Stabilization of a Class of Underactuated Systems*

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Abstract

In this paper the problem of stabilization of a class of underactuated systems by using backstepping is considered. To use the backstepping technique a suitable change of coordinates is constructed. Using backstepping a simple control law is proposed; this control law ensures the origin to be asymptotically stable. The acrobot system is used to illustrate the application of the results obtained.

1. Introduction.

Underactuated systems have been studied by many authors (see [1]-[4], [6]-[9], [11]). In fact, there are many real underactuated systems that have to be controlled; for instance, several laboratory prototypes such as the pendubot[7], the acrobot[8], the TORA[3], the inverted pendulum [11], the ball and beam system [2], etc., that are useful to prove new control techniques, belong to that class.

Underactuated systems are interesting because of their structural properties. For fully actuated mechanical systems a broad range of powerful techniques are used to improve their performance (optimal, robust, adaptive and learning control techniques). These techniques are possible because fully actuated systems possess a number of strong properties that facilitate control design, such as feedback linearizability, passivity and matching conditions. For underactuated systems one or more of the above properties are usually lost. Moreover, undesirable properties such as a high relative degree, nonminimum phase behavior and chaos are manifested.

To solve the control problem of underactuated systems there is not a unique theory useful for anyone, but we have to analyze their properties, choose and fit some common techniques or propose new techniques. For example, the passivity property of systems has been exploited to design control laws for underactuated systems. Backstepping is a powerful technique used to ensure stability based on passivity concepts. The backstepping technique is used on model systems with a special form[5], known as the cascade form, therefore it is not used directly on any systems, unless a change of coordinates has been applied. Then, one problem to apply backstepping is to represent the system model in the cascade form.

This paper has mainly three objectives. The first is to show the construction of a valid change of coordinates that allows to describe a class of underactuated systems in the cascade form. The second is to propose a control law for a class of underactuated systems; the design of the control law is based on the backstepping technique and also on the use of auxiliary systems; moreover, the control law ensures the origin to be asymptotically stable. Finally, the third objective is to use the acrobot model in order to illustrate the application of the change of coordinates and the design of a stabilizing control law.

To reach the proposed objectives, this paper is structured as follows: in section 2 we consider a class of underactuated systems. In the third section the control problem is stated. Section 4 deals with the construction of a change of coordinates to transform the class of systems considered below to the cascade form. In section 5 we give a theorem and its proof that allows to construct a control law to stabilize a class of underactuated systems. In the sixth section, we give an example of an underactuated system to illustrate the use of the change of coordinates and its stabilization by the proposed control law. Finally, we give some conclusions.

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2. A class of underactuated systems.

It is a common practice to use the Euler-Lagrange equation in order to obtain the dynamical model of electromechanical systems. Using the Euler-Lagrange equation, the differential equations that describe the dynamical behavior of n-DOF electromechanical systems, subjected to holonomic constraints, are given by, [6],

$$\frac{d}{dt}\left(\frac{\partial L}{\partial p_i}\right)-\frac{\partial L}{\partial q_i}=\tau_i,\ i=1,...,n, \tag{2.1}$$

where q_i is the generalized position, $p_i\,(=\dot{q}_i)$ is the generalized velocity, τ_i is the input generalized force, the function L is called the Lagrangian of the system, and is given by the difference between the kinetic energy, K, and the potential energy, V, of the system, that is $L=K(q_i,p_i,t)-V(q_i)$. The kinetic energy is a quadratic form of the velocities of the system, $K(q_i,p_i,t)=\frac{1}{2}\mathbf{p}^T D\mathbf{p}$, where $\mathbf{p}^T=\begin{pmatrix} p_1 & \ldots & p_n\end{pmatrix}$ and $D\in R^{n\times n}$ is a symmetric positive definite matrix that represents the generalized inertias. If any input τ_i is cero then the system is known as underactuated one.

In this paper 2–DOF underactuated systems are considered, with the following properties

P1. The inertia matrix D is only function of the q_2 coordinate, i.e. $D(q_2)$.

P2. The unique input generalized force is acting in the same direction of the second generalized coordinate, i.e. $\tau_1=0$.

P3. The level of the potential energy is chosen such that $\frac{\partial V(0,0)}{\partial(q_1,q_2)}=(0,0)$.

Under properties P1 and P2, the Lagrangian of the considered system is given by

$$L=\frac{1}{2}d_{11}(q_2)\,p_1^2+d_{12}(q_2)\,p_1p_2+\frac{1}{2}d_{22}(q_2)\,p_2^2-V(q_1,q_2)\,,$$

then by applying the Euler-Lagrange equation, the dynamical model can be represented as

$$\begin{gathered} d_{11}(q_2)\dot{p}_1+d_{12}(q_2)\dot{p}_2+c_{11}(q_2)\dot{q}_2p_1 \\ +c_{12}(q_2)\dot{q}_2p_2+h_1(q_1,q_2)=0 \end{gathered} \tag{2.2a}$$

$$\begin{gathered} d_{12}(q_2)\dot{p}_1+d_{22}(q_2)\dot{p}_2+c_{21}(q_2)p_1^2 \\ +c_{22}(q_2)(\dot{q}_2p_2-\tfrac{1}{2}p_2^2)+h_2(q_1,q_2)=\tau_2 \end{gathered} \tag{2.2b}$$

with

$$\begin{gathered} c_{11}(q_2)=\frac{\partial d_{11}(q_2)}{\partial q_2},\, c_{12}(q_2)=\frac{\partial d_{12}(q_2)}{\partial q_2}, \\ c_{21}(q_2)=-\frac{1}{2}\frac{\partial d_{11}(q_2)}{\partial q_2},\, c_{22}(q_2)=\frac{\partial d_{22}(q_2)}{\partial q_2}, \\ h_1(q_1,q_2)=\frac{\partial V}{\partial q_1},\, h_2(q_1,q_2)=\frac{\partial V}{\partial q_2}. \end{gathered} \tag{2.3}$$

Equation (2.2a) describes the dynamics of the unactuated joint, while (2.2b) describes the dynamics of the actuated joint.

3. Problem statement.

The stabilization of dynamical systems using backstepping is possible if they have a specific structure, known as the cascade form. So, this powerful technique cannot be applicable directly to any system, including some underactuated systems.

In order to stabilize a 2-DOF underactuated system (2.2), via backstepping, it is necessary to find a change of coordinates that represents the dynamics of the system in a convenient form.

System (2.2) can be described by using the state space vector (q_1,p_1,q_2,p_2) as, [10],

$$\begin{aligned} \dot{q}_1 &= p_1 && \text{(3.1a)} \\ \dot{p}_1 &= f_1(q,p)+g_1(q_2)\,u && \text{(3.1b)} \\ \dot{q}_2 &= p_2 && \text{(3.1c)} \\ \dot{p}_2 &= f_2(q,p)+g_2(q_2)\,u && \text{(3.1d)} \end{aligned}$$

where $q=(q_1,q_2)\,,p=(p_1,p_2)$ and $u=\tau_2$. System (3.1) has an equilibrium point at $(q,p)=\mathbf{0}$, with $u=0$, because of property P3.

To stabilize system (3.1) by using backstepping, it must have the following special cascade form

$$\begin{aligned} \dot{z} &= f(z,\xi_1) && \text{(3.2a)} \\ \dot{\xi}_1 &= \xi_2 && \text{(3.2b)} \\ \dot{\xi}_2 &= v, && \text{(3.2c)} \end{aligned}$$

where $z=(z_1,z_2)$.

Remark 1. *The dynamics of system (3.1) is partitioned into two subsystems: the dynamics of (z_1,z_2) define the subsystem Σ_z and the dynamics of (ξ_1,ξ_2) define the subsystem Σ_ξ. It is important to point out that ξ_1 is the virtual control of subsystem Σ_z. Moreover, the input control v, in the new coordinates, enters through the subsystem Σ_ξ. So, we can name subsystem Σ_ξ as the actuated subsystem, while subsystem Σ_z will be named as the unactuated subsystem.*

In this paper the following problem is considered: the system defined by (2.2) will be stabilized by using the backstepping technique, so a change of coordinates T should be found,

$$(z_1,z_2,\xi_1,\xi_2)=T(q_1,q_2,p_1,p_2)\,,$$

in order to transform system (2.2), or (3.1), to form (3.2).

4. The change of coordinates.

As we have mentioned in Remark 1, ξ_1 is the virtual control of the so called underactuated subsystem defined by Σ_z, and subsystem Σ_ξ is actuated by the control input v. Since system (2.2) has the control input applied in the direction of the coordinate q_2, then the dynamics of the coordinates (q_2, p_2) are related with the dynamics of the actuated subsystem Σ_ξ

$$(q_2, p_2) \longleftrightarrow (\xi_1, \xi_2), \tag{4.1}$$

in addition, the dynamics of the passive joint is related with the dynamics of the unactuated subsystem Σ_z,

$$(q_1, p_1) \longleftrightarrow (z_1, z_2). \tag{4.2}$$

It is important to note that the variable q_2 influences the dynamics of (q_1, p_1), (equation 2.2a), so we can consider q_2 as the virtual control of the underactuated subsystem. Then, the change of variables $(q_1, p_1, q_2, p_2) = T^{-1}(z_1, z_2, \xi_1, \xi_2)$, can be stated as follows

$$q_2 = \xi_1, \tag{4.3a}$$
$$p_2 = \xi_2, \tag{4.3b}$$
$$q_1 = \eta_1(z_1, z_2, \xi_1, \xi_2), \tag{4.3c}$$
$$p_1 = \eta_2(z_1, z_2, \xi_1, \xi_2). \tag{4.3d}$$

where η_1 and η_2 are functions to be determined.
In order to obtain a valid change of coordinate its Jacobian must be different from zero, i.e.

$$\frac{\partial(q_2, \dot{q}_2, q_1, \dot{q}_1)}{\partial(\xi_1, \xi_2, z_1, z_2)} = \frac{\partial\eta_1}{\partial z_1}\frac{\partial\eta_2}{\partial z_2} - \frac{\partial\eta_1}{\partial z_2}\frac{\partial\eta_2}{\partial z_1} \neq 0. \tag{4.4}$$

For simplicity, the argument of η_1 and η_2 will be omitted, and ξ_1 and ξ_2 will be written as q_2 and p_2, respectively.
Application of the change of coordinates (4.3) to (2.2), yields to

$$\begin{aligned} &d_{11}(q_2)\dot{\eta}_2 + d_{12}(q_2)\dot{p}_2 + c_{11}(q_2)\dot{q}_2\eta_2 \\ &\quad+ c_{12}(q_2)\dot{q}_2 p_2 + h_1(\eta_1, q_2) = 0, \end{aligned} \tag{4.5a}$$

$$\begin{aligned} &d_{12}(q_2)\dot{\eta}_2 + d_{22}(q_2)\dot{p}_2 + c_{21}(q_2)\eta_2^2 \\ &\quad+ \tfrac{1}{2}c_{22}(q_2)p_2^2 + h_2(\eta_1, q_2) = \tau_2. \end{aligned} \tag{4.5b}$$

The Eq. (4.5b), which is related with the dynamics of the actuated subsystem Σ_ξ, can be written in the form (3.2b,3.2c) using $v = \dot{p}_2$, $q_2 = \xi_1$ y $p_2 = \xi_2$.
Using the chain rule, the term $\dot{\eta}_2$ can be expressed as

$$\dot{\eta}_2(z_1, z_2, q_2, p_2) = \frac{\partial\eta_2}{\partial z_1}\dot{z}_1 + \frac{\partial\eta_2}{\partial z_2}\dot{z}_2 + \frac{\partial\eta_2}{\partial q_2}\dot{q}_2 + \frac{\partial\eta_2}{\partial p_2}\dot{p}_2, \tag{4.6}$$

by substituting (4.6) into (4.5b), we arrive to

$$\begin{aligned} 0 =\, &d_{11}(q_2)\tfrac{\partial\eta_2}{\partial z_1}\dot{z}_1 + d_{11}(q_2)\tfrac{\partial\eta_2}{\partial z_2}\dot{z}_2 + \left[d_{12}(q_2) + d_{11}(q_2)\tfrac{\partial\eta_2}{\partial p_2}\right]\dot{p}_2 \\ &+ \left[d_{11}(q_2)\tfrac{\partial\eta_2}{\partial q_2} + c_{11}(q_2)\eta_2 + c_{12}(q_2)p_2\right]\dot{q}_2 + h_1(\eta_1, q_2), \end{aligned} \tag{4.7}$$

Equation (4.7) is related with the dynamics of subsystem Σ_z, because of (4.2). In order to represent (4.7) in the form given by (3.2a), the terms $\dot{q}_2$ and $\dot{p}_2$ in (4.7) must be eliminated, that is the following equations

$$\frac{\partial\eta_2}{\partial p_2} = -\frac{d_{12}(q_2)}{d_{11}(q_2)}, \tag{4.8}$$

$$d_{11}(q_2)\frac{\partial\eta_2}{\partial q_2} + c_{11}(q_2)\eta_2 = -c_{12}(q_2)p_2. \tag{4.9}$$

must be satisfied. Furthermore, it is possible to neglect the $\dot{z}_1$ coefficient, that is $d_{11}(q_2)\frac{\partial\eta_2}{\partial z_1} = 0$, then $\eta_2 = \eta_2(z_2, q_2, p_2)$.
Using $c_{11}(q_2) = \frac{\partial d_{11}(q_2)}{\partial q_2}$, in equation (2.3), equation (4.9) can be expressed as

$$\frac{\partial}{\partial q_2}(\eta_2 d_{11}(q_2)) = -c_{12}(q_2)p_2. \tag{4.10}$$

Now, integrating (4.10) with respect to q_2, we get

$$\eta_2 = -\frac{p_2}{d_{11}(q_2)}\int c_{12}(q_2)\,dq_2 + \frac{1}{d_{11}(q_2)}\alpha_1(z_2, p_2),$$

where $\alpha_1(z_2, p_2)$ is an integration constant. From (2.3) we have $c_{12}(q_2) = \frac{\partial d_{12}(q_2)}{\partial q_2}$, then

$$\eta_2 = -\frac{p_2}{d_{11}(q_2)}d_{12}(q_2) + \frac{1}{d_{11}(q_2)}\alpha_1(z_2, p_2). \tag{4.11}$$

Deriving (4.11) with respect to p_2, and equalizing to (4.8) and simplifying we have

$$-\frac{d_{12}(q_2)}{d_{11}(q_2)} + \frac{1}{d_{11}(q_2)}\frac{\partial\alpha_1(z_2, p_2)}{\partial p_2} = -\frac{d_{12}(q_2)}{d_{11}(q_2)}$$

$$\frac{\partial\alpha_1(z_2, p_2)}{\partial p_2} = 0, \tag{4.12}$$

therefore $\alpha_1(z_2, p_2) = \alpha_1(z_2)$.
For simplicity we can take $\alpha_1(z_2) = z_2$, then (4.11) takes the form

$$\eta_2(z_2, q_2, p_2) = -\frac{d_{12}(q_2)}{d_{11}(q_2)}p_2 + \frac{z_2}{d_{11}(q_2)} \tag{4.13}$$

Now, we only need to calculate the function η_1. The function η_1 is proposed such that the expression for $\dot{z}_1$ possesses the form (3.2), moreover the condition (4.4) must be satisfied. To satisfy (4.4) it is sufficient to take $\eta_1 = \eta_1(z_1, q_2)$.
The structure of $\eta_1(z_1, q_2)$ can be chosen in the following semi-lineal form,

$$\eta_1(z_1, q_2) = az_1 + \phi(q_2), \tag{4.14}$$

where a and ϕ are quantities to be determined. Deriving (4.14) with respect to time and considering $\frac{d\eta_1}{dt} = \frac{dq_1}{dt} = p_1 = \eta_2$ and $\frac{dq_2}{dt} = p_2$ results in

$$\dot{z}_1 = \frac{1}{a}\left(\eta_2 - \frac{\partial\phi}{\partial q_2}p_2\right) = \frac{1}{a}\left[-\tfrac{d_{12}(q_2)}{d_{11}(q_2)}p_2 + \tfrac{z_2}{d_{11}(q_2)} - \tfrac{\partial\phi}{\partial q_2}p_2\right].$$

To ensure $\dot{z}_1$ be independent of p_2, the equation $\frac{\partial\phi}{\partial q_2} = -\frac{d_{12}(q_2)}{d_{11}(q_2)}$, must be satisfied, moreover it is possible to take $a = 1$.
Finally, the inverse transformation T^{-1}, (4.3), is given by

$$q_2 = \xi_1, \tag{4.15a}$$
$$p_2 = \xi_2, \tag{4.15b}$$

$$q_1 = \eta_1 = z_1 - \int\limits_0^{\xi_1} \frac{d_{12}(\zeta)}{d_{11}(\zeta)} d\zeta, \tag{4.15c}$$

$$p_1 = \eta_2 = -\frac{d_{12}(\xi_1)}{d_{11}(\xi_1)}\xi_2 + \frac{z_2}{d_{11}(\xi_1)}, \tag{4.15d}$$

and the transformation T is as follows

$$z_1 = q_1 + \int_0^{q_2} \frac{d_{12}(\zeta)}{d_{11}(\zeta)} d\zeta, \tag{4.16a}$$

$$z_2 = d_{11}(q_2)\, p_1 + d_{12}(q_2)\, p_2, \tag{4.16b}$$

$$\xi_1 = q_2, \tag{4.16c}$$

$$\xi_2 = p_2. \tag{4.16d}$$

Remark 2. *The transformation of coordinates T has the property $T(\mathbf{0}) = \mathbf{0}$. By making $\int \frac{d_{12}(\zeta)}{d_{11}(\zeta)} d\zeta = \Phi(\zeta)$, yields $z_1 = q_1 + \Phi(q_2) - \Phi(0)$. Hence, the property $T(\mathbf{0}) = \mathbf{0}$ can be easily verified.*

The underactuated system (2.2), in the new coordinates, is given by

$$\dot{z}_1 = \frac{z_2}{d_{11}(\xi_1)}, \tag{4.17a}$$

$$\dot{z}_2 = -h_1(z_1 - \Phi(\xi_1) + \Phi(0), \xi_1), \tag{4.17b}$$

$$\dot{\xi}_1 = \xi_2, \tag{4.17c}$$

$$\dot{\xi}_2 = v, \tag{4.17d}$$

where $v = \dot{p}_2$ is calculated from (4.5b).
As a conclusion, in this section we have proved the following theorem.

Theorem 4.1. *The system (2.2) can be transformable to the cascade form (3.2), using the transformations (4.16) and (4.15).* ■

5. Stabilization using backstepping.

The stabilization of (4.17) will be made by using the following theorem, that gives a control law based on the backstepping technique.

Theorem 5.1. *Suppose that for the Lyapunov function*

$$V_1(z_1, z_2) = \frac{1}{2}\left(z_1^2 + z_2^2\right). \tag{4.17r}$$

exists a control law $k_1(z_1, z_2)$ with $k_1(0,0) = 0$, such that the origin of (4.17a, 4.17b) is stable, then the control law

$$\begin{aligned} v = &-c_3(\xi_2 - w_2 x_2) - c_2 w_2(\xi_1 - w_1 x_1) \\ &+ w_1 w_2 (k_1 - w_1 x_1) - w_2^2 x_2. \end{aligned} \tag{4.17s}$$

makes the origin of system (4.17) to be asymptotically stable, where $c_2, c_3, w_1, w_2 > 0$ and x_1 and x_2 are solutions of the next auxiliary stable systems,

$$\dot{x}_1 = k_1 - w_1 x_1, \tag{4.17t}$$

$$\dot{x}_2 = k_2 - w_2 x_2, \tag{4.17u}$$

where $k_2 = -c_2(\xi_1 - w_1 x_1) + w_1(k_1 - w_1 x_1)$. ■

The proof is based on the backstepping technique, hence we need a Lyapunov function which ensures the stability of the origin of subsystem Σ_z. This Lyapunov function will be used to construct some others extended Lyapunov functions that ensure the stability of the origin of the complete system. The property of the origin to be asymptotically stable will be shown by using the LaSalle's invariance principle.

Proof. The derivative of V_1 along the trajectories of (4.17) is given by

$$\dot{V}_1(z_1, z_2) = z_2\left(\frac{z_1}{d_{11}(\xi_1)} - h_1(z_1 - \Phi(\xi_1) + \Phi(0), \xi_1)\right).$$

Theorem 5.1 assumes there exists a control law $k_1(z_1, z_2)$ satisfying

$$\frac{z_1}{d_{11}(k_1)} - h_1(z_1 - \Phi(k_1) + \Phi(0), k_1) = -c_1 z_2, \quad c_1 > 0 \tag{5.23}$$

which makes $\dot{V}_1$ to be negative semidefinite, that is

$$\dot{V}_1 = -c_1 z_2^2,$$

therefore the origin of (4.17a, 4.17b), with $\xi_1 = k_1(z_1, z_2)$ acting as a control variable, is stable.
Now, we need to stabilize the subsystem (z_1, z_2, ξ_1), i. e. (4.17a), (4.17b) and (4.17c), where ξ_2 is the virtual input control. If the following extended Lyapunov function is proposed,

$$V_2(z_1, z_2, \eta_1) = V_1(z_1, z_2) + \frac{1}{2}\eta_1^2, \tag{5.25}$$

where

$$\eta_1 = \xi_1 - w_1 x_1 \tag{5.26}$$

is an error function, with x_1 as the solution of (4.17t). By taking the time derivative of (5.25) along the trajectories of (4.17), we arrive to

$$\dot{V}_2(z_1, z_2, \xi_1) = \dot{V}_1(z_1, z_2) + \dot{\eta}_1 \eta_1,$$

if we require $\dot{V}_2$ to be negative semidefinite, we need to find $\xi_2 = k_2(z_1, z_2, \xi_1)$ such that

$$\dot{\eta}_1 = -c_2 \eta_1, \quad \textit{with } c_2 > 0, \tag{5.27}$$

substituting (5.26) in (5.27), results in

$$\dot{\xi}_1 - w_1 \dot{x}_1 = -c_2(\xi_1 - w_1 x_1),$$

now, substitution of (4.17c) and (4.17t) in the above equation results in

$$\xi_2 - w_1(k_1 - w_1 x_1) = -c_2(\xi_1 - w_1 x_1).$$

Then, the control law

$$\xi_2 = k_2(z_1, z_2, \xi_1) = -c_2(\xi_1 - w_1 x_1) + w_1(k_1 - w_1 x_1). \tag{5.28}$$

makes the origin of the subsystem (z_1, z_2, ξ_1) to be stable.

To stabilize the complete system (4.17) we use the extended Lyapunov function

$$V_3(z_1, z_2, \eta_1, \eta_2) = V_2(z_1, z_2, \eta_1) + \frac{1}{2}\eta_2^2,$$

where

$$\eta_2 = \xi_2 - w_2 x_2, \tag{5.29}$$

is an error function, with x_2 as the solution of (4.17u). Taking the time derivative of V_3 along the trajectories of (4.17), yields,

$$\dot{V}_3(z_1, z_2, \xi_1, \xi_2) = \dot{V}_2(z_1, z_2, \xi_1) + \dot{\eta}_2\eta_2,$$

once again we require $\dot{V}_3$ to be negative semidefinite, then we need to find $v = k_3(z_1, z_2, \xi_1, \xi_2)$ such that

$$\dot{\eta}_2 = -c_3\eta_2, \quad with\ c_3 > 0.$$

It is easy to find v as

$$v = -c_3(\xi_2 - w_2 x_2) + w_2(k_2 - w_2 x_2). \tag{5.30}$$

Substituting (5.28) into (5.30) we obtain (4.17s).

The control law (5.30) makes the origin of (4.17) to be stable, but using the LaSalle'invariance principle we can prove that it is also asymptotically stable. Hence, we need to show that the origin of (4.17, 4.17t, 4.17u) belongs to the next invariant set,

$$\Omega = \left\{(z_1, z_2, \xi_1, \xi_2, x_1, x_2) \in R^6 \mid \dot{V}_3 = 0\right\}.$$

The time derivative of V_3, with the control law (5.30), is as follows

$$\dot{V}_3 = \dot{V}_2 - c_3\eta_2^2 = \dot{V}_1 - c_2\eta_1^2 - c_3\eta_2^2 = -c_1 z_2^2 - c_2\eta_1^2 - c_3\eta_2^2,$$

using (5.26) and (5.29), in the above equation results in

$$\dot{V}_3 = -c_1 z_2^2 - c_2(\xi_1 - w_1 x_1)^2 - c_3(\xi_2 - w_2 x_2)^2.$$

The solutions of $\dot{V}_3 = 0$ are given by

$$z_2 = 0, \tag{5.34}$$

$$\xi_1 = w_1 x_1, \tag{5.35}$$

$$\xi_2 = w_2 x_2. \tag{5.36}$$

If $z_2 = 0$, by using (4.17b) and (5.35), we obtain

$$\dot{z}_2 = 0 = h_1(z_1 - \Phi(w_1 x_1) + \Phi(0), w_1 x_1),$$

because of h_1 is given by $h_1 = \frac{\partial V}{\partial q_1}$, and if $h_1 = 0$ we can use property P3, $[0 = h_1(0, 0) = h_1(z_1 - \Phi(w_1 x_1) + \Phi(0), w_1 x_1)]$ to obtain $z_1 = x_1 = 0$. With this result we obtain from (5.35) $\xi_1 = 0$, which, after substitution into (4.17c), gives $\xi_2 = 0$, similarly, substituting $\xi_2 = 0$ into (5.36), we obtain $x_2 = 0$. So, we have proved that no solution can stay forever in Ω, except the trivial solution. Then, the origin is asymptotically stable. ■

Remark 3. *Theorem 5.1 gives a control law without implicit differentiation of k_1 and k_2, this is a difference with respect to the works given in [5], therefore the partial control law $k_1(z_1, z_2)$ can be a continuous or a discontinuous function, because of the use of auxiliary systems (4.17t) and (4.17u).*

6. An Application Example: The Acrobot.

There are some systems belonging to the class of underactuated systems defined in section 2; for example, the systems known as the acrobot, the TORA, the ball and beam, etc. In order to illustrate the stabilization of an underactuated system we have applied the previous techniques to the acrobot system.

The acrobot is a manipulator of 2-DOF whose links are moving in the vertical plane, and the only actuator is between the links. We suppose the positions and velocities can be measured. Applying the Euler-Lagrange equation, (2.1), the dynamical equations of the acrobot can be written as

$$\begin{gathered}
0 = \left(m_1 l_{c1}^2 + m_2 l_1^2 + m_2 l_{c2}^2 + I_{zz1} + I_{zz2}\right)\ddot{q}_1 + m_2 l_{c2}^2\ddot{q}_2 \\
+2m_2 l_1 l_{c2}\cos(q_2)\ddot{q}_1 + I_{zz2}\ddot{q}_2 + m_2 l_1 l_{c2}\cos(q_2)\ddot{q}_2 \\
-m_2 l_1 l_{c2}\dot{q}_2^2\sin q_2 - 2m_2 l_1\dot{q}_1 l_{c2}\dot{q}_2\sin q_2 \\
-m_2 g l_{c2}\sin(q_2 + q_1) - m_1 g l_{c1}\sin q_1 - m_2 g l_1\sin q_1, \\
\tau_2 = \left(m_2 l_1 l_{c2}\cos q_2 + m_2 l_{c2}^2 + I_{zz2}\right)\ddot{q}_1 + m_2 l_{c2}^2\ddot{q}_2 \\
+I_{zz2}\ddot{q}_2 + m_2 l_1\dot{q}_1^2 l_{c2}\sin q_2 - m_2 g l_{c2}\sin(q_2 + q_1),
\end{gathered}$$

where m_i denotes the mass of link i, l_i denotes the length of link i, l_{ci} denotes the center of mass of link i with respect to its rotation axis, I_{zzi} is the inertia of link i with respect to its rotation axis, q_1 denotes the position of the first link measured from the up vertical and q_2 denotes the position of the second link measured from the first link. The model of the acrobot does not have the cascade form, but it satisfies the properties given in section 2. If the dimensions of the acrobot are related by $I_{zz1} = I_{zz2} - \frac{m_2 l_{c2}^2}{8}$, $l_{c1} = \frac{l_{c2}}{2}$, $l_1 = l_{c2}$, $m_1 = \frac{m_2}{2}$, the inverse transformation (4.15), T^{-1}, is simplified as

$$q_2 = \xi_1, \tag{6.38}$$

$$p_2 = \xi_2, \tag{6.39}$$

$$q_1 = z_1 - \frac{\xi_1}{2} \tag{6.40}$$

$$p_1 = -\frac{\xi_2}{2} + \frac{1}{2}\frac{z_2}{m_2\, l_{c2}^2\cos(\xi_1) + m_2\, l_{c2}^2 + I_{zz2}} \tag{6.41}$$

while the direct transformation T is as follows

$$z_1 = q_1 + \frac{q_2}{2}, \tag{6.42a}$$

$$z_2 = (m_2 l_{c2}^2\cos(q_2) + m_2 l_{c2}^2 + I_{zz_2})(2p_1 + p_2), \tag{6.42b}$$

$$\xi_1 = q_2, \tag{6.42c}$$

$$\xi_2 = p_2. \tag{6.42d}$$

If the acrobot has the following dimensions, $I_{zz_2} = 0.016749\ kgm^2, l_2 = 0.38417m, l_{c2} = 0.19208m, m_2 = 0.3346kg$, the dynamics of the acrobot in the new coordinates has the form

$$\dot{z}_1 = \frac{z_2}{0.0246899 \cos \xi_1 + 0.0581879}, \tag{6.43a}$$
$$\dot{z}_2 = 0.6304884 \sin(\tfrac{\xi_1}{2} + z_1) + 0.7881105 \sin(z_1 - \tfrac{\xi_1}{2}), \tag{6.43b}$$
$$\dot{\xi}_1 = \xi_2, \tag{6.43c}$$
$$\dot{\xi}_2 = v. \tag{6.43d}$$

where $v = \ddot{q}_2$.

We can now find the control law $k_1(z_1, z_2)$ to stabilize the subsystem Σ_z, (6.43a, 6.43b), i.e. we have to find k_1 from (5.23), that is we must solve $\frac{z_1}{0.0246899 \cos k_1 + 0.0581879} + 0.6304884 \sin(\frac{k_1}{2} + z_1) + 0.7881105 \sin(z_1 - \frac{k_1}{2}) = -c_1 z_2$, with $c_1 > 0$. It is impossible to find a real solution of the above equation, so if we note that $d_{11}(k_1) > 0$, we can substitute $\frac{1}{d_{11}(k_1)}$ by a $c > 0$, resulting in $cz_1 - h_1(\eta_1, k_1) = -c_1 z_2$, then we have

$$cz_1 + 0.6304884 \sin(\frac{k_1}{2} + z_1) + 0.7881105 \sin(z_1 - \frac{k_1}{2}) = -c_1 z_2. \tag{6.44}$$

Solving for k_1, we obtain two solutions, but we choose the one satisfying $k_1(0, 0) = 0$. This is given by

$$k_1(z_1, z_2) = 4 \text{ arctan} \frac{15214 \cos(z_1) - 2\sqrt{\Psi}}{96522cz_1 - 136926 \sin(z_1) + 96522 c_1 z_2} \tag{6.45}$$

where $\Psi = 57866449 \cos(z_1)^2 - 2329124121 c^2 z_1^2 - 4658248242 c z_1 c_1 z_2 + 4687182369 \sin(z_1)^2 - 2329124121 c_1^2 z_2^2$.

Then, with control law (4.17s) we can stabilize (6.43), of course we must first construct the partial control law (5.28). The control input τ_2 to the acrobot must be determined from the inverse model of the acrobot and using $\dot{p}_2 = v$, (4.17s) and (6.42).

Figure 1 shows the dynamic behavior of the acrobot under the initial conditions $(q_1^o, q_2^o, p_1^o, p_2^o) = (\pi, 0, 0, 0)$, with the next simulation parameters $c = 0.05$, $c_1 = 1.9$, $c_2 = 0.5$, $c_3 = 0.5$, $w_1 = 20$, $w_2 = 25$, and integration step equal to 0.001.

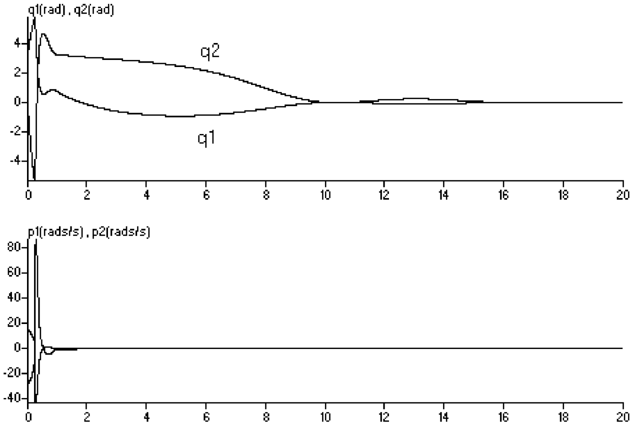

Fig. 1 Acrobot positions and velocities.

7. Conclusions.

We have analized the model of a class of underactuated systems in order to construct a change of coordinates that allows to represent some underactuated systems in a special cascade form. Systems in cascade form are stabilized by using the control law given by Theorem 5.1. This proposed control law is simpler than that obtained by using standar backstepping, because of the avoidance of the use of analytical differentiation. It is important to point out that the existence and simplicity of the control law (4.17s) depends on the existence and structure of the partial control law $k_1(z_1, z_2)$. The results obtained were applied to the acrobot system.

References

[1] Bloch, A. M., N.E. Leonard and J. E. Marsden (1997). *Stabilization of Mechanical Systems Using Controlled Lagrangians.* Proceedings of IEEE of the 36th CDC, San Diego, California, USA.

[2] Hauser, J., S. Sastry and P. Kokotović (1992). *Nonlinear Control Via Approximate Input-Output Linearization:The Ball and Beam Example.* IEEE TAC, Vol.37, No. 3.

[3] Janković, M., D. Fontaine, and P. V. Kokotović (1996). *TORA Example: Cascade- and Passivity-Based Control Designs.* IEEE TCST, Vol. 4, No. 3, pp. 292-297.

[4] Lozano, R. and I. Fantoni (1996). *Passivity Based Control of the Inverted Pendulum.* IFAC 13th Triennial World Congress, USA, pp. 145-150.

[5] Sepulchre, R., M. Janković and P.V. Kokotović(1997). *Constructive Nonlinear Control.*

[6] Spong, M. W., M. Vidyasagar (1989). *Robot dynamics and control.* John Wiley & Sons, USA.

[7] Spong, M. W. (1994). *The control of underactuated mechanical systems.* 1st Int. Conf. Mech. Mexico.

[8] Spong, M. W.(1995).*The swingup control problem for the acrobot.*IEEE CSM. Vol.15, No.1, pp.49-55.

[9] Spong, M. W. (1996). *Energy Based Control of a Class of underactuated mechanical systems.* IFAC 13th Triennial World Congress, USA, pp. 431-435.

[10] Sontag, E. and H. J. Sussmann (1985). *Remarks on the time-optimal control of two-link mainipulators.* Proc. of IEEE of the 24th CDC, pp. 1646-1652.

[11] Sugie, T. and K. Fujimoto (1994). *Control of Inverted Pendulum Systems Based on Approximate Linerization: Design and Experiment.* Proc. of IEEE of the 33rd CDC.