Performance Limitations in the Robust Servomechanism Problem for Discrete Time Periodic Systems

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Abstract

Fundamental limitations for error tracking/regulation are obtained for the robust servomechanism problem (RSP) for discrete time periodic systems. In studying this problem, the RSP for a multi-input/multi-output discrete time system is considered. Application of these results is then made to the "periodic system robust servomechanism problem", and explicit expressions for the limiting costs for error tracking regulation are obtained. These limitations can be characterized completely by the number and location of the non-minimum phase transmission zeros.

Keywords Discrete periodic systems; proper systems; cheap control problem; optimal control; performance limitations; servomechanism problem; non-minimum phase; lifting thechnique;

1 Introduction

The analysis and control of linear discrete-time periodic systems have recently been receiving a good deal of attention because linear periodic systems, such as cyclostationary processes [1, 2], and multirate digital control systems [3, 4] arise very often in nature and engineering.

Common techniques used in carrying out such analysis studies are the lifting technique and the cyclic technique, techniques which transform the linear discrete-time periodic system into a linear time invariant system [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] while preserving the algebraic structure [15, 8, 14] and norms of the system [8]. In view of the bijective relationship between the periodic discrete time realization and its time-invariant reformulation, structural properties such as observability, reachability, detectability, and stabilizability are equivalently analyzed by making reference to the time-invariant reformulation [9, 16]. Lifting techniques, in particular, have been used to study zeros [7], robust stabilization [6, 8], pole assignment [9], and state and output feedback stabilization [17, 18, 19] for discrete-time linear periodic systems.

The tracking and disturbance rejection problem for discrete-time linear periodic systems has been considered in [19, 20, 21, 22, 23, 24, 25, 26]. In [19, 20] only periodic and constant reference signals are considered, while in [21] general classes of disturbance functions and reference signals are considered. In [22], the disturbance rejection problem has been studied using a geometric approach, and necessary and sufficient conditions to solve this problem are provided. In [23, 24, 25, 26], necessary and sufficient conditions for the existence of state and/or output dead-beat controllers with the requirement of output dead-beat regulation under periodic disturbances are obtained.

The optimal periodic control problem and optimal prediction problem have been considered in [27, 28] and in [29, 30, 31] respectively. In these papers, a solution for the difference periodic Riccati equation has been analyzed, and necessary and sufficient conditions for a unique solution to exist have been given; in addition, a one-to-one correspondence between this solution and the solution of the discrete time invariant Riccati equation has been established, and a numerical method for computing the periodic solution has been presented.

Fundamental performance limitations for tracking/regulation, in the robust servomechanism problem (RSP) for discrete systems, when no penalty is imposed on the control signal energy of the closed loop system, have been obtained in [32, 33]. In this paper, performance limitations in the RSP is considered for periodic discrete time systems.

2 Preliminary results and Development

Consider the following linear discrete-time periodic system Σ described by

$$\begin{array}{rcccc} x(t+1) & = & A(t)x(t) & + & B(t)(u(t)+\omega(t)) \\ y(t) & = & C(t)x(t) & + & \eta(t) \\ e(t) & = & y(t) & - & y_{ref} \end{array} \quad (1)$$

where the control $u(t)$, the state $x(t)$, the output $y(t)$, the input disturbance $\omega(t)$, and the output disturbance $\eta(t)$ are periodic real vectors in $\mathbf{R}^m, \mathbf{R}^n, \mathbf{R}^m, \mathbf{R}^m$ and $\mathbf{R}^m$ with period $T \in \mathbf{N}$ respectively, and where the matrices $A(t), B(t)$, and $C(t)$ are periodic with period T, i.e., for all t, $A(t) = A(t+T)$, $B(t) = B(t+T)$, and $C(t) = C(t+T)$. Assume that the system (1) is point-wise detectable and stabilizable, i.e., it the matrix pair $[A(t),\ B(t)]$ is stabilizible and the matrix pair $[A(t),\ C(t)]$ is detectable for all fixed $t \in [0,\ T]$.

2.1 Associated Lifted Time-Invariant Systems

Consider the linear periodic discrete time system Σ (1); then for any initial time $\tau \in \mathbf{Z}$, the output response of system Σ for $k \geq \tau$, for a given initial state $x(\tau)$ and input function $u(.)$, can be obtained from the time-invariant associated system [4, 8, 15] at time τ, Σ_τ where Σ_τ is given by:

$$\begin{array}{rclcl} \bar{x}_{k+1} & = & \bar{A}\bar{x}_k & + & \bar{B}(\bar{u}_k + \bar{\omega}_k) \\ \bar{y}_k & = & \bar{C}_k\bar{x}_k & + & \bar{D}\bar{u}_k + \bar{\eta}_k \\ \bar{e}_k & = & \bar{y}_k & - & \bar{y}_{ref} \end{array} \quad (2)$$

where $k \in \mathbf{N}$, $\bar{x}_k = x(kT+\tau)$,

$$\bar{u}_k = \left[\, u(kT+\tau) \quad u(kT+1+\tau) \quad \ldots \quad u(kT+T-1+\tau) \,\right]'$$

$$\bar{y}_k = \left[\, y(kT+\tau)' \quad y(kT+1+\tau)' \quad \ldots \quad y(kT+T-1+\tau)' \,\right]'$$

$$\bar{e}_k = \left[\, e(kT+\tau)' \quad e(kT+1+\tau)' \quad \ldots \quad e(kT+T-1+\tau)' \,\right]'$$

$$\bar{A} = \Phi(T+\tau, \tau)$$

$$\bar{B} = \left[\ \Phi(T+\tau, 1+\tau)B(\tau) \quad \Phi(T+\tau, 2+\tau)B(1+\tau) \quad \ldots \quad \Phi(T+\tau, T-1+\tau)B(T-2+\tau) \quad B(T-1+\tau)\ \right]$$

$$\bar{C} = \left[\ C(\tau) \quad \Phi(1+\tau,\tau)'C(1+\tau) \quad \ldots \quad \Phi(T-2+\tau,\tau)'C(T-2+\tau) \quad \Phi(T-1+\tau,\tau)'C(T-1+\tau)\ \right]'$$

$\bar{D} = d_{ij}$ where

$$d_{ij} = \begin{cases} C(i-1+\tau)'\,\Phi(i-1+\tau, j+\tau)B(j-1+\tau) & i > j \\ 0 & i \leq j. \end{cases} \quad (3)$$

where $\Phi(t, t_o)$ is the system transition matrix given by:

$$\begin{array}{rcl} \Phi(t, t_o) & := & A(t-1)A(t-2)\ldots A(t_o) \\ \Phi(t,t) & := & I. \end{array}$$

2.2 Stability

The eigenvalues of $\bar{\Phi} := \Phi(T+\tau, \tau)$, called the monodromy matrix, are independent of τ and are called the *characteristic multipliers of A(.)* [34, 35]. The periodic system (1) is asymptotically stable if and only if its characteristic multipliers have modulus less than 1 [16]. Hence, system (1) is asymptotically stable if and only if system (2) is asymptotically stable.

2.3 Stabilizability and Detectability of Periodic Systems

In view of the bijective relationship between the periodic discrete time realization (1) and its time-invariant reformulation (2), structural properties such as observability, reachability, detectability, and stabilizability can be equivalently analyzed by making reference to the time-invariant reformulation [9, 16]. Thus, we have:

Lemma 2.1 *1) The pair $(A(.), B(.))$ is stabilizable if and only if the pair $(\bar{A}, \bar{B})$ is stabilizable.*

2) The pair $(A(.), C(.))$ is detectable if and only if the pair $(\bar{A}, \bar{C})$ is detectable.

It is to be noted that stabilizability and detectability does not depend on the initial sampling time τ.

2.4 State Feedback

Suppose that the state of system (1) is measurable, and consider a control law

$$\bar{u}(k) = \bar{K}\bar{x}(k) \quad (4)$$

for system (2), where $\bar{K} = \left[\ \bar{K}_1' \quad \bar{K}_2' \quad \ldots \quad \bar{K}_T'\ \right]'$, $\bar{K}_i \in \mathbf{R}^{m\times n}$, $i = 1, 2, \ldots, T$.

Recalling the meaning of $\bar{u}(k)$, and $\bar{x}(k)$, and on letting $K(.):\ \mathbf{Z} \rightarrow \mathbf{R}^{m\times n}$ be a T-periodic matrix function such that:

$$K(\tau + i) = \bar{K}_{i+1} \quad i = 0, 1, 2, \ldots, T-1 \quad (5)$$

the corresponding control law for system (1) is given as follows [18]:

$$u(t) = K(t)x(\tau+kT),\ t \in [\tau+kT, \ldots, \tau+kT+T-1] \quad (6)$$

2.5 Transmission Zeros

The definition and the properties of the transmission zeros of discrete periodic systems have been studied in [7, 36, 37].

Lemma 2.2 *[36, 7] If the periodic system (1) is reachable and observable at time k, the transmission zeros at time k, with their multiplicities, coincide with the transmission zeros of the corresponding lifted system (2).*

The time dependence of the transmission zeros of (1) has been studied in [36, 37], and the results are obtained as follows:

Lemma 2.3 *[36, 37] The nonzero transmission zeros of (1) at time k are independent of k, together with their multiplicities.*

3 RSP for LTI Systems

Consider the square LTI system modeled by (2), and let r be the number of outputs of the system; the following results are now obtained:

Lemma 3.1 *[38] There exists a solution to the RSP for (2) iff the following conditions are all satisfied:*
(i) $(\bar{C}, \bar{A}, \bar{B})$ *is stabilizable and detectable*

(ii) $rank \begin{bmatrix} \bar{A}-I & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} = n + r$

(iii) y_k *is measurable.*

Given the linear time invariant system Σ_τ (2), assume that the conditions in lemma 3.1 are all satisfied; then a solution to the RSP for Σ_τ (2) can be obtained by finding a state feedback controller which minimizes the following cheap performance index:

$$\bar{J}_\epsilon = h \sum_{k=o}^{\infty} \left\{ \bar{e}_{k-1}' \bar{e}_{k-1} + \epsilon (\bar{u}_k - \bar{u}_{k-1})' (\bar{u}_k - \bar{u}_{k-1}) \right\} \quad (7)$$

where $\bar{e}_k := \bar{y}_k - \bar{y}_{ref}$ is the error in the system Σ_τ, and $\epsilon > 0$, subject to the constraint that:

$$\begin{aligned} \begin{bmatrix} \bar{x}_{k+1} - \bar{x}_k \\ \bar{e}_k \end{bmatrix} &= \begin{bmatrix} \bar{A} & 0 \\ \bar{C} & I \end{bmatrix} \begin{bmatrix} \bar{x}_k - \bar{x}_{k-1} \\ \bar{e}_{k-1} \end{bmatrix} + \begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix} (\bar{u}_k - \bar{u}_{k-1}) \\ \bar{e}(k-1) &= \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \bar{x}_k - \bar{x}_{k-1} \\ \bar{e}_{k-1} \end{bmatrix} \end{aligned} \quad (8)$$

Let this state feedback controller be given by:

$$\bar{u}(k) = \bar{K}[(\bar{x}_k - \bar{x}_{k-1})' \bar{e}_{k-1}']' \quad (9)$$

where $\bar{K}$ is the optimal cheap control gain for the state space representation (8).

The following preliminary result is needed to obtain an explicit expression for the limiting optimal "cheap control" cost for non-minimum phase systems.

Lemma 3.2 *[39] A non-minimum phase right-invertible transfer function matrix* F *can always be factorized as* $F = F_1 F_2$ *such that* F_1 *is inner,* F_2 *is minimum phase and right-invertible, and such that the unstable poles of* F_2 *are equal to the unstable poles of* F*. Moreover if* F *is proper, then* F_2 *is proper.*

In lemma 3.2, given a transfer function matrix F then F_2 is called the minimum phase counterpart of F.

Lemma 3.3 *Given a discrete time non-minimum phase system* $(\bar{C}, \bar{A}, \bar{B}, \bar{D})$ *(2) which has* p *non-minimum phase transmission zeros* λ_i, $i = 1, 2, \ldots, p$ *(including multiplicities), then a state space presentation of the minimum phase counterpart [39] of* $(\bar{C}, \bar{A}, \bar{B}, \bar{D})$ *is given by the* $(C_m, \bar{A}, \bar{B}, D_m)$ *system, where* C_m *and* D_m *are given by:*

$$\begin{aligned} C_m^0 &= \bar{C}, \ D_m^0 = \bar{D} \\ C_m^i &= C_m^{i-1} - \left(\frac{\lambda_i \bar{\lambda}_i - 1}{\bar{\lambda}_i + 1} \right) \bar{v}_i u_i' (\bar{A} + I) \\ D_m^i &= (I + \frac{\lambda_i \bar{\lambda}_i - 1}{\bar{\lambda}_i + 1} \bar{v}_i v_i') D_m^{i-1}, \ i = 1, 2, \ldots, p \\ C_m &= C_m^p, \ \text{and } D_m = D_m^p, \end{aligned} \quad (10)$$

where the vector v_i *is given by:*

$$[u_i' \ v_i'] \begin{bmatrix} \lambda_i I - \bar{A} & -\bar{B} \\ -C_m^{i-1} & -D_m^{i-1} \end{bmatrix} = 0 \quad (11)$$

and v_i *is normalized, i.e.* $\bar{v}_i' v_i = 1$.

The following preliminary result is needed to obtain an explicit expression for the limiting optimal cheap control cost for systems when $\bar{D}$ is not invertible.

Lemma 3.4 *Given a linear time invariant system* $(\bar{C}, \bar{A}, \bar{B}, \bar{D})$ *(2), assume that* $\bar{D} \in \mathbf{R}^{m \times m}$ *is not invertible; then there exist input/output unitary transformations* T_1 *and* T_2 *such that the system matrices of the resulting system have the following structure:*

$$\tilde{A} = \bar{A}, \ \tilde{B} = [\tilde{B}_1 \ \ \tilde{B}_2], \ \tilde{C} = \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix}, \ \text{and } \tilde{D} = \begin{bmatrix} \tilde{D}_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $\tilde{D}_1 \in \mathbf{R}^{m_1 \times m_1}$ *and* $\tilde{D}_1$ *is invertible, and where* $\tilde{C}_1 \in \mathbf{R}^{m_1 \times n}$, $\tilde{C}_2 \in \mathbf{R}^{(r-m_1) \times n}$, $\tilde{B}_1 \in \mathbf{R}^{n \times m_1}$, $\tilde{B}_2 \in \mathbf{R}^{n \times (r-m_1)}$ *, and* $m_1 < r$.

Assume now that the discrete LTI system (2) has $\rho \in [1, n]$ transmission zeros, with $p \in [0, \rho]$ non-minimum phase transmission zeros given by λ_i, $i = 1, 2, ..., p$, and that $\bar{D}$ is not invertible. Let $(C_m, \bar{A}, \bar{B}, D_m)$ be the minimum phase counterpart of the discrete LTI system (2) defined in lemma 3.3. Then input/output unitary transformations T_1, and T_2 can be applied to the $(C_m, \bar{A}, \bar{B}, D_m)$ system so that the resulting system has the structure given in lemma 3.4, i.e.

$$\begin{aligned} \tilde{A} &= \bar{A}, & \tilde{B} &= [\tilde{B}_1 \ \ \tilde{B}_2] := \bar{B} T_1', \\ \tilde{C} &= \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix} := T_2 C_m, & \tilde{D} &= \begin{bmatrix} \tilde{D}_1 & 0 \\ 0 & 0 \end{bmatrix} := T_2 D_m T_1', \end{aligned} \quad (12)$$

where $\tilde{D}_1 \in \mathbf{R}^{m_1 \times m_1}$ and $\tilde{D}_1$ is invertible, and where $\tilde{C}_1 \in \mathbf{R}^{m_1 \times n}$, $\tilde{C}_2 \in \mathbf{R}^{(m-m_1) \times n}$, $\tilde{B}_1 \in \mathbf{R}^{n \times m_1}$, $\tilde{B}_2 \in \mathbf{R}^{n \times (m-m_1)}$, and $m_1 < m$.

Theorem 3.1 *Given a square linear time invariant discrete time system* $(\bar{C}, \bar{A}, \bar{B}, \bar{D})$ *(2), with* r *outputs, which has the property that there exists a solution to the RSP and is non-minimum phase, consider the optimal controller which minimizes the performance index (7); then:*

(a) if $\bar{x}(0) = 0$, $\bar{y}_{ref} \neq 0$, $\bar{\eta} \neq 0$, *and* $\bar{\omega} = 0$, *the optimal cost* $\bar{J}_\epsilon$ *(7) as* $\epsilon \to 0$ *is given by:*

$$\bar{J}_{opt} = (\bar{y}_{ref} - \bar{\eta})' M (\bar{y}_{ref} - \bar{\eta}) \quad (13)$$

where M *is a constant matrix with*

$$trace(M) = n + r - \rho + \sum_{i=1}^{p} \frac{(\lambda_i + 1)}{(\lambda_i - 1)}$$

(b) *if $\bar{x}(0) = 0$, $\bar{y}_{ref} = 0$, $\bar{\eta} = 0$, and $\bar{\omega} \neq 0$, the optimal cost $\bar{J}_\epsilon$ (7) as $\epsilon \to 0$ is given by:*

$$\bar{J}_{opt} = \bar{\omega}^{'}(D_m' D_m + \bar{B}^{'}\tilde{C}_2^{'}\tilde{C}_2\bar{B})\bar{\omega}, \qquad (14)$$

where D_m is given by (10) and $(\tilde{C}_2\tilde{B}_2)$, given by (12), is invertible.

Remark 3.1 *The limiting optimal performance costs, given by trace(M) obtained in theorem 3.1 are of interest, in that they indicate the "best possible" achievable performance as measured by $J_{opt} = E(\sum_0^\infty \bar{e}_{k-1}^{'}\bar{e}_{k-1})$, where E is the expectation operator, which can be achieved.*

4 RSP for Discrete Time Periodic Systems

It is desired now to find a "high performance" controller to solve the RSP for Σ (1) in the presence of constant disturbances, ω, and constant set-points, y_{ref}. In order to achieve this goal, the following " cheap performance index" will be minimized:

$$J_\epsilon = \sum_{t=0}^{\infty} e(t-1)^{'}e(t-1) + \epsilon(u(t)-u(t-T))^{'}(u(t)-u(t-T)) \qquad (15)$$

where $e_t := y_t - y_{ref}$ is the error in the system, and $\epsilon \to 0$.

The following existence result is now obtained re a solution to the RSP for discrete time periodic systems (1):

Lemma 4.1 *There exists a solution to the RSP for the periodic system Σ(1) if there exists a solution to the RSP for any of the associated linear time invariant system Σ_τ, $\tau = 0, 1, \ldots, T-1$, i.e. there exists an initial time $\tau \in \{0, 1, \ldots, T-1\}$ such that the conditions in lemma 3.1 are all satisfied for the associated linear time invariant system Σ_τ (2).*

Consider now the discrete periodic system Σ (1), and let Σ_0 (2) be the associated linear time invariant representations of Σ at $\tau = 0$. Assume that there exists a solution to the RSP for the associated linear time invariant system Σ_0 [1], and let $\bar{K}$ (16) be the optimal controller that minimizes $\bar{J}$(7). The optimal controller which solves the RSP for Σ (1) and which minimizes J (15) as $\epsilon \to 0$ can now be obtained by solving the RSP for the associated linear time invariant systems Σ_0. The following results are now obtained.

[1] If there is no solution to the RSP for Σ_0, then we consider Σ_1(2), and etc..

Theorem 4.1 *Assume that there exists a solution to the RSP for the associated linear time invariant systems Σ_0, and let $\bar{K}$ (16) be the optimal controller gain that minimizes $\bar{J}$ (7):*

$$\bar{K} = \begin{bmatrix} \bar{K}_{1,0} & \bar{K}_{1,1} & \ldots & \bar{K}_{1,T} \\ \bar{K}_{2,0} & \bar{K}_{2,1} & \ldots & \bar{K}_{2,T} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{K}_{T,0} & \bar{K}_{T,1} & \ldots & \bar{K}_{T,T} \end{bmatrix} \qquad (16)$$

Then there exits a solution to the RSP for the periodic system Σ (1), the performance index J_ϵ (15) is given by:

$$J_\epsilon = \bar{J}_\epsilon \qquad (17)$$

and the optimal corresponding periodic controller that minimizes J_ϵ is given by:

$$\begin{aligned} u(t) &= u(t-T) + K_0(t)(x(kT) - x((k-1)T)) + \\ &K_1(t)y((k-1)T) + K_2(t)y((k-1)T+1) + \ldots + \\ &K_T(t)y(kT-1) - (K_0(t) + K_1(t) + \ldots + K_T(t))y_{ref} \end{aligned} \qquad (18)$$

where $K_j(t)$ is a T-periodic matrix function such that:

$$K_j(i) := \bar{K}_{i+1,j}, \ i = 0, 1, \ldots, T-1, \ \text{and} \ j = 0, 1, \ldots, T. \qquad (19)$$

The results in the following corollary follows directly from theorem theorem 3.1 and theorem 4.1.

Corollary 4.1 *Given a stabilizable and detectable discrete time periodic system Σ (1) which has the property that there exists a solution to the RSP, let J_{opt} (15) be the limiting optimal performance index of J_ϵ (15) as $\epsilon \to 0$. Let Σ_0 be the associated time invariant representation (2) that has $\bar{J}_{opt}$ (7) as the associated limiting performance index of $\bar{J}_\epsilon$ (7) as $\epsilon \to 0$. Assume that Σ has $\rho \in [1, n]$ transmission zeros [36, 37] with $p \in [0, \rho]$ non-minimum phase transmission zeros given by λ_i, $i = 1, 2, \ldots, p$, then:*

(a) *if $x(0) = 0$, $y_{ref} \neq 0$, $\eta \neq 0$, and $\omega = 0$, the optimal cost J_ϵ (15) as $\epsilon \to 0$ is given by:*

$$J_{opt} = (\bar{y}_{ref} - \bar{\eta})^{'}M(\bar{y}_{ref} - \bar{\eta})) \qquad (20)$$

where M is a constant matrix with

$$trace(M) = n + mT - \rho + \sum_{i=1}^{p} \frac{(\lambda_i + 1)}{(\lambda_i - 1)}$$

(b) *if $x(0) = 0$, $y_{ref} = 0$, $\eta = 0$, and $\omega \neq 0$, the optimal cost J_ϵ (15) as $\epsilon \to 0$ is given by:*

$$J_{opt} = \bar{\omega}^{'}(D_m' D_m + \bar{B}^{'}\tilde{C}_2^{'}\tilde{C}_2\bar{B})\bar{\omega}, \qquad (21)$$

where D_m is given by (10) and $(\tilde{C}_2\tilde{B}_2)$, given by (12), is invertible.

5 Examples

Two examples will be considered. In the first example, the periodic system has only stable transmission zeros, and in the second example, the system has unstable transmission zeros, and the LTI representation of the periodic system results in a system with a higher number of inputs than the states.

Example 5.1 Consider the following discrete time periodic system given by:

$$A_k=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a_{1,k} & a_{2,k} & a_{3,k} & a_{4,k} & a_{5,k} \end{bmatrix}, B_k=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ b_k \end{bmatrix}, C_k=\begin{bmatrix} 0 \\ 0 \\ 0 \\ c_{1,k} \\ c_{2,k} \end{bmatrix}' \quad (22)$$

where $a_{1,k}=sin(2\pi(k+T)/T+0.9)$, $a_{2,k}=sin(2\pi(k+T)/T+0.7)$, $a_{3,k} = 2sin(2\pi(k+T)/T-0.5)$, $a_{4,k} = sin(2\pi(k+T)/T-0.1)$, $a_{5,k} = sin(2\pi(k+T)/T+0.3)$, $b_k=cos(2\pi(k{+}T)/T)$, $c_{1,k}=0.5cos(2\pi(k{+}T)/T{+}0.2)$, and $c_{2,k}=cos(2\pi(k{+}T)/T{+}0.1)$, and $T=3$.

All Σ_τ (2), $\tau=0,1,2$ associated with the periodic system (22) have the property that there exits a solution to the RSP, and have the property that they have the same transmission zeros {0, 0, 0, -0.108}.

If $x(0)=0$, $\omega=0$, then it follows from (13) that the limiting optimal cost for Σ_0, as $\epsilon\to 0$, is given by:

$$\bar{J}_{opt} = (\bar{y}_{ref}-\bar{\eta})' M(\bar{y}_{ref}-\bar{\eta})$$

where $trace(M)=4$, and it follows from (20) that the limiting optimal cost for Σ, as $\epsilon\to 0$, is given by:

$$J_{opt} = (\bar{y}_{ref}-\bar{\eta})' M(\bar{y}_{ref}-\bar{\eta})$$

where $trace(M)=4$.

If $x(0)=0$, $y_{ref}=0$, $\eta=0$, $\omega=1$ then it follows from (14) that the limiting optimal cost, as $\epsilon\to 0$, is given by:

$$\bar{J}_{opt} = 2.3314$$

A simulation of the resultant closed loop periodic system, using the robust servomechanism controller (18), for the case of zero initial conditions is given in Fig. 1, for the case when $\omega=0$, $y_{ref}=1$, and $y_{ref}=0$, $\omega=1$. It is seen in Fig. 1 that the "optimal" transient error costs obtained, do confirm the results obtained in (20), (21), i.e. in the case of Fig.1 (a), Fig.1(b), $J_{opt} = \sum_0^\infty e_{k-1}' e_{k-1} = 4 = [1\,1\,1]M[1\,1\,1]'$, where $M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $J_{opt} = 2.3314 = [1\,1\,1](D_m' D_m + \bar{B}'\tilde{C}_2'\tilde{C}_2\bar{B})[1\,1\,1]'$ respectively. Note that

$$(\bar{D}'\bar{D} + \bar{B}'\tilde{C}_2'\tilde{C}_2\bar{B}) = \begin{bmatrix} 1.8192 & -0.3624 & 0.5642 \\ -0.3624 & 0.0999 & -0.1195 \\ 0.5642 & -0.1195 & 0.2475 \end{bmatrix}$$

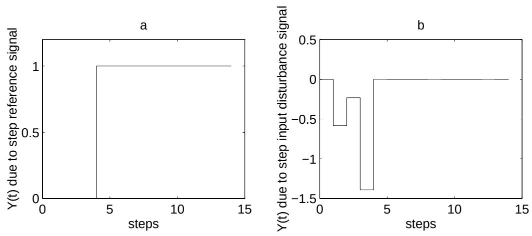

Figure 1: Closed loop response with RSC (a): $y_{ref}=1$, $\omega=0$, (b): $y_{ref}=0$, $\omega=1$.

Example 5.2 Consider the following discrete time periodic system given by:

$$A_k=\begin{bmatrix} A_{1,k} & A_{2,k} \\ 0 & A_{3,k} \end{bmatrix}, B_k=\begin{bmatrix} B_{1,k} & B_{2,k} \end{bmatrix}, C_k=\begin{bmatrix} C_{1,k} \\ C_{2,k} \end{bmatrix} \quad (23)$$

where

$$A_{1,k}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_{1,k} & a_{2,k} & a_{3,k} \end{bmatrix}, A_{2,k}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}, A_{3,k}=\begin{bmatrix} 0 & 1 \\ a_{4,k} & a_{5,k} \end{bmatrix},$$

$$B_{1,k}=\begin{bmatrix} 0 \\ 0 \\ b_{1,k} \\ 0 \\ 0 \end{bmatrix}, B_{2,k}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ b_{2,k} \end{bmatrix}, C_{1,k}'=\begin{bmatrix} 0.1 \\ 0 \\ c_{1,k} \\ 0 \\ c_{1,k} \end{bmatrix}, C_{2,k}'=\begin{bmatrix} 0 \\ 0 \\ c_{3,k} \\ 0 \\ c_{4,k} \end{bmatrix},$$

where $a_{1,k}=sin(2\pi(k{+}T)/T{+}0.9)$, $a_{2,k}=sin(2\pi(k+T)/T+0.7)$, $a_{3,k} = 2sin(2\pi(k+T)/T-0.5)$, $a_{4,k} = sin(2\pi(k+T)/T-0.1)$, $a_{5,k} = sin(2\pi(k+T)/T+0.3)$, $b_{1,k}=cos(2\pi(k{+}T)/T+1.3)$, $b_{2,k} = cos(2pi(i+T)/T)$, $c_{1,k} = cos(2\pi(k+T)/T+0.2)$, and $c_{2,k} = 0.1cos(2\pi(k{+}T)/T{+}0.1)$, $c_{3,k}=1$, $c_{4,k}=cos(2\pi(k+T)/T{+}0.1)^2$, and $T=3$.

All Σ_τ (2), $\tau=0,1,2$ associated with the periodic system (22) have the property that there exits a solution to the RSP, and have the property that they have the same transmission zeros {-7.1441, -0.2578, -0.6607}. Also, they have the property that they have more inputs than states; in this case, the number of states in the system (8) can be increased by adding an appropriate number of uncontrollable unobservable stable modes in order to obtain the optimal cheap control gain (16).

If $x(0)=0$, $\omega=[0,0]'$ then it follows from (13) that the limiting optimal cost for Σ_0, as $\epsilon\to 0$, is given by:

$$\bar{J}_{opt} = (\bar{y}_{ref}-\bar{\eta})' M(\bar{y}_{ref}-\bar{\eta})$$

where $trace(M)=8.7544$, and it follows from (20) that the limiting optimal cost for Σ, as $\epsilon\to 0$, is given by:

$$J_{opt} = (\bar{y}_{ref}-\bar{\eta})' M(\bar{y}_{ref}-\bar{\eta})$$

where $trace(M)=8.7544$.

If $x(0) = 0$, $y_{ref} = [0,\, 0]'$, $\eta = [0,\, 0]'$, $\omega = [1,\, 1]'$ it then follows from (14) that the limiting optimal cost, as $\epsilon \to 0$, is given by:

$$\bar{J}_{opt} = 115.4637$$

A simulation of the resultant closed loop periodic system, using the robust servomechanism controller (18), for the case of zero initial conditions is given in Fig. 2, for the case when $\omega = [0,\, 0]'$, $y_{ref} = [1,\, 1]'$, and $y_{ref} = [0,\, 0]'$, $\omega = [1,\, 1]'$. It is seen in Fig. 2 that the "optimal" transient error costs obtained, do confirm the results obtained in (20), (21), i.e. in the case of Fig. 2 (a), Fig. 2 (b), $J_{opt} = \sum_0^\infty e'_{k-1} e_{k-1} = 8.7544 = \bar{y}'_{ref} M \bar{y}_{ref}$, and $J_{opt} = 115.4637 = \bar{\omega}'(D'_m D_m + \bar{B}'\tilde{C}'_2 \tilde{C}_2 \bar{B})\bar{\omega}$ respectively.

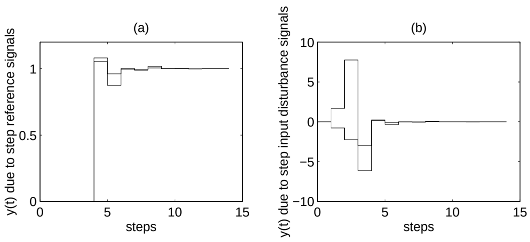

Figure 2: Closed loop response with RSC (a): $y_{ref} = [1\,1]'$, $\omega = [0\,0]'$, (b): $y_{ref} = [0\,0]'$ $\omega = [1\,1]'$.

6 Conclusions

In this paper, the robust servomechanism controller for a discrete time periodic system is studied by solving the robust servomechanism problem (RSP) for the system's associated lifted LTI discrete time system. In this case, it is shown that the fundamental performance limitations on solving the RSP for a discrete time periodic system is independent of the order of the plant, and can be characterized by the number and the location of the non-minimum phase transmission zeros of the system's associated lifted system. These limitations can be used to evaluate whether a discrete time periodic system is "inherently hard to control", and to assess a given closed loop system design, i.e. to determine how near a closed loop system's performance is from the best possible attainable.

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