + Q_{22} = 0, \quad (9a)$$

$$A_{22}Y + YA_{22}^T - YQ_{22}Y + S_{22} = 0. \quad (9b)$$

**Remark 1**. Different from Ref. 7, it is worth to note that $T_4^{-1}$ exists only if Assumption 1 is satisfied. However, Assumption 1 is not sufficient for $T_4$ to be invertible. Moreover, the existence of a real symmetric solution $X$ to (9a) is equivalent to the existence of a real symmetric solution $Y$ to (9b) if $Q_{22} > 0$, which means that the system (1) is causally observable, a more restricted condition than that in Assumption 1.

Since $T_4^{-1}$ exists, we can calculate

$$\begin{bmatrix} A_r & -S_r \\ Q_r & A_r^T \end{bmatrix} = T_1 - T_2 T_4^{-1} T_3, \tag{10}$$

directly by some numerical methods. However, for the purpose of theoretical analysis, we need the explicit expressions of $A_r, S_r, Q_r$.

**Lemma 1**. Suppose that Assumptions 1-3 are satisfied. Then, there exist matrices $B_r^1 \in R^{r\times m}$, $B_r^2 \in R^{r\times l}$ such that $S_r = B_r^1 B_r^{1T} - B_r^2 B_r^{2T}$. Moreover, $Q_r$ is a positive semidefinite matrix.

**Proof**. See Appendix A for the proof.

We are now in the position to construct the linear quadratic dynamic game for the reduced-order state space system. Because of Lemma 1, we can formulate the standard linear quadratic dynamic game for the reduced-order state space systems as follows.

**Reduced-Order Dynamic Game**. Find the linear feedback saddle-point solution of the cost function

$$J_r = \frac{1}{2} z_N^{1T} Q_{11N} z_N^1 + \frac{1}{2}\sum_{k=0}^{N-1}\{z_k^{1T} Q_r z_k^1 + u_k^{1T} u_k^1 - u_k^{2T} u_k^2\}, \tag{11}$$

subject to the system equation

$$z_{k+1}^1 = A_r z_k^1 + B_r^1 u_k^1 + B_r^2 u_k^2, \tag{12}$$

where $[z_k^{1T}\ z_k^{2T}]^T = H^{-1} x_k$, hence $z_0^1$ is also known by both players.

**Theorem 1**. The linear quadratic zero-sum dynamic game described by (11),(12) admits a unique linear feedback saddle-point solution if, and only if,

$$I + B_r^{1T} P_{k+1} B_r^1 > 0,\ (k \in \mathbf{K}) \tag{13a}$$

$$I - B_r^{2T} P_{k+1} B_r^2 > 0,\ (k \in \mathbf{K}) \tag{13b}$$

in which case the unique equilibrium strategies are given by

$$\gamma_k^{1*}(z_k^1) = -B_r^{1T} L_k^1 z_k^1,\ (k \in \mathbf{K}) \tag{14a}$$

$$\gamma_k^{2*}(z_k^1) = B_r^{2T} L_k^1 z_k^1,\ (k \in \mathbf{K}) \tag{14b}$$

and the corresponding unique state trajectory $\{z_k^{1*}; k \in \mathbf{K}\}$ satisfies the difference equation

$$z_{k+1}^{1*} = Z_k^1 z_k^{1*}, \tag{15}$$

where

$$L_k^1 = P_{k+1}(I + S_r P_{k+1})^{-1} A_r, \tag{16}$$

$$Z_k^1 = (I + S_r P_{k+1})^{-1} A_r. \tag{17}$$

$P_{k+1}$ satisfies the discrete-time Riccati equation

$$P_k = Q_r + A_r^T P_{k+1}(I + S_r P_{k+1})^{-1} A_r;\ P_N = Q_{11N}. \tag{18}$$

**Proof**. See Başar and Olsder(Ref. 2) for the proof.

Since the unique linear saddle-point solution exists, the upper value and the lower value of the dynamic game must be equal to the value of the dynamic game. Hence, we arrive at the following conclusions.

**Corollary 1**. Suppose that the unique linear saddle-point solution exists. Then, the following relations hold.

(i) Two discrete-time Riccati equations

$$P_k = Q_r + L_k^{1T} B_r^1 B_r^{1T} L_k^1 + (A_r - B_r^1 B_r^{1T} L_k^1)^T P_{k+1}(I - B_r^2 B_r^{2T} P_{k+1})^{-1}(A_r - B_r^1 B_r^{1T} L_k^1),\ P_N = Q_{11N}; \tag{19a}$$

$$P_k = Q_r + L_k^{1T} B_r^2 B_r^{2T} L_k^1 + (A_r + B_r^2 B_r^{2T} L_k^1)^T P_{k+1}(I + B_r^1 B_r^{1T} P_{k+1})^{-1}(A_r + B_r^2 B_r^{2T} L_k^1),\ P_N = Q_{11N}, \tag{19b}$$

have the same solution $P_k$ as that of (18).

(ii)

$$(I + B_r^1 B_r^{1T} P_{k+1})^{-1}(A_r + B_r^2 B_r^{2T} L_k^1) = (I + S_r P_{k+1})^{-1} A_r, \tag{20a}$$

$$(I - B_r^2 B_r^{2T} P_{k+1})^{-1}(A_r - B_r^1 B_r^{1T} L_k^1) = (I + S_r P_{k+1})^{-1} A_r. \tag{20b}$$

**Proof**. To prove the relations given above, we only need to fix $u_k^1 = \gamma_k^{1*}(z_k^1)$(or, $u_k^2 = \gamma_k^{2*}(z_k^1)$) of (11),(12) and solve the corresponding maximizing (minimizing) problem obtained from (11),(12). The details are omitted here.

## 4 Dynamic Game for Descriptor Systems

In this section, using the results obtained in Section 3, we will solve the dynamic game for descriptor systems. Before doing that, we first define the following notations.

$$\begin{bmatrix} Z_k^2 \\ L_k^2 \end{bmatrix} = -T_4^{-1} T_3 \begin{bmatrix} I \\ L_k^1 \end{bmatrix}. \tag{21}$$

Moreover, we have

$$L_k^2 - X Z_k^2 = N_{1r}^T L_k^1 - N_{2r}^T. \tag{22}$$

The reader is referred to Appendix B for the explicit expressions of (21) and the derivations of (22).

**Theorem 2**. For the linear quadratic zero-sum dynamic game of the discrete time descriptor systems, suppose that Assumptions 1-3 are satisfied. Then,

(i) The dynamic game admits a linear feedback saddle-point solution if, and only if conditions (13a),(13b) of Theorem 1 are satisfied.

(ii) Under conditions (13a), (13b), there exist uncountably many linear feedback saddle-point solutions, with the family of the equilibrium strategies given by

$$\gamma_k^{1*}(x_k) = -B^{1T}M^T \begin{bmatrix} L_k^1 & 0 \\ L_k^2 - F_k^1 Z_k^2 & F_k^1 \end{bmatrix} H^{-1}x_k, \quad (k \in \mathbf{K}) \tag{23a}$$

$$\gamma_k^{2*}(x_k) = B^{2T}M^T \begin{bmatrix} L_k^1 & 0 \\ L_k^2 - F_k^2 Z_k^2 & F_k^2 \end{bmatrix} H^{-1}x_k, \quad (k \in \mathbf{K}) \tag{23b}$$

where $F_k^1$, $F_k^2$ are arbitrary two $(n-r)\times(n-r)$ matrices making $A_{22} - B_2^1 B_2^{1T} F_k^1 + B_2^2 B_2^{2T} F_k^2$ invertible.

**Proof**. First, let $F_k^1 = F_k^2 = X$ in (23). Then, $A_{22} - B_2^1 B_2^{1T} X + B_2^2 B_2^{2T} X = \hat{A}_{22}$ is invertible. Substituting $\gamma_k^{1*}(x_k)$ into (1),(2) and making transformations of (5),(6) yield

$$z_{k+1}^1 = A_{11}^{max} z_k^1 + A_{12}^{max} z_k^2 + B_1^2 u_k^2, \tag{24a}$$

$$0 = A_{21}^{max} z_k^1 + A_{22}^{max} z_k^2 + B_2^2 u_k^2, \tag{24b}$$

and

$$J = \frac{1}{2} z_N^{1T} Q_{11N} z_N^1 + \frac{1}{2} \sum_{k=0}^{N-1} \{ \begin{bmatrix} z_k^1 \\ z_k^2 \end{bmatrix}^T \begin{bmatrix} Q_{11}^{max} & Q_{12}^{max} \\ Q_{12}^{maxT} & Q_{22}^{max} \end{bmatrix} \times \begin{bmatrix} z_k^1 \\ z_k^2 \end{bmatrix} - u_k^{2T} u_k^2 \}, \tag{25}$$

where the reader is referred to Appendix C for the expressions of the related terms.

We now solve the maximizing problem of Player 2 described by (24),(25). Introducing the codescriptor vectors $\lambda_k^1$, $\lambda_k^2$ , the necessary conditions for (25) to be maximized are (Refs. 6,8)

$$z_{k+1}^1 = A_{11}^{max} z_k^1 + A_{12}^{max} z_k^2 + B_1^2 u_k^2, \tag{26a}$$

$$0 = A_{21}^{max} z_k^1 + A_{22}^{max} z_k^2 + B_2^2 u_k^2, \tag{26b}$$

$$\lambda_k^1 = A_{11}^{maxT} \lambda_{k+1}^1 + A_{21}^{maxT} \lambda_{k+1}^2 + Q_{11}^{max} z_k^1 + Q_{12}^{max} z_k^2, \tag{26c}$$

$$0 = A_{12}^{maxT} \lambda_{k+1}^1 + A_{22}^{maxT} \lambda_{k+1}^2 + Q_{21}^{max} z_k^1 + Q_{22}^{max} z_k^2, \tag{26d}$$

$$u_k^2 = B_1^{2T} \lambda_{k+1}^1 + B_2^{2T} \lambda_{k+1}^2. \tag{26e}$$

Substituting (26e) into (26a) and (26b) yields

$$\begin{bmatrix} z_{k+1}^1 \\ \lambda_k^1 \end{bmatrix} = \begin{bmatrix} A_{11}^{max} & B_1^2 B_1^{2T} \\ Q_{11}^{max} & A_{11}^{maxT} \end{bmatrix} \begin{bmatrix} z_k^1 \\ \lambda_{k+1}^1 \end{bmatrix} + \begin{bmatrix} A_{12}^{max} & B_1^2 B_2^{2T} \\ Q_{12}^{max} & A_{21}^{maxT} \end{bmatrix} \begin{bmatrix} z_k^2 \\ \lambda_{k+1}^2 \end{bmatrix}, \tag{27a}$$

$$0 = \begin{bmatrix} A_{21}^{max} & B_2^2 B_1^{2T} \\ -Q_{12}^{maxT} & -A_{12}^{maxT} \end{bmatrix} \begin{bmatrix} z_k^1 \\ \lambda_{k+1}^1 \end{bmatrix} + \begin{bmatrix} A_{22}^{max} & B_2^2 B_2^{2T} \\ -Q_{22}^{max} & -A_{22}^{maxT} \end{bmatrix} \begin{bmatrix} z_k^2 \\ \lambda_{k+1}^2 \end{bmatrix}. \tag{27b}$$

Define

$$\hat{T}_1 = \begin{bmatrix} A_{11}^{max} & B_1^2 B_1^{2T} \\ Q_{11}^{max} & A_{11}^{maxT} \end{bmatrix}, \tag{28a}$$

$$\hat{T}_2 = \begin{bmatrix} A_{12}^{max} & B_1^2 B_2^{2T} \\ Q_{12}^{max} & A_{21}^{maxT} \end{bmatrix}, \tag{28b}$$

$$\hat{T}_3 = \begin{bmatrix} A_{21}^{max} & B_2^2 B_1^{2T} \\ -Q_{12}^{maxT} & -A_{12}^{maxT} \end{bmatrix}, \tag{28c}$$

$$\hat{T}_4 = \begin{bmatrix} A_{22}^{max} & B_2^2 B_2^{2T} \\ -Q_{22}^{max} & -A_{22}^{maxT} \end{bmatrix}. \tag{28d}$$

Since

$$\begin{bmatrix} A_{22}^{max} & B_2^2 B_2^{2T} \\ -Q_{22}^{max} & -A_{22}^{maxT} \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} \hat{A}_{22} & B_2^2 B_2^{2T} \\ 0 & -\hat{A}_{22}^T \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \tag{29}$$

exists, we can calculate

$$\begin{bmatrix} \hat{A}_r & \hat{S}_r \\ \hat{Q}_r & \hat{A}_r^T \end{bmatrix} = \hat{T}_1 - \hat{T}_2 \hat{T}_4^{-1} \hat{T}_3 \tag{30}$$

using (28),(29) directly. Furthermore, we can prove

$$\hat{A}_r = A_r - B_r^1 B_r^{1T} L_k^1, \tag{31a}$$

$$\hat{S}_r = B_r^2 B_r^{2T}, \tag{31b}$$

$$\hat{Q}_r = Q_r + L_k^{1T} B_r^1 B_r^{1T} L_k^1. \tag{31c}$$

Therefore, we obtain

$$\begin{bmatrix} z_{k+1}^1 \\ \lambda_k^1 \end{bmatrix} = \begin{bmatrix} A_r - B_r^1 B_r^{1T} L_k^1 & B_r^2 B_r^{2T} \\ Q_r + L_k^{1T} B_r^1 B_r^{1T} L_k^1 & (A_r - B_r^1 B_r^{1T} L_k^1)^T \end{bmatrix} \begin{bmatrix} z_k^1 \\ \lambda_{k+1}^1 \end{bmatrix}, \tag{32}$$

from (27). Let $\lambda_k^1 = \hat{P}_k z_k^1$, we have the discrete-time Riccati equation

$$\hat{P}_k = Q_r + L_k^{1T} B_r^1 B_r^{1T} L_k^1 + (A_r - B_r^1 B_r^{1T} L_k^1)^T \hat{P}_{k+1} \times (I - B_r^2 B_r^{2T} \hat{P}_{k+1})^{-1} (A_r - B_r^1 B_r^{1T} L_k^1), \ \hat{P}_N = Q_{11N}. \tag{33}$$

Using the solution $\hat{P}_k$ of (33), we arrive at the following equa tions

$$z_{k+1}^1 = \hat{Z}_k^1 z_k^1, \tag{34}$$

$$\lambda_{k+1}^1 = \hat{L}_k^1 z_k^1, \tag{35}$$

where

$$\hat{Z}_k^1 = (I - B_r^2 B_r^{2T} \hat{P}_{k+1})^{-1}(A_r - B_r^1 B_r^{1T} L_k^1), \tag{36}$$

$$\hat{L}_k^1 = \hat{P}_{k+1}(I - B_r^2 B_r^{2T} \hat{P}_{k+1})^{-1}(A_r - B_r^1 B_r^{1T} L_k^1). \tag{37}$$

Taking into account the facts of (19),(20), we readily have

$$\hat{P}_k = P_k,\ \hat{Z}_k^1 = Z_k^1,\ \hat{L}_k^1 = L_k^1. \tag{38}$$

Furthermore, substituting $\lambda_{k+1}^1$ into (27b) yields

$$z_k^2 = \hat{Z}_k^2 z_k^1, \tag{39}$$

$$\lambda_{k+1}^2 = \hat{L}_k^2 z_k^1, \tag{40}$$

which are obtained from the formula

$$\begin{bmatrix} \hat{Z}_k^2 \\ \hat{L}_k^2 \end{bmatrix} = -\hat{T}_4^{-1}\hat{T}_3 \begin{bmatrix} I \\ \hat{L}_k^1 \end{bmatrix}. \tag{41}$$

We can also prove

$$\hat{L}_k^2 - X\hat{Z}_k^2 = L_k^2 - XZ_k^2, \tag{42}$$

$$\hat{Z}_k^2 = Z_k^2. \tag{43}$$

Therefore, the maximizing problem of Player 2 admits a solution

$$\gamma_k^{2*}(x_k) = B^{2T} M^T \begin{bmatrix} L_k^1 & 0 \\ L_k^2 - XZ_k^2 & X \end{bmatrix} H^{-1} x_k. \ (k \in \mathbf{K}) \tag{44}$$

Symmetrically, substituting $\gamma_k^{2*}(x_k)$ of (44) into (1),(2), we can obtain a minimizing problem of Player 1. Solving the minimizing problem in a similar way as above gives the result that

$$\gamma_k^{1*}(x_k) = -B^{1T} M^T \begin{bmatrix} L_k^1 & 0 \\ L_k^2 - XZ_k^2 & X \end{bmatrix} H^{-1} x_k, \tag{45}$$

constitutes a minimizing solution to the problem. Therefore, we conclude that the dynamic game admits a linear feedback saddle-point solution which is given by (44),(45). The corresponding unique state trajectory is $\{x_k^* = H[z_k^{1*T}, z_k^{2*T}]^T; k \in \mathbf{K}\}$, where $z_k^{1*}$ satisfies the difference equation (34)(or (15)) and $z_k^{2*}$ satisfies the equation (39).

In the following, we will show that the linear feedback saddle-point solutions are not unique. To do this, it suffices to show that the strategies (23) and the strategies (44),(45) lead to the same state trajectory. Substituting (23) into (1) and making some transformations give the closed-loop system

$$z_{k+1}^1 = A_{11}^c z_k^1 + A_{12}^c z_k^2, \tag{46a}$$

$$0 = A_{21}^c z_k^1 + A_{22}^c z_k^2, \tag{46b}$$

where

$$A_{11}^c = A_{11} - B_1^1 B_1^{1T} L_k^1 - B_1^1 B_2^{1T}(L_k^2 - F_k^1 Z_k^2)$$
$$+ B_1^2 B_1^{2T} L_k^1 + B_1^2 B_2^{2T}(L_k^2 - F_k^2 Z_k^2), \tag{47a}$$

$$A_{12}^c = A_{12} - B_1^1 B_2^{1T} F_k^1 + B_1^2 B_2^{2T} F_k^2, \tag{47b}$$

$$A_{21}^c = A_{21} - B_2^1 B_1^{1T} L_k^1 - B_2^1 B_2^{1T}(L_k^2 - F_k^1 Z_k^2)$$
$$+ B_2^2 B_1^{2T} L_k^1 + B_2^2 B_2^{2T}(L_k^2 - F_k^2 Z_k^2), \tag{47c}$$

$$A_{22}^c = A_{22} - B_2^1 B_2^{1T} F_k^1 + B_2^2 B_2^{2T} F_k^2. \tag{47d}$$

Since $A_{22} - B_2^1 B_2^{1T} F_k^1 + B_2^2 B_2^{2T} F_k^2$ is invertible, (46) admits a unique solution, for example, $(\hat{z}_k^1, \hat{z}_k^2)$. On the other hand, it is easy to verify that $(z_k^{1*}, z_k^{2*})$ are also the solutions of (46) because of $z_k^{2*} = Z_k^2 z_k^{1*}$. Hence, $\hat{z}_k^1 = z_k^{1*}$ and $\hat{z}_k^2 = z_k^{2*}$, which means that (23) is a linear feedback saddle-point solution of the problem. Thereby, we have finished the proof of the theorem.

## 5 Conclusions

In this paper, we have investigated the linear quadratic zero-sum dynamic game for discrete time descriptor systems. This problem is solved through the solution of the reduced-order linear quadratic zero-sum dynamic game for standard discrete time state space system. Checkable conditions are given such that such a reduced-order zero-sum dynamic game is available. These conditions are described in term of the solvability of two dual algebraic Riccati equations and the invertible condition of the Hamiltonian matrix. Furthermore, the sufficient conditions for the existence of the linear feedback saddle-point solutions are obtained which are the same as the conditions of the linear quadratic zero-sum dynamic game for the reduced-order discrete time state space system. Similar to the counterpart results of continuous time descriptor systems, we show that the dynamic game in discrete time descriptor systems admits uncountably many linear feedback saddle-point solutions. All these solutions have the same existence conditions and achieve the same value of the dynamic game. Therefore, the nonunique feature of the linear feedback saddle-point solutions in this paper is different from the so-called informational nonuniqueness in state space systems, where players have access to closed-loop state information (with memory) and different saddle-point equilibria do not necessarily require the same existence conditions (Ref. 2).

**Appendix A.** Proof of Lemma 1.

Under Assumptions 2,3, we have

$$\begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} \hat{A}_{22}^{-1} & -\hat{A}_{22}^{-1} S_{22} \hat{A}_{22}^{-T} \\ 0 & -\hat{A}_{22}^{-T} \end{bmatrix} \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}. \tag{48}$$

where $\hat{A}_{22} = A_{22} - S_{22}X$. Using (48) in the calculation of (10 ) yields

$$A_r = A_{11} + N_{1r}A_{21} + N_{1r}S_{22}N_{2r}^T + S_{12}N_{2r}^T, \quad (49a)$$

$$S_r = S_{11} + N_{1r}S_{12}^T + N_{1r}S_{22}N_{1r}^T + S_{12}N_{1r}^T, \quad (49b)$$

$$Q_r = Q_{11} - N_{2r}A_{21} - N_{2r}S_{22}N_{2r}^T - A_{21}^T N_{2r}^T, \quad (49c)$$

where

$$N_{1r} = -\hat{A}_{12}\hat{A}_{22}^{-1},\ N_{2r} = \hat{Q}_{12}\hat{A}_{22}^{-1}, \quad (50a)$$

$$\hat{A}_{12} = A_{12} - S_{12}X,\ \hat{Q}_{12} = Q_{12} + A_{21}^T X. \quad (50b)$$

Hence,

$$S_r = B_r^1 B_r^{1T} - B_r^2 B_r^{2T}, \quad (51)$$

where

$$B_r^1 = B_1^1 + N_{1r}B_2^1,\ B_r^2 = B_1^2 + N_{1r}B_2^2. \quad (52)$$

On the other hand,

$$\begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix}^{-1} = \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix}$$

$$\times \begin{bmatrix} \tilde{A}_{22}^{-T} & 0 \\ -\tilde{A}_{22}^{-1}Q_{22}\tilde{A}_{22}^{-T} & -\tilde{A}_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}. \quad (53)$$

where $\tilde{A}_{22} = A_{22}^T - Q_{22}Y$. Using (53) in the calculation of (10) gives

$$A_r = A_{11} + A_{12}M_{1r}^T + M_{2r}Q_{12}^T + M_{2r}Q_{22}M_{1r}^T, \quad (54a)$$

$$S_r = S_{11} - A_{12}M_{2r}^T - M_{2r}A_{12}^T - M_{2r}Q_{22}M_{2r}^T, \quad (54b)$$

$$Q_r = Q_{11} + Q_{12}M_{1r}^T + M_{1r}Q_{12}^T + M_{1r}Q_{22}M_{1r}^T, \quad (54c)$$

where

$$M_{1r} = -\tilde{A}_{21}\tilde{A}_{22}^{-1},\ M_{2r} = \tilde{S}_{12}\tilde{A}_{22}^{-1}, \quad (55a)$$

$$\tilde{A}_{21} = A_{21}^T - Q_{12}Y,\ \tilde{S}_{12} = S_{12} + A_{12}^T Y. \quad (55b)$$

Therefore,

$$Q_r = (C_1 + C_2M_{1r}^T)^T(C_1 + C_2M_{1r}^T) \geq 0. \quad (56)$$

**Appendix B.** Derivations of $Z_k^2$, $L_k^2$ in Section 4.

Using (48) to (21) yields

$$Z_k^2 = (\hat{A}_{22}^{-1}S_{12}^T + \hat{A}_{22}^{-1}S_{22}N_{1r}^T)L_k^1 - \hat{A}_{22}^{-1}A_{21} - \hat{A}_{22}^{-1}S_{22}N_{2r}^T, \quad (57)$$

$$L_k^2 = X(\hat{A}_{22}^{-1}S_{12}^T + \hat{A}_{22}^{-1}S_{22}N_{1r}^T)L_k^1$$

$$-X(\hat{A}_{22}^{-1}A_{21} + \hat{A}_{22}^{-1}S_{22}N_{2r}^T) + N_{1r}^T L_k^1 - N_{2r}^T. \quad (58)$$

Hence,

$$L_k^2 - XZ_k^2 = N_{1r}^T L_k^1 - N_{2r}^T. \quad (59)$$

**Appendix C.** Derivations of the related terms in (24),(25)

$$A_{11}^{max} = A_{11} - B_1^1[B_1^{1T}L_k^1 + B_2^{1T}(L_k^2 - XZ_k^2)],$$

$$A_{12}^{max} = A_{12} - B_1^1 B_2^{1T} X,$$

$$A_{21}^{max} = A_{21} - B_2^1[B_1^{1T}L_k^1 + B_2^{1T}(L_k^2 - XZ_k^2)],$$

$$A_{22}^{max} = A_{22} - B_2^1 B_2^{1T} X. \quad (60)$$

$$\begin{bmatrix} Q_{11}^{max} & Q_{12}^{max} \\ Q_{12}^{maxT} & Q_{22}^{max} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}$$

$$+ \begin{bmatrix} L_k^{1T} & L_k^{2T} - Z_k^{2T}X \\ 0 & X \end{bmatrix} \begin{bmatrix} B_1^1 B_1^{1T} & B_1^1 B_2^{1T} \\ B_2^1 B_1^{1T} & B_2^1 B_2^{1T} \end{bmatrix}$$

$$\times \begin{bmatrix} L_k^1 & 0 \\ L_k^2 - XZ_k^2 & X \end{bmatrix}. \quad (61)$$

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