

Controller Design for Time-Delay Systems Using Discretized Lyapunov Functional Approach*

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Abstract

This paper investigates the controller synthesis of uncertain linear time-delay systems. Stabilizability criteria are derived based on a *discretized Lyapunov functional* approach. A new controller design method is developed. Numerical examples show that the results using the proposed method are less conservative than some existing ones.

1. Introduction

Many practical systems are subject to uncertainty and time-delay. Therefore, robust control of uncertain time-delay systems has received significant attentions recently. Some researchers have focused on the problem of designing robust memoryless controllers for uncertain time-delay systems. A number of results on this topic have been reported. Most of the early Lyapunov function (or functional) based approaches yield *delay-independent* stabilization criteria (Han and Mehdi, 1999a; Phoojaruenchanachai and Furuta, 1992; Shen *et al.*, 1991). These results are often overly conservative and can not deal with systems whose stabilizability depends on the size of the time delay. Razumikhin type of stabilizability criteria can indeed yield *delay-dependent* stabilizability results (Han and Mehdi, 1999b; Li and de Souza 1997; Niculescu *et al.*, 1994). They can also allow time-varying delay. However, experience shows that the results are still rather conservative for many practical applications. This can be seen by applying these type of criteria to a constant time-delay system without uncertainty and comparing them with the analytical result (Gu, 1997).

Gu (1997, 1999a) has proposed a *discretized Lyapunov functional* approach to check the stability of linear uncertain systems with constant time-delay. The criteria have shown significant improvements over the existing

results even under very coarse discretization. For uncertainty-free systems, the analytical results can be approached with fine discretization. Gu and Han (2000) has extended the result in Gu (1997) to the case where the delay may be *time-varying*. The result in Gu and Han (2000) has been improved in Han and Gu (2000) by using a *generalized discretized Lyapunov functional* approach.

In this paper, based on the *discretized Lyapunov functional* approach in Gu (1997), we address the problem of robust control design for uncertain linear time-delay systems. The case of a single, constant time-delay is considered. We develop a design algorithm of a linear memoryless static state feedback controller for uncertain time-delay systems.

Notations

$W > 0$: symmetric positive definite matrix;
 $W < 0$: symmetric negative definite matrix;
 $W \geq 0$: symmetric positive semi-definite matrix;
 $W \leq 0$: symmetric negative semi-definite matrix;
 \mathbb{C} : the set of continuous \mathfrak{R}^n valued functions on $[-r, 0]$;
 r : time-delay;

2. Problem Statement

Consider the uncertain time-delay system

$$\Sigma^{(1)} : \dot{x}(t) = A^{(1)}(t)x(t) + A_d^{(1)}(t)x(t-r) + B^{(1)}(t)u(t) \quad (2.1)$$

where $x(t) \in \mathfrak{R}^n$ is the state vector, $u(t) \in \mathfrak{R}^m$ is the control vector, $A^{(1)}(t) \in \mathfrak{R}^{n \times n}$, $A_d^{(1)}(t) \in \mathfrak{R}^{n \times n}$ and $B^{(1)}(t) \in \mathfrak{R}^{n \times m}$ are uncertain matrices, which are unknown and possibly time-varying, but known to be bounded by some compact set Ω , i.e.

$$[A^{(1)}(t), A_d^{(1)}(t), B^{(1)}(t)] \in \Omega \subset \mathfrak{R}^{n \times (2n+m)}, \text{ for all } t \in [0, \infty)$$

For a $t \in [0, \infty)$ define

$$x_t \in \mathbb{C}, \quad x_t(\theta) = x(t+\theta), \quad \theta \in [-r, 0] \quad (2.2)$$

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The objective is to design a static memoryless state feedback controller to stabilize the above system. For this purpose, consider an initial system of system (2.1)

$$\Sigma^{(2)}: \dot{x}(t) = A^{(2)}x(t) + A_d^{(2)}x(t-r) + B^{(2)}u(t) \quad (2.3)$$

which can be stabilized by a static memoryless state feedback controller.

Now consider a class of systems

$$\Sigma: = \lambda\Sigma^{(1)} + (1-\lambda)\Sigma^{(2)}, \lambda \in [0, 1]$$

That is

$$\begin{aligned} \Sigma: \dot{x}(t) = & [A^{(2)} + \lambda(A^{(1)}(t) - A^{(2)})]x(t) \\ & + [A_d^{(2)} + \lambda(A_d^{(1)}(t) - A_d^{(2)})]x(t-r) \\ & + [B^{(2)} + \lambda(B^{(1)}(t) - B^{(2)})]u(t) \end{aligned} \quad (2.4)$$

Let the static state feedback control law be

$$u(t) = Fx(t) \quad (2.5)$$

Then the closed-loop system is

$$\begin{aligned} \Sigma_{CL}: \dot{x}(t) = & \{ (A^{(2)} + B^{(2)}F) + \lambda[(A^{(1)}(t) \\ & - A^{(2)}) + (B^{(1)}(t) - B^{(2)})F] \} x(t) \\ & + [A_d^{(2)} + \lambda(A_d^{(1)}(t) - A_d^{(2)})]x(t-r) \end{aligned} \quad (2.6)$$

3. Stability of Time-Delay Systems

Consider the stability problem of time-delay system

$$\dot{x}(t) = A(t)x(t) + A_d(t)x(t-r) \quad (3.1)$$

The stability of system (3.1) can be investigated by choosing a quadratic Lyapunov functional candidate

$$V(x_t): \mathbb{C} \mapsto \mathfrak{R}$$

$$\begin{aligned} V(\phi) = & \frac{1}{2}\phi^T(0)P\phi(0) + \phi^T(0)\int_{-r}^0 Q(\xi)\phi(\xi)d\xi \\ & + \frac{1}{2}\int_{-r}^0 \int_{-r}^0 \phi^T(\xi)R(\xi-\eta)\phi(\eta)d\eta d\xi \\ & + \frac{1}{2}\int_{-r}^0 \phi^T(\xi)S(\xi)\phi(\xi)d\xi \end{aligned} \quad (3.2)$$

where

$$\begin{aligned} P & \in \mathfrak{R}^{n \times n}, \quad P = P^T \\ Q: [-r, 0] & \rightarrow \mathfrak{R}^{n \times n} \\ S: [-r, 0] & \rightarrow \mathfrak{R}^{n \times n}, \quad S^T(\xi) = S(\xi) \\ R: [-r, r] & \rightarrow \mathfrak{R}^{n \times n}, \quad R(-\xi) = R^T(\xi) \end{aligned}$$

Let the delay interval $[-r, 0]$ be divided into N segments $[\delta_{i-1}, \delta_i]$ of equal length h , where

$$\begin{aligned} \delta_i & = -r_m + ih, \quad i = 0, 1, 2, \dots, N \\ h & = r/N \end{aligned}$$

Choose Q, R, S and T to be continuous piecewise linear, i.e.

$$Q^i(\alpha) = Q(\delta_{i-1} + \alpha h) = (1-\alpha)Q_{i-1} + \alpha Q_i \quad (3.3a)$$

$$S^i(\alpha) = S(\delta_{i-1} + \alpha h) = (1-\alpha)S_{i-1} + \alpha S_i \quad (3.3b)$$

$$T^i(\alpha) = T(\delta_{i-1} + \alpha h) = (1-\alpha)T_{i-1} + \alpha T_i \quad (3.3c)$$

$$T_{kj}^i(\alpha) = T_{kj}(\delta_{i-1} + \alpha h) = (1-\alpha)T_{kj,i-1} + \alpha T_{kj,i} \quad (3.3d)$$

$$R^i(\alpha) = R(\delta_{i-1} + \alpha h) = (1-\alpha)R_{i-1} + \alpha R_i \quad (3.3e)$$

for $0 \leq \alpha \leq 1$. Then, the following results were proven in Gu (1997).

Lemma 1 (Gu,1997). For piecewise linear Q, S and R as described by (3.3), the Lyapunov functional satisfies

$$V(\phi) \geq \varepsilon \phi^T(0)\phi(0) \quad (3.4)$$

for some $\varepsilon > 0$ if

$$S_i \geq 0, \quad i = 0, 1, 2, \dots, N \quad (3.5)$$

$$\begin{bmatrix} \hat{R} & \hat{Q}^T \\ \hat{Q} & P \end{bmatrix} > 0 \quad (3.6)$$

where

$$\begin{aligned} \hat{R} & = \begin{bmatrix} R_0 & R_{-1} & \dots & R_{-N} \\ R_1 & R_0 & \dots & R_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ R_N & R_{N-1} & \dots & R_0 \end{bmatrix} \\ \hat{Q} & = [Q_0 \quad Q_1 \quad \dots \quad Q_N] \end{aligned}$$

Lemma 2 (Gu, 1997). For piecewise linear Q, S, R and T as described by (3.3), the derivative of the Lyapunov functional along the solution of (3.1) satisfies

$$\dot{V}(\phi) \leq -\varepsilon \phi^T(0)\phi(0) \quad (3.7)$$

for some $\varepsilon > 0$ if the following inequality is satisfied

$$\begin{bmatrix} \Delta^1 + T_{11,i+j-1} & -\Delta^2 + T_{12,i+j-1} & \Delta_{ij}^3 \\ -\Delta^{2T} + T_{12,i+j-1}^T & S_0 + T_{22,i+j-1} & \Delta_{ij}^4 \\ \Delta_{ij}^{3T} & \Delta_{ij}^{4T} & \frac{1}{N}(S_i - S_{i-1}) \end{bmatrix} > 0 \quad (3.8)$$

for all $i = 1, 2, \dots, N, j = 0, 1, [A(t), A_d(t)] \in \Omega_1$, where

$$\begin{aligned}\Delta^1 &= -[PA(t) + A^T(t)P + S_N + Q_N + Q_N^T] \\ \Delta^2 &= PA_d - Q_0 \\ \Delta_{ij}^3 &= hA^T(t)Q_{i-1+j} - (Q_i - Q_{i-1}) + hR_{i-N-1+j}^T \\ \Delta_{ij}^4 &= hA_d^T Q_{i-1+j} - hR_{i-1+j}^T\end{aligned}$$

and T_i ($i = 0, 1, 2, \dots, N$) satisfying

$$T_0 + T_N + 2 \sum_{i=1}^{N-1} T_i = 0 \quad (3.9)$$

4. Controller Synthesis

Choosing the same Lyapunov functional as (3.3) for system (2.6), by Lemmas 1 and 2, one can easily obtain the following results.

Corollary 1. The Lyapunov functional satisfies

$$V(\phi) \geq \varepsilon \phi^T(0) \phi(0) \quad (4.1)$$

for some $\varepsilon > 0$ if

$$S_i \geq 0, \quad i = 0, 1, 2, \dots, N \quad (4.2)$$

$$M(Z) = \begin{bmatrix} \hat{R} & \hat{Q}^T \\ \hat{Q} & P \end{bmatrix} > 0 \quad (4.3)$$

where $Z = [P, Q, R, S, T]^T$.

Corollary 2. The derivative of the Lyapunov functional along the solution of (2.6) satisfies

$$\dot{V}(\phi) \leq -\varepsilon \phi^T(0) \phi(0) \quad (4.4)$$

for some $\varepsilon > 0$ if the following inequality is satisfied

$$H(Z, F, \lambda) = \begin{bmatrix} \Delta_{\Sigma}^1 + T_{11,i+j-1} & -\Delta_{\Sigma}^2 + T_{12,i+j-1} & \Delta_{\Sigma ij}^3 \\ -\Delta_{\Sigma}^{2T} + T_{12,i+j-1}^T & S_0 + T_{22,i+j-1} & \Delta_{\Sigma ij}^4 \\ \Delta_{\Sigma ij}^{3T} & \Delta_{\Sigma ij}^{4T} & \frac{1}{N}(S_i - S_{i-1}) \end{bmatrix} > 0 \quad (4.5)$$

for all $i = 1, 2, \dots, N$, $j = 0, 1$, $[A^{(1)}(t), A_d^{(1)}(t), B^{(1)}(t)] \in \Omega$,

where

$$\begin{aligned}\Delta_{\Sigma}^1 &= -\{P(A^{(2)} + B^{(2)}F) + \lambda P[(A^{(1)}(t) - A^{(2)}) \\ &\quad + (B^{(1)}(t) - B^{(2)}F) + (A^{(2)T} + F^T B^{(2)T})P \\ &\quad + \lambda[(A^{(1)T}(t) - A^{(2)T}) + F^T(B^{(1)T}(t) - B^{(2)T})]P \\ &\quad + S_N + Q_N + Q_N^T\}\end{aligned}$$

$$\begin{aligned}\Delta_{\Sigma}^2 &= PA_d^{(2)} + \lambda P[A_d^{(1)}(t) - A_d^{(2)}] - Q_0 \\ \Delta_{\Sigma ij}^3 &= h(A^{(2)} + B^{(2)}F)^T Q_{i-1+j} + \lambda h[(A^{(1)}(t) - A^{(2)}) \\ &\quad + (B^{(1)}(t) - B^{(2)}F)^T] Q_{i-1+j} - (Q_i - Q_{i-1}) + hR_{i-N-1+j}^T \\ \Delta_{\Sigma ij}^4 &= hA_d^{(2)T} Q_{i-1+j} + \lambda h[A_d^{(1)T}(t) - A_d^{(2)T}] Q_{i-1+j} - hR_{i-1+j}^T\end{aligned}$$

and T_i ($i = 0, 1, 2, \dots, N$) satisfying

$$T_0 + T_N + 2 \sum_{i=1}^{N-1} T_i = 0 \quad (4.6)$$

Now, by Corollaries 1 and 2, we have the stabilization result for system (2.4).

Theorem 1. System (2.4) is robustly stabilizable via a linear state feedback controller (2.5) if there exist real matrices P , \hat{Q} , and \hat{R} satisfying (4.3) and $S_0 \geq 0$, and T_i $i = 0, 1, 2, \dots, N$ satisfying (4.6) such that (4.5) is satisfied.

Remark 1. Condition (4.2) (i.e. $S_i \geq 0$, $i = 0, 1, 2, \dots, N$) is implied by $S_0 \geq 0$ and (4.5).

Remark 2. From (2.4), it is apparent to see that if the parameter $\lambda \rightarrow 1$, system (2.4) approaches the original system (2.1).

From Theorem 1 and Remark 2, for the robust stabilization of system (2.1)-(2.2), the following result is easily obtained.

Theorem 2. System (2.1)-(2.2) is robustly stabilizable via a linear state feedback controller (2.5) if there exist real matrices P , \hat{Q} , and \hat{R} satisfying (4.3) and $S_0 \geq 0$, and T_i $i = 0, 1, 2, \dots, N$ satisfying (4.6) such that the following inequality is satisfied

$$H(Z, F, 1) = \begin{bmatrix} \Delta_{\Sigma 1}^1 + T_{11,i+j-1} & -\Delta_{\Sigma 1}^2 + T_{12,i+j-1} & \Delta_{\Sigma 1 ij}^3 \\ -\Delta_{\Sigma 1}^{2T} + T_{12,i+j-1}^T & S_0 + T_{22,i+j-1} & \Delta_{\Sigma 1 ij}^4 \\ \Delta_{\Sigma 1 ij}^{3T} & \Delta_{\Sigma 1 ij}^{4T} & \frac{1}{N}(S_i - S_{i-1}) \end{bmatrix} > 0 \quad (4.7)$$

for all $i = 1, 2, \dots, N$, $j = 0, 1$, $[A^{(1)}(t), A_d^{(1)}(t), B^{(1)}(t)] \in \Omega$,

where

$$\begin{aligned}\Delta_{\Sigma 1}^1 &= -\{P[A^{(1)}(t) + B^{(1)}(t)F] + [A^{(1)T}(t) + F^T B^{(1)T}(t)]P \\ &\quad + S_N + Q_N + Q_N^T\} \\ \Delta_{\Sigma 1}^2 &= PA_d^{(1)}(t) - Q_0 \\ \Delta_{\Sigma 1 ij}^3 &= h[A^{(1)}(t) + B^{(1)}(t)F]^T Q_{i-1+j} - (Q_i - Q_{i-1})\end{aligned}$$

$$\Delta_{\Sigma 1ij}^4 = hA_d^{(1)T}(t)Q_{i-1+j} - hR_{i-1+j}^T + hR_{i-N-1+j}^T$$

The LMIs derived need to hold for all the possible uncertain system matrices. Here, we assume that the uncertainty set is polytopic

$$\Omega = Co\{(A_j, B_j) \mid j=1, 2, \dots, n_v\} \quad (4.8)$$

Then it is easy to see that one only needs to check the satisfaction of LMIs at all the vertices.

Now we give the idea for designing the static state feedback controller $u(t) = Fx(t)$. First we set $\lambda = 0$. From the initial system (2.3) (it is supposed to be stabilized) one obtains a proper gain matrix $F^{(0)}$ such that the closed-loop system of system (2.3) with $u(t) = F^{(0)}x(t)$ is asymptotically stabilized. From (4.2)-(4.3) and (4.5), a group of matrices $Z^{(0)} = [P^{(0)}, Q^{(0)}, R^{(0)}, S^{(0)}, T^{(0)}]^T$ are determined. Second for the $F^{(k)}$ and $\lambda^{(k)}$ given in the previous step, one can construct a generalized eigenvalue problem to determine new $Z^{(k+1)}$ and $\Delta\lambda^{(k)1} > 0$. If 1 belongs to the interval $[\lambda^{(k)}, \lambda^{(k)1}]$, where $\lambda^{(k)1} = \lambda^{(k)} + \Delta\lambda^{(k)1}$, the static state feedback controller $u(t) = F^{(k)}x(t)$ robustly stabilizes system (2.1)-(2.2). Third if it is not the case, fixing the new derived $Z^{(k+1)}$ and $\lambda^{(k)1}$, one can find new $F^{(k+1)}$ and $\Delta\lambda^{(k)2} > 0$ that solve a generalized eigenvalue problem. If 1 belongs to the new interval $[\lambda^{(k)1}, \lambda^{(k+1)}]$, where $\lambda^{(k+1)} = \lambda^{(k)1} + \Delta\lambda^{(k)2}$, the static state feedback controller $u(t) = F^{(k)}x(t)$ for system (2.1)-(2.2) is obtained. Otherwise repeat the above process until one get a static state feedback controller. If the interval $[\lambda^{(k)}, \lambda^{(k)1}]$ or $[\lambda^{(k)1}, \lambda^{(k+1)}]$ that includes 1 is not obtained, the procedure fails.

Based on Remark 2, Theorem 2 and the above discussion, we will give a design procedure. First of all, we introduce some notations as follows.

Let $\lambda^{(k+1)} - \lambda^{(k)} = \Delta\lambda^{(k)} = \Delta\lambda^{(k)1} + \Delta\lambda^{(k)2}$, where $\Delta\lambda^{(k)1} > 0$ and $\Delta\lambda^{(k)2} > 0$

Denote

$$M(Z^{(k)}) = \begin{bmatrix} \hat{R}^{(k)} & \hat{Q}^{(k)T} \\ \hat{Q}^{(k)} & P^{(k)} \end{bmatrix}$$

$$G(Z^{(k)}, F^{(k)}, \lambda^{(k)}) = \begin{bmatrix} G^{(k)1} & G^{(k)2} & G_{ij}^{(k)3} \\ G^{(k)2T} & 0 & G_{ij}^{(k)4} \\ G_{ij}^{(k)3T} & G_{ij}^{(k)4T} & 0 \end{bmatrix}$$

$$H(Z^{(k)}, F^{(k)}, \lambda^{(k)}) =$$

$$\begin{bmatrix} \Delta_{\Sigma}^{(k)1} + T_{11,i+j-1} & -\Delta_{\Sigma}^{(k)2} + T_{12,i+j-1} & \Delta_{\Sigma ij}^{(k)3} \\ -\Delta_{\Sigma}^{(k)2T} + T_{12,i+j-1}^T & S_0^{(k)} + T_{22,i+j-1} & \Delta_{\Sigma ij}^{(k)4} \\ \Delta_{\Sigma ij}^{(k)3T} & \Delta_{\Sigma ij}^{(k)4T} & \frac{1}{N}(S_i^{(k)} - S_{i-1}^{(k)}) \end{bmatrix}$$

where

$$\hat{R}^{(k)} = \begin{bmatrix} R_0^{(k)} & R_{-1}^{(k)} & \dots & R_{-N}^{(k)} \\ R_1^{(k)} & R_0^{(k)} & \dots & R_{-N+1}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ R_N^{(k)} & R_{N-1}^{(k)} & \dots & R_0^{(k)} \end{bmatrix}$$

$$\hat{Q}^{(k)} = [Q_0^{(k)} \quad Q_1^{(k)} \quad \dots \quad Q_N^{(k)}]$$

and

$$\Delta_{\Sigma}^{(k)1} = -\{P^{(k)}(A^{(2)} + B^{(2)}F^{(k)}) + \lambda^{(k)}P^{(k)}[(A^{(1)}(t) - A^{(2)}) + (B^{(1)}(t) - B^{(2)})F^{(k)}] + (A^{(2)T} + F^{(k)T}B^{(2)T})P^{(k)} + \lambda^{(k)}[(A^{(1)T}(t) - A^{(2)T}) + F^{(k)T}(B^{(1)T}(t) - B^{(2)T})]P^{(k)} + S_N^{(k)} + Q_N^{(k)} + Q_N^{(k)T}\}$$

$$\Delta_{\Sigma}^{(k)2} = P^{(k)}A_d^{(2)} + \lambda^{(k)}P^{(k)}[A_d^{(1)}(t) - A_d^{(2)}] - Q_0^{(k)}$$

$$\Delta_{\Sigma ij}^{(k)3} = h(A^{(2)} + B^{(2)}F^{(k)})^T Q_{i-1+j}^{(k)} + \lambda^{(k)}h[(A^{(1)}(t) - A^{(2)}) + (B^{(1)}(t) - B^{(2)})F^{(k)}]^T Q_{i-1+j}^{(k)} - (Q_i^{(k)} - Q_{i-1}^{(k)}) + hR_{i-N-1+j}^{(k)T}$$

$$\Delta_{\Sigma ij}^{(k)4} = hA_d^{(2)T} Q_{i-1+j}^{(k)} + \lambda^{(k)}h[A_d^{(1)T}(t) - A_d^{(2)T}] Q_{i-1+j}^{(k)} - hR_{i-1+j}^{(k)T}$$

and

$$G^{(k)1} = P^{(k)}[(A^{(1)}(t) - A^{(2)}) + (B^{(1)}(t) - B^{(2)})F^{(k)}] + [(A^{(1)T}(t) - A^{(2)T}) + F^{(k)T}(B^{(1)T}(t) - B^{(2)T})]P^{(k)}$$

$$G^{(k)2} = P^{(k)}[A_d^{(1)}(t) - A_d^{(2)}]$$

$$G_{ij}^{(k)3} = -h[(A^{(1)}(t) - A^{(2)}) + (B^{(1)}(t) - B^{(2)})F^{(k)}]^T Q_{i-1+j}^{(k)}$$

$$G_{ij}^{(k)4} = -h[A_d^{(1)T}(t) - A_d^{(2)T}] Q_{i-1+j}^{(k)}$$

Now we state the algorithm for designing the static state feedback controller $u(t) = Fx(t)$ as follows.

Algorithm 4.1

Step 1. Let $\lambda^{(0)} = 0$, design $F^{(0)}$ to stabilize $\Sigma^{(2)}$. From $0 \leq S_0$, $0 < M(Y^{(0)})$ and $0 < H(Z^{(0)}, F^{(0)}, \lambda^{(0)})$, one obtains $Z^{(0)} = [P^{(0)}, Q^{(0)}, R^{(0)}, S^{(0)}, T^{(0)}]^T$.

Step 2. For $F^{(k)}$ and $\lambda^{(k)}$ given in the previous step, find new $Z^{(k+1)}$ and $\Delta\lambda^{(k)1}$ that solve the generalized eigenvalue problem

$$\min_{Z^{(k)}} \frac{1}{\Delta\lambda^{(k)1}} \quad \text{subject to:}$$

$$0 \leq S_0; 0 < M(Z^{(k)}); 0 < H(Z^{(k)}, F^{(k)}, \lambda^{(k)})$$

$$G(Z^{(k)}, F^{(k)}, \lambda^{(k)}) < \frac{1}{\Delta\lambda^{(k)1}} H(Z^{(k)}, F^{(k)}, \lambda^{(k)})$$

If $1 \in [\lambda^{(k)1}, \lambda^{(k+1)}]$, where $\lambda^{(k+1)} = \lambda^{(k)} + \Delta\lambda^{(k)1}$, one has the static state feedback control law $u(t) = F^{(k)}x(t)$, then stop. Otherwise go to Step 3.

Step 3. For $Z^{(k+1)}$ and $\lambda^{(k+1)}$ derived in Step 2, find new $F^{(k+1)}$ and $\Delta\lambda^{(k)2}$ that solve the generalized eigenvalue problem

$$\begin{aligned} \min_{F^{(k)}} \frac{1}{\Delta\lambda^{(k)2}} \quad \text{subject to:} \\ 0 \leq S_0; 0 < M(Z^{(k+1)}) \\ 0 < H(Z^{(k+1)}, F^{(k)}, \lambda^{(k)1}) \end{aligned}$$

$$G(Z^{(k+1)}, F^{(k)}, \lambda^{(k)1}) < \frac{1}{\Delta\lambda^{(k)2}} H(Z^{(k+1)}, F^{(k)}, \lambda^{(k)1})$$

If $1 \in [\lambda^{(k)1}, \lambda^{(k+1)}]$, where $\lambda^{(k+1)} = \lambda^{(k)1} + \Delta\lambda^{(k)2} = \lambda^{(k)} + \Delta\lambda^{(k)1} + \Delta\lambda^{(k)2}$, the static state feedback control law $u(t) = F^{(k+1)}x(t)$ is obtained, then stop. Otherwise let $k = k + 1$, go to Step 2.

Remark 3. Concerning the choice of the initial system, we can choose a rough approximation of the original system, and a well-established method may be used to design the controller of the initial system, for example, choosing the Pade approximation.

Remark 4. By the results in Gu (1999b), Algorithm 4.1 can be easily extended to synthesize the static state feedback controller for uncertain linear systems with multiple time-delays.

5. Illustrate Examples

Example 1. Consider the following time-delay system

$$\dot{x}(t) = A^{(1)}(t)x(t) + A_d^{(1)}(t)x(t-r) + B^{(1)}(t)u(t) \quad (5.1)$$

where

$$\begin{aligned} A^{(1)}(t) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_d^{(1)}(t) = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix} \\ B^{(1)}(t) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad r = 1.45 \end{aligned}$$

Note that $A^{(1)}(t) + A_d^{(1)}(t) = \begin{bmatrix} -1 & -1 \\ 0 & 0.1 \end{bmatrix}$ is unstable and $\{A^{(1)}(t), B^{(1)}(t)\}$ is not stabilizable, system (5.1) can not be stabilized independent of delay using the state feedback controller. Therefore, the delay-independent stabilization methods in Phoojaruenchanachai and Furuta (1992) and Shen *et al.* (1991) can not be applied to this system.

Choose the initial system of (5.1) as

$$\dot{x}(t) = A^{(2)}x(t) + A_d^{(2)}x(t-r) + B^{(2)}u(t) \quad (5.2)$$

where

$$\begin{aligned} A^{(2)} &= A^{(1)} + A_d^{(1)} = \begin{bmatrix} -1 & -1 \\ 0 & 0.1 \end{bmatrix} \\ A_d^{(2)} &= 0, \quad B^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad r = 1.45 \end{aligned}$$

Using the pole assignment technique, one easily find the controller gain matrix $F^{(0)} = [-0.6 \quad -1]$ for system (5.2). Applying Algorithm 4.1, we obtain the state feedback control law $u(t) = Fx(t) = [-0.6017 \quad -1.0069]x(t)$ that asymptotically stabilizes system (5.1).

Example 2. Consider the following uncertain time-delay system provided in de Souza and Li (1999) (with a slight modification)

$$\dot{x}(t) = A^{(1)}(t)x(t) + A_d^{(1)}(t)x(t-r) + B^{(1)}(t)u(t) \quad (5.3)$$

where

$$\begin{aligned} A_d^{(1)}(t) &= \begin{bmatrix} -1 - \rho(t) & -1 - \rho(t) \\ -\rho(t) & -0.9 - \rho(t) \end{bmatrix} \\ A^{(1)}(t) &= \begin{bmatrix} \rho(t) & \rho(t) \\ \rho(t) & 1 + \rho(t) \end{bmatrix}, \quad B^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad r = 1 \end{aligned}$$

where

$$|\rho(t)| \leq 0.1, \quad \forall t$$

The system matrices $A^{(1)}(t)$ and $A_d^{(1)}(t)$ can be written as

$$A^{(1)}(t) = A^{(1)} + \Delta A^{(1)}(t), \quad A_d^{(1)}(t) = A_d^{(1)} + \Delta A_d^{(1)}(t)$$

and

$$A^{(1)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Delta A^{(1)}(t) = \begin{bmatrix} \rho(t) & \rho(t) \\ \rho(t) & \rho(t) \end{bmatrix}$$

$$A_d^{(1)} = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}, \quad \Delta A_d^{(1)}(t) = \begin{bmatrix} -\rho(t) & -\rho(t) \\ -\rho(t) & -\rho(t) \end{bmatrix}$$

where the uncertain matrices $\Delta A^{(1)}(t)$ and $\Delta A_d^{(1)}(t)$ are a case of any unknown matrices $\Delta A^{(1)}(t)$ and $\Delta A_d^{(1)}(t)$ satisfying $\|\Delta A^{(1)}(t)\| \leq 0.2$ and $\|\Delta A_d^{(1)}(t)\| \leq 0.2$, $\forall t$, given in de Souza and Li (1999).

For any unknown matrices $\Delta A^{(1)}(t)$ and $\Delta A_d^{(1)}(t)$ satisfying $\|\Delta A^{(1)}(t)\| \leq 0.2$ and $\|\Delta A_d^{(1)}(t)\| \leq 0.2$, $\forall t$, using the result of de Souza and Li (1999), system (5.3) is

robustly stabilizable for $0 \leq r \leq 0.3346$, whereas by the results in Li and de Souza (1997), Niculescu *et al.* (1994) bound for time-delay is 0.2250 and 0.0929, respectively. However, for the time-delay is $r=1$ in system (5.3), no conclusion can be made by the criteria in de Souza and Li (1999), Li and de Souza (1997), Niculescu *et al.* (1994).

Now we apply the design algorithm in this paper to solve this problem. System (5.3) can be modeled as a polytopic system, with $n_v = 2$ and

$$A_1^{(1)} = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 1.1 \end{bmatrix}, A_2^{(1)} = \begin{bmatrix} -0.1 & -0.1 \\ -0.1 & 0.9 \end{bmatrix}$$

$$A_{d1}^{(1)} = \begin{bmatrix} -1.1 & -1.1 \\ -0.1 & -1 \end{bmatrix}, A_{d2}^{(1)} = \begin{bmatrix} -0.9 & -0.9 \\ 0.1 & -0.8 \end{bmatrix}$$

$$B_1^{(1)} = B_2^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We will also choose system (5.2) as the initial system of system (5.3) with the controller gain matrix $F^{(0)} = [-0.6 \quad -1]$. Applying Algorithm 4.1, it is found that system (5.3) is robustly stabilizable via the state feedback control law $u(t) = Fx(t) = [-0.6338, -1.0619] \times x(t)$. For this example, the controller design method of this paper provides a less conservative result than those obtained via the methods of de Souza and Li (1999), Li and de Souza (1997), Niculescu *et al.* (1994).

Remark 5. The controllers are obtained for $N=1$ in Examples 1 and 2. As N increases, one will get a better result.

6. Conclusion

The problem of robust stabilization of uncertain time-delay system has been addressed. Based on the *discretized Lyapunov functional* approach, we have developed a new controller design method. Numerical examples have been presented to demonstrate the effectiveness of the proposed method.

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