Nonlinear Robust Control for Nonlinear Uncertain Impulsive Dynamical Systems

Wassim M. Haddad, Nataša A. Kablar, and VijaySekhar Chellaboina†

School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150
†Mechanical and Aerospace Engineering, University of Missouri, Columbia, MO 65211

Abstract

In this paper we develop an optimality-based framework for addressing the problem of nonlinear optimal robust hybrid control for nonlinear uncertain impulsive dynamical systems.

1. Introduction

Although the theory of impulsive dynamical systems is mature [1], robust analysis and control design techniques for nonlinear uncertain impulsive dynamical systems remain relatively undeveloped. In this paper we extend the analysis and design framework for nonlinear impulsive dynamical systems developed in [2,3] to address robustness considerations in impulsive dynamical systems. Specifically, in [3] a unified framework for hybrid feedback optimal and inverse optimal control involving a hybrid nonlinear-nonquadratic performance functional was developed. In this paper we build on these results to develop an optimality-based framework for addressing the problem of nonlinear-nonquadratic optimal hybrid control for *uncertain* nonlinear impulsive dynamical systems with structured parametric uncertainity. Specifically, using a Lyapunov bounding framework the robust nonlinear hybrid control problem is transformed into an optimal hybrid control problem by modifying a nonlinear-nonquadratic hybrid cost functional to account for system parametric uncertainty.

The main contribution of this paper is a methodology for designing nonlinear hybrid controllers which provide robust stability and robust performance over a prescribed range of impulsive system uncertainty. The present framework extends the guaranteed cost control approach to nonlinear impulsive dynamical systems by utilizing a hybrid performance bound to provide robust performance in addition to robust stability. In particular, the performance bound can be evaluated in closed-form as long as the nonlinear-nonquadratic hybrid cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees robust stability over a prescribed uncertainity set. This Lyapunov function is shown to be a solution to the steady-state form of the hybrid Hamilton-Jacobi-Bellman equation for the nominal impulsive dynamical system and plays a key role in constructing the optimal nonlinear robust hybrid control law. Hence, the overall framework provides for a generalization of the hybrid Hamilton-Jacobi-Bellman conditions [3] for addressing the design of robust optimal hybrid controllers for nonlinear uncertain impulsive dynamical systems.

Finally, in this paper we use the following standard notation. Let $\mathbb{R}$ denote the set of real numbers, let $\mathcal{N}$ denote the set of nonnegative integers, let $\mathbb{R}^{n\times m}$ denote the set of real $n \times m$ matrices, let $\mathbb{S}^n$ denote the set of $n \times n$ symmetric matrices, and let $\mathbb{N}^n$ (resp., $\mathbb{P}^n$) denote the set of $n \times n$ nonnegative (resp., positive) definite matrices. Furthermore, $A \geq 0$ (resp., $A > 0$) denotes the fact that the Hermitian matrix is nonnegative (resp., positive) definite and $A \geq B$ (resp., $A > B$) denotes the fact that $A - B \geq 0$ (resp., $A - B > 0$). In addition, we write $V'(x)$ for the Fréchet derivative of $V(\cdot)$ at x and, for a subset $\mathcal{S} \subset \mathbb{R}^n$, we write $\partial\mathcal{S}$, $\overset{\circ}{\mathcal{S}}$, $\bar{\mathcal{S}}$ for the boundary, the interior, and the closure of $\mathcal{S}$, respectively.

2. Robust Stability Analysis of Nonlinear Uncertain Impulsive Dynamical Systems

In this section we present sufficient conditions for robust stability for a class of nonlinear uncertain impulsive dynamical systems. Here we restrict our attention to nonlinear state-dependent uncertain impulsive dynamical systems $\mathcal{G}$ given by

$$\dot{x}(t) = f_{\rm c}(x(t)), \qquad x(0) = x_0, \quad x(t) \notin \mathcal{Z}, \tag{1}$$

$$\Delta x(t) = f_{\rm d}(x(t)), \qquad x(t) \in \mathcal{Z}, \tag{2}$$

where $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f_{\rm c}(\cdot) \in \mathcal{F}_{\rm c} \subset \{f_{\rm c} : \mathcal{D} \to \mathbb{R}^n : f_{\rm c}(0) = 0\}$, where $f_{\rm c}(\cdot)$ is Lipschitz continuous, $f_{\rm d}(\cdot) \in \mathcal{F}_{\rm d} \subset \{f_{\rm d} : \mathcal{D} \to \mathbb{R}^n : f_{\rm d}(0) = 0\}$, where $f_{\rm d}(\cdot)$ is continuous, and $\mathcal{Z} \subset \mathcal{D}$ is the *resetting set.* We refer to the differential equation (1) as the *continuous-time dynamics*, and we refer to the difference equation (2) as the *resetting law.* Furthermore, $\mathcal{F}_{\rm c}$ and $\mathcal{F}_{\rm d}$ denote the class of nonlinear uncertain impulsive dynamical systems with $f_{\rm c0}(\cdot) \in \mathcal{F}_{\rm c}$ and $f_{\rm d0}(\cdot) \in \mathcal{F}_{\rm d}$ defining the nominal nonlinear impulsive dynamical system for the continuous-time and the resetting dynamics, respectively. Note that since the resetting set $\mathcal{Z}$ is a subset of the state space $\mathcal{D}$ and is independent of time, state-dependent impulsive dynamical systems are time-invariant. In this paper we assume that existence and uniqueness properties of a given state-dependent impulsive dynamical system are satisfied in forward time. For details see [1].

For a particular trajectory $x(t)$, we let $\tau_k(x_0)$ denote the $k^{\rm th}$ instant of time at which $x(t)$ intersects $\mathcal{Z}$, and we call the times $\tau_k(x_0)$ the *resetting times.* Thus the trajectory of the system (1), (2) from the initial condition $x(0) = x_0$ is given by $x(t, x_0)$ for $0 < t \leq \tau_1(x_0)$. If and when the trajectory reaches a state $x_1 \triangleq x(\tau_1(x_0))$ satisfying $x_1 \in \mathcal{Z}$, then the state is instantaneously transferred to $x_1^+ \triangleq x_1 + f_{\rm d}(x_1)$ according to the resetting law (2). The trajectory $x(t)$, $\tau_1(x_0) < t \leq \tau_2(x_0)$, is then given by $x(t - \tau_1(x_0), x_1^+)$, and so on. Note that the solution $x(t)$ of (1), (2) is left-continuous; that is, it is continuous everywhere except at the resetting times $\tau_k(x_0)$, and

$$x_k \triangleq x(\tau_k(x_0)) = \lim_{\varepsilon \to 0^+} x(\tau_k(x_0) - \varepsilon), \tag{3}$$

$$x_k^+ \triangleq x(\tau_k(x_0)) + f_{\rm d}(x(\tau_k(x_0))), \tag{4}$$

for $k = 1, 2, \ldots$.

This research was supported in part by NSF under Grant ECS-9496249, AFOSR under Grant F49620-96-1-0125, and ARO under Grant DAAH04-96-1-0008.

We make the following additional assumptions:

A1. If $x(t) \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$, $x(\delta, x(t)) \notin \mathcal{Z}$.

A2. If $x \in \mathcal{Z}$, then $x + f_{\rm d}(x) \notin \mathcal{Z}$, $f_{\rm d}(\cdot) \in \mathcal{F}_{\rm d}$.

Assumption A1 ensures that if a trajectory reaches the closure of $\mathcal{Z}$ at a point that does not belong to $\mathcal{Z}$, then the trajectory must be directed away from $\mathcal{Z}$, that is, a trajectory cannot enter $\mathcal{Z}$ through a point that belongs to the closure of $\mathcal{Z}$ but not to $\mathcal{Z}$. Furthermore, A2 ensures that when a trajectory intersects the resetting set $\mathcal{Z}$, it instantaneously exits $\mathcal{Z}$. Finally, we note that if $x_0 \in \mathcal{Z}$, then the system initially resets to $x_0^+ = x_0 + f_{\rm d}(x_0)$ which serves as the initial condition for the continuous dynamics (1). It follows from A1 and A2 that $\partial\mathcal{Z} \cap \mathcal{Z}$ is closed and hence the resetting times $\tau_k(x_0)$ are well defined and distinct. Furthermore, it follows from A2 that if $x^* \in \mathcal{D}$ satisfies $f_{\rm d}(x^*) = 0$, then $x^* \notin \mathcal{Z}$. In particular, we note $0 \notin \mathcal{Z}$. For further insights on Assumptions A1 and A2 the interested reader is referred to [2].

For the following result let $L_{\rm c} : \mathcal{D} \to \mathbb{R}$ and $L_{\rm d} : \mathcal{D} \to \mathbb{R}$. Within the context of robustness analysis, it is assumed that the zero solution $x(t) \equiv 0$ of the nominal nonlinear impulsive dynamical system (1), (2) is asymptotically stable. Note that in addressing the stability of the zero solution of an impulsive dynamical system the usual stability definitions are valid. Furthermore, we assume that an infinite number of resettings occur. For the following result and the remainder of the paper we denote the resetting times $\tau_k(x_0)$ by t_k and define $\mathcal{F} \triangleq \mathcal{F}_{\rm c} \times \mathcal{F}_{\rm d}$ and $\mathcal{N}_{[0,t)} \triangleq \{k : 0 \leq t_k < t\}$.

Theorem 2.1. Consider the nonlinear uncertain impulsive dynamical system $\mathcal{G}$ given by (1), (2), where $(f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$, with the hybrid performance functional

$$J_{(f_{\rm c},f_{\rm d})}(x_0) \triangleq \int_0^\infty L_{\rm c}(x(t)){\rm d}t + \sum_{k\in\mathcal{N}_{[0,\infty)}} L_{\rm d}(x(t_k)). \quad (5)$$

Furthermore, assume there exist functions $\Gamma_{\rm c} : \mathcal{D} \to \mathbb{R}$, $\Gamma_{\rm d} : \mathcal{D} \to \mathbb{R}$, and $V : \mathcal{D} \to \mathbb{R}$, where $V(\cdot)$ is a C^1 function, such that

$$V(0) = 0, \quad (6)$$

$$V(x) > 0, \quad x \in \mathcal{D}, \quad x \neq 0, \quad (7)$$

$$V'(x)f_{\rm c}(x) \leq V'(x)f_{\rm c0}(x) + \Gamma_{\rm c}(x), \ x \notin \mathcal{Z}, \ f_{\rm c}(\cdot) \in \mathcal{F}_{\rm c}, \quad (8)$$

$$V'(x)f_{\rm c0}(x) + \Gamma_{\rm c}(x) < 0, \quad x \notin \mathcal{Z}, \quad x \neq 0, \quad (9)$$

$$V(x + f_{\rm d}(x)) \leq V(x + f_{\rm d0}(x)) + \Gamma_{\rm d}(x), \ x \in \mathcal{Z}, f_{\rm d}(\cdot) \in \mathcal{F}_{\rm d}, \quad (10)$$

$$V(x + f_{\rm d0}(x)) - V(x) + \Gamma_{\rm d}(x) \leq 0, \quad x \in \mathcal{Z}, \quad (11)$$

$$L_{\rm c}(x) + V'(x)f_{\rm c0}(x) + \Gamma_{\rm c}(x) = 0, \quad x \notin \mathcal{Z}, \quad (12)$$

$$L_{\rm d}(x) + V(x + f_{\rm d0}(x)) - V(x) + \Gamma_{\rm d}(x) = 0, \quad x \in \mathcal{Z}, \quad (13)$$

where $(f_{\rm c0}(\cdot), f_{\rm d0}(\cdot)) \in \mathcal{F}$ defines the nominal nonlinear impulsive dynamical system. Then there exists a neighborhood $\mathcal{D}_0 \subset \mathcal{D}$ of the origin such that if $x_0 \in \mathcal{D}_0$, then the zero solution $x(t) \equiv 0$ to (1), (2) is locally asymptotically stable for all $(f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$, and

$$\sup_{(f_{\rm c}(\cdot),f_{\rm d}(\cdot))\in\mathcal{F}} J_{(f_{\rm c},f_{\rm d})}(x_0) \leq \mathcal{J}(x_0) = V(x_0), \quad (14)$$

where

$$\begin{aligned} \mathcal{J}(x_0) \ &\triangleq \int_0^\infty [L_{\rm c}(x(t)) + \Gamma_{\rm c}(x(t))]{\rm d}t \\ &\quad + \sum_{k\in\mathcal{N}_{[0,\infty)}} [L_{\rm d}(x(t_k)) + \Gamma_{\rm d}(x(t_k))], \quad (15) \end{aligned}$$

and where $x(t)$, $t \geq 0$, is a solution to (1), (2) with $(f_{\rm c}(x(t)), f_{\rm d}(x(t_k))) = (f_{\rm c0}(x(t)), f_{\rm d0}(x(t_k)))$. Finally, if $\mathcal{D} = \mathbb{R}^n$ and

$$V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty, \quad (16)$$

then the zero solution $x(t) \equiv 0$ to (1), (2) is globally asymptotically stable for all $(f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$.

Proof. Let $(f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$ and $x(t)$, $t \geq 0$, satisfy (1), (2). Then,

$$\begin{aligned} \dot{V}(x(t)) \triangleq \frac{\rm d}{{\rm d}t}V(x(t)) = V'(x(t))f_{\rm c}(x(t)), \quad x(t) \notin \mathcal{Z}, \\ t_k < t \leq t_{k+1}. \quad (17) \end{aligned}$$

Hence, it follows from (8) and (9) that

$$\dot{V}(x(t)) < 0, \quad x(t) \notin \mathcal{Z}, \ x(t) \neq 0, \ t_k < t \leq t_{k+1}. \quad (18)$$

Furthermore,

$$\Delta V(x(t_k)) \triangleq V(x(t_k) + f_{\rm d}(x(t_k))) - V(x(t_k)), \ x(t_k) \in \mathcal{Z}. \quad (19)$$

Hence, it follows from (10) and (11) that

$$\Delta V(x(t_k)) \leq 0, \qquad x(t_k) \in \mathcal{Z}. \quad (20)$$

Thus, using (6), (7), (18), and (20) it follows from Theorem 3.2 of [2] that $V(\cdot)$ is a Lyapunov function for (1), (2), which proves local asymptotic stability of the zero solution $x(t) \equiv 0$ to (1), (2) for all $(f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$. Consequently, $x(t) \to 0$ as $t \to \infty$ for all initial conditions $x_0 \in \mathcal{D}_0$ for some neighborhood $\mathcal{D}_0 \subset \mathcal{D}$ of the origin. Now, (18) implies that

$$0 = -\dot{V}(x(t)) + V'(x(t))f_{\rm c}(x(t)), \ x(t) \notin \mathcal{Z}, t_k < t \leq t_{k+1}, \quad (21)$$

and hence, using (8) and (12),

$$\begin{aligned} L_{\rm c}(x(t)) &= -\dot{V}(x(t)) + L_{\rm c}(x(t)) + V'(x(t))f_{\rm c}(x(t)) \\ &\leq -\dot{V}(x(t)) + L_{\rm c}(x(t)) + V'(x(t))f_{\rm c0}(x(t)) + \Gamma_{\rm c}(x(t)) \\ &= -\dot{V}(x(t)), \qquad x(t) \notin \mathcal{Z}, \quad t_k < t \leq t_{k+1}. \quad (22) \end{aligned}$$

Similarly, (19) implies that

$$\begin{aligned} 0 = -\Delta V(x(t_k)) + V(x(t_k) + f_{\rm d}(x(t_k))) - V(x(t_k)), \\ x(t_k) \in \mathcal{Z}, \quad (23) \end{aligned}$$

and hence, using (10) and (13),

$$\begin{aligned} L_{\rm d}(x(t_k)) \ &= \ -\Delta V(x(t_k)) + L_{\rm d}(x(t_k)) + V(x(t_k) \\ &\quad + f_{\rm d}(x(t_k))) - V(x(t_k)) \\ &\leq \ -\Delta V(x(t_k)) + L_{\rm d}(x(t_k)) + V(x(t_k) \\ &\quad + f_{\rm d0}(x(t_k))) - V(x(t_k)) + \Gamma_{\rm d}(x(t_k)) \\ &= \ -\Delta V(x(t_k)), \qquad x(t_k) \in \mathcal{Z}. \quad (24) \end{aligned}$$

Now, integrating over the interval $[0, t)$ with $\mathcal{N}_{[0,t)} = \{1, 2, \ldots, i\}$, (22) and (24) yield

$$\begin{aligned} &\int_0^t L_{\rm c}(x(s)){\rm d}s + \sum_{k\in\mathcal{N}_{[0,t)}} L_{\rm d}(x(t_k)) \\ &= \int_0^{t_1} L_{\rm c}(x(s)){\rm d}s + L_{\rm d}(x(t_1)) + \int_{t_1^+}^{t_2} L_{\rm c}(x(s)){\rm d}s + L_{\rm d}(x(t_2)) \\ &\quad + \ldots + \int_{t_{i-1}^+}^{t_i} L_{\rm c}(x(s)){\rm d}s + L_{\rm d}(x(t_i)) + \int_{t_{i+}}^{t} L_{\rm c}(x(s)){\rm d}s \\ &\leq -V(x(t_1)) + V(x_0) - V(x(t_1) + f_{\rm d}(x(t_1))) + V(x(t_1)) \\ &\quad -V(x(t_2)) + V(x(t_1^+)) - V(x(t_2) + f_{\rm d}(x(t_2))) \\ &\quad +V(x(t_2)) + \ldots - V(x(t_i)) + V(x(t_{i-1}^+)) \\ &\quad -V(x(t_i) + f_{\rm d}(x(t_i))) + V(x(t_i)) - V(x(t)) + V(x(t_i^+)) \\ &\leq -V(x(t_1)) + V(x_0) - V(x(t_1^+)) + V(x(t_1)) - V(x(t_2)) \\ &\quad +V(x(t_1^+)) - V(x(t_2^+)) + V(x(t_2)) + \ldots - V(x(t_i)) \end{aligned}$$

$$+V(x(t_{i-1}^+))-V(x(t_i^+))+V(x(t_i))-V(x(t))+V(x(t_i^+)) \leq -V(x(t))+V(x_0). \quad (25)$$

Letting $t \to \infty$ and noting that $V(x(t)) \to 0$ for all $x_0 \in \mathcal{D}_0$ yields $J_{(f_{\rm c},f_{\rm d})}(x_0) \leq V(x_0)$. Next, let $x(t)$, $t \geq 0$, satisfy (1), (2) with $(f_{\rm c}(x(t)), f_{\rm d}(x(t_k))) = (f_{\rm c0}(x(t)), f_{\rm d0}(x(t_k)))$. Then it follows from (12) that

$$\begin{aligned} L_{\rm c}(x(t))+\Gamma_{\rm c}(x(t)) &= -\dot{V}(x(t))+L_{\rm c}(x(t)) \\ &\quad +V'(x(t))f_{\rm c0}(x(t))+\Gamma_{\rm c}(x(t)) \\ &= -\dot{V}(x(t)),\ \ x(t)\notin\mathcal{Z},\ t_k<t\leq t_{k+1}. \quad (26) \end{aligned}$$

Similarly, it follows from (13) that

$$\begin{aligned} L_{\rm d}(x(t_k))+\Gamma_{\rm d}(x(t_k)) &= -\Delta V(x(t_k))+L_{\rm d}(x(t_k))+V(x(t_k) \\ &\quad +f_{\rm d0}(x(t_k)))-V(x(t_k))+\Gamma_{\rm d}(x(t_k)) \\ &= -\Delta V(x(t_k)), \qquad x(t_k)\in\mathcal{Z}. \quad (27) \end{aligned}$$

Now, integrating over the interval $[0,t)$ with $\mathcal{N}_{[0,t)} = \{1,2,\ldots,i\}$, (26) and (27) yield

$$\int_0^t [L_{\rm c}(x(t))+\Gamma_{\rm c}(x(t))]{\rm d}t + \sum_{k\in\mathcal{N}_{[0,t)}} [L_{\rm d}(x(t_k))+\Gamma_{\rm d}(x(t_k))] = -V(x(t))+V(x_0). \quad (28)$$

Letting $t \to \infty$ and noting that $V(x(t)) \to 0$ for all $x_0 \in \mathcal{D}_0$ yields $\mathcal{J}(x_0) = V(x_0)$. Finally, for $\mathcal{D} = \mathbb{R}^n$ and for all $(f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$, global asymptotic stability of the zero solution $x(t) \equiv 0$ to (1), (2) is a direct consequence of Theorem 3.2 of [2] using the radially unbounded condition (16) on $V(x)$, $x \in \mathbb{R}^n$. □

Remark 2.1. Theorem 2.1 provides sufficient conditions for robust stability of a class of nonlinear uncertain impulsive dynamical systems $(f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$. Specifically, (6) and (7) assume that $V(x)$ is a Lyapunov function candidate for the nonlinear uncertain impulsive dynamical system (1), (2). Conditions (8)–(11) imply $\dot{V}(x(t)) < 0$, $x(t) \notin \mathcal{Z}$, $t > 0$, and $\Delta V(x(t_k)) \leq 0$, $x(t_k) \in \mathcal{Z}$, $k \in \mathcal{N}$, for $x(\cdot)$ satisfying (1), (2) for all $(f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$, and hence $V(\cdot)$ is a Lyapunov function guaranteeing robust stability of the nonlinear uncertain impulsive dynamical system (1), (2). It is important to note that Conditions (9) and (11) are *verifiable* conditions since they are independent of the uncertain system parameters $(f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$. To apply Theorem 2.1 we specify the bounding functions $\Gamma_{\rm c}(\cdot)$ and $\Gamma_{\rm d}(\cdot)$ for the uncertain set $\mathcal{F}_{\rm c} \times \mathcal{F}_{\rm d}$ such that $\Gamma_{\rm c}(\cdot)$ and $\Gamma_{\rm d}(\cdot)$ bound $\mathcal{F}_{\rm c} \times \mathcal{F}_{\rm d}$. For further details see [4]. If $\mathcal{F}$ consists only of the nominal nonlinear impulsive dynamical system $(f_{\rm c0}(\cdot), f_{\rm d0}(\cdot))$, then $\Gamma_{\rm c}(x) = 0$ and $\Gamma_{\rm d}(x) = 0$ for all $x \in \mathcal{D}$ satisfy (8) and (10), respectively, and hence $J_{(f_{\rm c0},f_{\rm d0})}(x_0) = \mathcal{J}(x_0)$. Finally, a worst-case upper bound to the nonlinear-nonquadratic hybrid performance functional is given in terms of a Lyapunov function which can be interpreted in terms of an auxiliary cost defined for the nominal impulsive dynamical system.

Next, we specialize Theorem 2.1 to nonlinear uncertain impulsive dynamical systems of the form

$$\dot{x}(t) = f_{\rm c0}(x(t))+\Delta f_{\rm c}(x(t)), \quad x(0)=x_0,\ x(t)\notin\mathcal{Z}, \quad (29)$$

$$\Delta x(t) = f_{\rm d0}(x(t))+\Delta f_{\rm d}(x(t)), \quad x(t)\in\mathcal{Z}, \quad (30)$$

where $t \geq 0$, $f_{\rm c0} : \mathcal{D} \to \mathbb{R}^n$ and satisfies $f_{\rm c0}(0) = 0$, $f_{\rm d0} : \mathcal{D} \to \mathbb{R}^n$ and satisfies $f_{\rm d0}(0) = 0$, and $(\Delta f_{\rm c}, \Delta f_{\rm d}) \in \mathcal{F} = \mathcal{F}_{\rm c} \times \mathcal{F}_{\rm d}$, where

$$\mathcal{F}_{\rm c} \subset \{\Delta f_{\rm c} : \mathcal{D} \to \mathbb{R}^n : \Delta f_{\rm c}(0) = 0\}, \quad (31)$$

$$\mathcal{F}_{\rm d} \subset \{\Delta f_{\rm d} : \mathcal{D} \to \mathbb{R}^n : \Delta f_{\rm d}(0) = 0\}. \quad (32)$$

Corollary 2.1. Consider the nonlinear uncertain impulsive dynamical system (29), (30) with the hybrid performance functional (5). Furthermore, assume there exist functions $\Gamma_{\rm c} : \mathcal{D} \to \mathbb{R}$, $\Gamma_{\rm d} : \mathcal{D} \to \mathbb{R}$, $P_{1f_{\rm d}} : \mathcal{D} \to \mathbb{R}^{1\times n}$, $P_{2f_{\rm d}} : \mathcal{D} \to \mathbb{N}^n$, and $V : \mathcal{D} \to \mathbb{R}$, where $V(\cdot)$ is a C^1 function, such that (6), (7), (9), and (11)–(13) hold, and

$$V'(x)\Delta f_{\rm c}(x) \leq \Gamma_{\rm c}(x), \quad x\notin\mathcal{Z},\ \Delta f_{\rm c}(\cdot)\in\mathcal{F}_{\rm c}, \quad (33)$$

$$P_{1f_{\rm d}}(0) = 0, \quad (34)$$

$$\Delta f_{\rm d}^{\rm T}(x)P_{1f_{\rm d}}(x)+P_{1f_{\rm d}}(x)\Delta f_{\rm d}(x)+\Delta f_{\rm d}^{\rm T}(x)P_{2f_{\rm d}}(x)\Delta f_{\rm d}(x) \leq \Gamma_{\rm d}(x), \quad x\in\mathcal{Z},\ \Delta f_{\rm d}(\cdot)\in\mathcal{F}_{\rm d}, \quad (35)$$

$$\begin{aligned} V(x+f_{\rm d0}(x)+\Delta f_{\rm d}(x)) &= V(x+f_{\rm d0}(x))+\Delta f_{\rm d}^{\rm T}(x)P_{1f_{\rm d}}^{\rm T}(x) \\ &\quad +P_{1f_{\rm d}}(x)\Delta f_{\rm d}(x)+\Delta f_{\rm d}^{\rm T}(x)P_{2f_{\rm d}}(x)\Delta f_{\rm d}(x), \quad x\in\mathcal{Z}, \\ &\qquad \Delta f_{\rm d}(\cdot)\in\mathcal{F}_{\rm d}. \quad (36) \end{aligned}$$

Then there exists a neighborhood $\mathcal{D}_0 \subset \mathcal{D}$ of the origin such that if $x_0 \in \mathcal{D}_0$, then the zero solution $x(t) \equiv 0$ to (29), (30) is locally asymptotically stable for all $(\Delta f_{\rm c}(\cdot), \Delta f_{\rm d}(\cdot)) \in \mathcal{F}$, and the hybrid performance functional (5) satisfies

$$\sup_{(\Delta f_{\rm c}(\cdot),\Delta f_{\rm d}(\cdot))\in\mathcal{F}} J_{(\Delta f_{\rm c},\Delta f_{\rm d})}(x_0) \leq \mathcal{J}(x_0) = V(x_0), \quad (37)$$

where

$$\mathcal{J}(x_0) \triangleq \int_0^\infty [L_{\rm c}(x(t))+\Gamma_{\rm c}(x(t))]{\rm d}t + \sum_{k\in\mathcal{N}_{[0,\infty)}} [L_{\rm d}(x(t_k))+\Gamma_{\rm d}(x(t_k))], \quad (38)$$

and where $x(t)$, $t \geq 0$, is a solution to (29), (30) with $(\Delta f_{\rm c}(x(t)), \Delta f_{\rm d}(x(t_k))) = (0,0)$. Finally, if $\mathcal{D} = \mathbb{R}^n$ and $V(x)$, $x \in \mathbb{R}^n$, satisfies (16), then the zero solution $x(t) \equiv 0$ to (29), (30) is globally asymptotically stable for all $(\Delta f_{\rm c}(\cdot), \Delta f_{\rm d}(\cdot)) \in \mathcal{F}$.

Proof. The result is direct consequence of Theorem 2.1 with $f_{\rm c}(x) = f_{\rm c0}(x)+\Delta f_{\rm c}(x)$, $f_{\rm d}(x) = f_{\rm d0}(x)+\Delta f_{\rm d}(x)$, and $V(x+f_{\rm d}(x))$ given by (36). □

The following corollary specializes Theorem 2.1 to a class of linear uncertain impulsive dynamical systems. Specifically, we consider $\mathcal{F} = \mathcal{F}_{\rm c} \times \mathcal{F}_{\rm d}$ to be the set of linear uncertain impulsive dynamical systems given by

$$\mathcal{F}_{\rm c} = \{(A_{\rm c}+\Delta A_{\rm c})x : x\in\mathbb{R}^n,\ A_{\rm c}\in\mathbb{R}^{n\times n},\ \Delta A_{\rm c}\in\boldsymbol{\Delta}_{A_{\rm c}}\},$$

$$\mathcal{F}_{\rm d} = \{(A_{\rm d}+\Delta A_{\rm d})x : x\in\mathbb{R}^n,\ A_{\rm d}\in\mathbb{R}^{n\times n},\ \Delta A_{\rm d}\in\boldsymbol{\Delta}_{A_{\rm d}}\},$$

where $(\boldsymbol{\Delta}_{A_{\rm c}}, \boldsymbol{\Delta}_{A_{\rm d}}) \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ are given bounded uncertainity sets of uncertain perturbations $\Delta A_{\rm c}$ and $\Delta A_{\rm d}$ of the nominal system matrices $A_{\rm c}$ and $A_{\rm d}$ such that $0 \in \boldsymbol{\Delta}_{A_{\rm c}}$ and $0 \in \boldsymbol{\Delta}_{A_{\rm d}}$. For the statement of the following result define $\boldsymbol{\Delta} \triangleq \boldsymbol{\Delta}_{A_{\rm c}} \times \boldsymbol{\Delta}_{A_{\rm d}}$.

Corollary 2.2. Let $R_{\rm c} \in \mathbb{P}^n$ and $R_{\rm d} \in \mathbb{N}^n$. Consider the linear state-dependent uncertain impulsive dynamical system

$$\dot{x}(t) = (A_{\rm c}+\Delta A_{\rm c})x(t), \quad x(0)=x_0,\ t\geq 0,\ x(t)\notin\mathcal{Z}, \quad (39)$$

$$\Delta x(t) = (A_{\rm d}+\Delta A_{\rm d}-I_n)x(t), \quad x(t)\in\mathcal{Z}, \quad (40)$$

with the hybrid quadratic performance functional

$$J_{(\Delta A_{\rm c},\Delta A_{\rm d})}(x_0) \triangleq \int_0^\infty x^{\rm T}(t)R_{\rm c}x(t){\rm d}t + \sum_{k\in\mathcal{N}_{[0,\infty)}} x^{\rm T}(t_k)R_{\rm d}x(t_k), \quad (41)$$

where $(\Delta A_{\rm c}, \Delta A_{\rm d}) \in \boldsymbol{\Delta}$. Let $\Omega_{\rm c} : \mathcal{N}_{\rm P} \subseteq \mathbb{S}^n \to \mathbb{N}^n$ and

$\Omega_{\rm d} : \mathcal{N}_{\rm P} \subseteq \mathbb{S}^n \to \mathbb{N}^n$ be such that

$$x^{\rm T}(\Delta A_{\rm c}^{\rm T}P + P\Delta A_{\rm c})x \le x^{\rm T}\Omega_{\rm c}(P)x,\ x \notin \mathcal{Z},\ \Delta A_{\rm c} \in \mathbf{\Delta}_{A_{\rm c}}, \tag{42}$$

$$x^{\rm T}(\Delta A_{\rm d}^{\rm T}PA_{\rm d} + A_{\rm d}^{\rm T}P\Delta A_{\rm d} + \Delta A_{\rm d}^{\rm T}P\Delta A_{\rm d})x \le x^{\rm T}\Omega_{\rm d}(P)x, \quad x \in \mathcal{Z}, \quad \Delta A_{\rm d} \in \mathbf{\Delta}_{A_{\rm d}}, \tag{43}$$

where $P \in \mathcal{N}_{\rm P}$. Furthermore, suppose there exists $P \in \mathbb{P}^n$ satisfying

$$0 = x^{\rm T}(A_{\rm c}^{\rm T}P + PA_{\rm c} + \Omega_{\rm c}(P) + R_{\rm c})x, \quad x \notin \mathcal{Z}, \tag{44}$$

$$0 = x^{\rm T}(A_{\rm d}^{\rm T}PA_{\rm d} - P + \Omega_{\rm d}(P) + R_{\rm d})x, \quad x \in \mathcal{Z}. \tag{45}$$

Then the zero solution $x(t) \equiv 0$ to (39), (40) is globally asymptotically stable for all $(\Delta A_{\rm c}, \Delta A_{\rm d}) \in \mathbf{\Delta}$, and the hybrid quadratic performance functional (41) satisfies

$$\sup_{(\Delta A_{\rm c},\Delta A_{\rm d})\in\mathbf{\Delta}} J_{(\Delta A_{\rm c},\Delta A_{\rm d})}(x_0) \le \mathcal{J}(x_0) = x_0^{\rm T}Px_0,\ x_0 \in \mathbb{R}^n, \tag{46}$$

where

$$\mathcal{J}(x_0) \triangleq \int_0^\infty x^{\rm T}(t)(\Omega_{\rm c}(P) + R_{\rm c})x(t){\rm d}t + \sum_{k\in\mathcal{N}_{[0,\infty)}} x^{\rm T}(t_k)(\Omega_{\rm d}(P) + R_{\rm d})x(t_k), \tag{47}$$

and where $x(t)$, $t \ge 0$, is a solution to (39), (40) with $(\Delta A_{\rm c}, \Delta A_{\rm d}) = (0, 0)$.

Proof. The result is a direct consequence of Theorem 2.1 with $f_{\rm c}(x) = (A_{\rm c} + \Delta A_{\rm c})x$, $f_{\rm c0}(x) = A_{\rm c}x$, $L_{\rm c}(x) = x^{\rm T}R_{\rm c}x$, $\Gamma_{\rm c}(x) = x^{\rm T}\Omega_{\rm c}(P)x$, $f_{\rm d}(x) = (A_{\rm d} + \Delta A_{\rm d} - I_n)x$, $f_{\rm d0}(x) = (A_{\rm d} - I_n)x$, $L_{\rm d}(x) = x^{\rm T}R_{\rm d}x$, $\Gamma_{\rm d}(x) = x^{\rm T}\Omega_{\rm d}(P)x$, $V(x) = x^{\rm T}Px$, and $\mathcal{D} = \mathbb{R}^n$. □

3. Optimal Robust Control for Nonlinear Uncertain Impulsive Dynamical Systems

To address the robust optimal control problem let $\mathcal{D} \subset \mathbb{R}^n$ be an open set with $0 \in \mathcal{D}$ and let $\mathcal{C}_{\rm c} \subset \mathbb{R}^{m_{\rm c}}$ and $\mathcal{C}_{\rm d} \subset \mathbb{R}^{m_{\rm d}}$, where $0 \in \mathcal{C}_{\rm c}$ and $0 \in \mathcal{C}_{\rm d}$. Furthermore, let $\mathcal{F}_{\rm c} \subset \{F_{\rm c} : \mathcal{D} \times \mathcal{C}_{\rm c} \to \mathbb{R}^n : F_{\rm c}(0, 0) = 0\}$ and $\mathcal{F}_{\rm d} \subset \{F_{\rm d} : \mathcal{D} \times \mathcal{C}_{\rm d} \to \mathbb{R}^n : F_{\rm d}(0, 0) = 0\}$. Next, consider the nonlinear uncertain impulsive controlled dynamical system

$$\dot{x}(t) = F_{\rm c}(x(t), u_{\rm c}(t)), \quad x(0) = 0,\ x(t) \notin \mathcal{Z}_x,\ u_{\rm c}(t) \in \mathcal{U}_{\rm c}, \tag{48}$$

$$\Delta x(t) = F_{\rm d}(x(t), u_{\rm d}(t)), \quad x(t) \in \mathcal{Z}_x,\ u_{\rm d}(t) \in \mathcal{U}_{\rm d}, \tag{49}$$

where $t \ge 0$, $(F_{\rm c}(\cdot,\cdot), F_{\rm d}(\cdot,\cdot)) \in \mathcal{F}$, where $\mathcal{F} \triangleq \mathcal{F}_{\rm c} \times \mathcal{F}_{\rm d}$, $\mathcal{Z}_x \subset \mathcal{D}$, and $(u_{\rm c}(t), u_{\rm d}(t_k)) \in \mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d} \subset \mathcal{C}_{\rm c} \times \mathcal{C}_{\rm d}$, $k \in \mathcal{N}$, is the hybrid control input where the control constraint sets $\mathcal{U}_{\rm c}$ and $\mathcal{U}_{\rm d}$ are given. We assume $(0, 0) \in \mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d}$, $F_{\rm c} : \mathcal{D} \times \mathcal{U}_{\rm c} \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $F_{\rm c}(0, 0) = 0$, and $F_{\rm d} : \mathcal{D} \times \mathcal{U}_{\rm d} \to \mathbb{R}^n$ is continuous and satisfies $F_{\rm d}(0, 0) = 0$. To address the robust optimal nonlinear hybrid feedback control problem let $\phi_{\rm c} : \mathcal{D} \to \mathcal{U}_{\rm c}$ be such that $\phi_{\rm c}(0) = 0$ and let $\phi_{\rm d} : \mathcal{D} \to \mathcal{U}_{\rm d}$ be such that $\phi_{\rm d}(0) = 0$. If $(u_{\rm c}(t), u_{\rm d}(t_k)) = (\phi_{\rm c}(x(t)), \phi_{\rm d}(x(t_k)))$, where $x(t)$, $t \ge 0$, satisfies (48), (49), then $(u_{\rm c}(\cdot), u_{\rm d}(\cdot))$ is a *hybrid feedback control*. Given the hybrid feedback control $(u_{\rm c}(t), u_{\rm d}(t_k)) = (\phi_{\rm c}(x(t)), \phi_{\rm d}(x(t_k)))$, the closed-loop state-dependent impulsive dynamical system has the form

$$\dot{x}(t) = F_{\rm c}(x(t), \phi_{\rm c}(x(t))), \quad x(0) = x_0,\ t \ge 0,\ x(t) \notin \mathcal{Z}_x, \tag{50}$$

$$\Delta x(t) = F_{\rm d}(x(t), \phi_{\rm d}(x(t))), \quad x(t) \in \mathcal{Z}_x, \tag{51}$$

for all $(F_{\rm c}(\cdot,\cdot), F_{\rm d}(\cdot,\cdot)) \in \mathcal{F}$.

Next we present sufficient conditions for characterizing robust nonlinear hybrid feedback controllers that guarantee robust stability over a class of nonlinear uncertain impulsive dynamical systems and minimize an auxiliary hybrid performance functional. For the statement of this result let $L_{\rm c} : \mathcal{D} \times \mathcal{U}_{\rm c} \to \mathbb{R}$, $L_{\rm d} : \mathcal{D} \times \mathcal{U}_{\rm d} \to \mathbb{R}$, and define the set of asymptotically stabilizing hybrid controllers for the nominal nonlinear impulsive dynamical system $(F_{\rm c0}(\cdot,\cdot), F_{\rm d0}(\cdot,\cdot))$ by

$$\mathcal{C}(x_0) \triangleq \{(u_{\rm c}(\cdot), u_{\rm d}(\cdot)) : (u_{\rm c}(\cdot), u_{\rm d}(\cdot)) \text{ is admissible and the zero solution } x(t) \equiv 0 \text{ to (48), (49) is asymptotically stable with } (F_{\rm c}(\cdot,\cdot), F_{\rm d}(\cdot,\cdot)) = (F_{\rm c0}(\cdot,\cdot), F_{\rm d0}(\cdot,\cdot))\}.$$

Theorem 3.1. Consider the nonlinear uncertain impulsive dynamical system (48), (49) with the hybrid performance functional

$$J_{(F_{\rm c},F_{\rm d})}(x_0, u_{\rm c}(\cdot), u_{\rm d}(\cdot)) = \int_0^\infty L_{\rm c}(x(t), u(t)){\rm d}t + \sum_{k\in\mathcal{N}_{[0,\infty)}} L_{\rm d}(x(t_k), u_{\rm d}(t_k)), \tag{52}$$

where $(F_{\rm c}(\cdot,\cdot), F_{\rm d}(\cdot,\cdot)) \in \mathcal{F}$ and $(u_{\rm c}(\cdot), u_{\rm d}(\cdot))$ is an admissible hybrid control. Assume there exist functions $V : \mathcal{D} \to \mathbb{R}$, $\Gamma_{\rm c} : \mathcal{D} \times \mathcal{U}_{\rm c} \to \mathbb{R}$, $\Gamma_{\rm d} : \mathcal{D} \times \mathcal{U}_{\rm d} \to \mathbb{R}$, and a hybrid control law $\phi_{\rm c} : \mathcal{D} \to \mathcal{U}_{\rm c}$ and $\phi_{\rm d} : \mathcal{D} \to \mathcal{U}_{\rm d}$, where $V(\cdot)$ is a C^1 function, such that

$$V(0) = 0, \tag{53}$$

$$V(x) > 0, \quad x \in \mathcal{D}, \quad x \ne 0, \tag{54}$$

$$\phi_{\rm c}(0) = 0, \tag{55}$$

$$\phi_{\rm d}(0) = 0, \tag{56}$$

$$V'(x)F_{\rm c}(x, \phi_{\rm c}(x)) \le V'(x)F_{\rm c0}(x, \phi_{\rm c}(x)) + \Gamma_{\rm c}(x, \phi_{\rm c}(x)), \quad x \notin \mathcal{Z}_x, \quad F_{\rm c}(\cdot,\cdot) \in \mathcal{F}_{\rm c}, \tag{57}$$

$$V'(x)F_{\rm c0}(x, \phi_{\rm c}(x)) + \Gamma_{\rm c}(x, \phi_{\rm c}(x)) < 0,\ x \notin \mathcal{Z}_x,\ x \ne 0, \tag{58}$$

$$V(x + F_{\rm d}(x, \phi_{\rm d}(x))) \le V(x + F_{\rm d0}(x, \phi_{\rm d}(x))) + \Gamma_{\rm d}(x, \phi_{\rm d}(x)), \quad x \in \mathcal{Z}_x, \quad F_{\rm d}(\cdot,\cdot) \in \mathcal{F}_{\rm d}, \tag{59}$$

$$V(x + F_{\rm d0}(x, \phi_{\rm d}(x))) - V(x) + \Gamma_{\rm d}(x, \phi_{\rm d}(x)) \le 0,\ x \in \mathcal{Z}_x, \tag{60}$$

$$H_{\rm c}(x, \phi_{\rm c}(x)) = 0, \quad x \notin \mathcal{Z}_x, \tag{61}$$

$$H_{\rm d}(x, \phi_{\rm d}(x)) = 0, \quad x \in \mathcal{Z}_x, \tag{62}$$

$$H_{\rm c}(x, u_{\rm c}(x)) \ge 0, \quad x \notin \mathcal{Z}_x, \quad u_{\rm c} \in \mathcal{U}_{\rm c}, \tag{63}$$

$$H_{\rm d}(x, u_{\rm d}(x)) \ge 0, \quad x \in \mathcal{Z}_x, \quad u_{\rm d} \in \mathcal{U}_{\rm d}, \tag{64}$$

where $(F_{\rm c0}(\cdot,\cdot), F_{\rm d0}(\cdot,\cdot)) \in \mathcal{F}$ defines the nominal impulsive dynamical system and

$$H_{\rm c}(x, u_{\rm c}) \triangleq L_{\rm c}(x, u_{\rm c}) + V'(x)F_{\rm c0}(x, u_{\rm c}) + \Gamma_{\rm c}(x, u_{\rm c}), \tag{65}$$

$$H_{\rm d}(x, u_{\rm d}) \triangleq L_{\rm d}(x, u_{\rm d}) + V(x + F_{\rm d0}(x, u_{\rm d})) - V(x) + \Gamma_{\rm d}(x, u_{\rm d}). \tag{66}$$

Then, with the hybrid feedback control $(u_{\rm c}(\cdot), u_{\rm d}(\cdot)) = (\phi_{\rm c}(x(\cdot)), \phi_{\rm d}(x(\cdot)))$, there exists a neighborhood of the origin $\mathcal{D}_0 \subset \mathcal{D}$ such that if $x_0 \in \mathcal{D}_0$, the zero solution $x(t) \equiv 0$ of the closed-loop system (50), (51) is locally asymptotically stable for all $(F_{\rm c}(\cdot,\cdot), F_{\rm d}(\cdot,\cdot)) \in \mathcal{F}$. Furthermore,

$$\sup_{(F_{\rm c}(\cdot,\cdot),F_{\rm d}(\cdot,\cdot))\in\mathcal{F}} J_{(F_{\rm c},F_{\rm d})}(x_0, \phi_{\rm c}(x(\cdot)), \phi_{\rm d}(x(\cdot))) \le \mathcal{J}(x_0, \phi_{\rm c}(\cdot), \phi_{\rm d}(\cdot)) = V(x_0), \quad x_0 \in \mathcal{D}_0, \tag{67}$$

where

$$\mathcal{J}(x_0, u_{\rm c}(\cdot), u_{\rm d}(\cdot)) \triangleq \int_0^\infty [L_{\rm c}(x(t), u_{\rm c}(t)) + \Gamma_{\rm c}(x(t), u_{\rm c}(t))]{\rm d}t + \sum_{k\in\mathcal{N}_{[0,\infty)}} [L_{\rm d}(x(t_k), u_{\rm d}(t_k)) + \Gamma_{\rm d}(x(t_k), u_{\rm d}(t_k))], \tag{68}$$

and where $(u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot))$ is an admissible hybrid control and $x(t)$, $t \geq 0$, solves (48), (49) with $(F_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t)), F_{\mathrm{d}}(x(t_k), u_{\mathrm{d}}(t_k))) = (F_{\mathrm{c}0}(x(t), u_{\mathrm{c}}(t)), F_{\mathrm{d}0}(x(t_k), u_{\mathrm{d}}(t_k)))$. In addition, if $x_0 \in \mathcal{D}_0$ then the hybrid feedback control $(u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot)) = (\phi_{\mathrm{c}}(x(\cdot)), \phi_{\mathrm{d}}(x(\cdot)))$ minimizes $J(x_0, u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot))$ in the sense that

$$\mathcal{J}(x_0, \phi_{\mathrm{c}}(x(\cdot)), \phi_{\mathrm{d}}(x(\cdot))) = \min_{(u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot)) \in \mathcal{C}(x_0)} \mathcal{J}(x_0, u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot)). \tag{69}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $\mathcal{U}_{\mathrm{c}} = \mathbb{R}^{m_{\mathrm{c}}}$, $\mathcal{U}_{\mathrm{d}} = \mathbb{R}^{m_{\mathrm{d}}}$, and $V(x) \to \infty$ as $\|x\| \to \infty$, then the zero solution $x(t) \equiv 0$ of the closed-loop system (50), (51) is globally asymptotically stable for all $(F_{\mathrm{c}}(\cdot,\cdot), F_{\mathrm{d}}(\cdot,\cdot)) \in \mathcal{F}$.

Proof. Local and global asymptotic stability are a direct consequence of (53)–(60) by applying Theorem 2.1 to the closed-loop system (50), (51). Furthermore, using (61) and (62), condition (67) is a restatement of (14) as applied to the closed-loop system (50), (51). Next, let $(u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot)) \in \mathcal{C}(x_0)$ and let $x(\cdot)$ be the solution of (48), (49) with $(F_{\mathrm{c}}(\cdot,\cdot), F_{\mathrm{d}}(\cdot,\cdot)) = (F_{\mathrm{c}0}(\cdot,\cdot), F_{\mathrm{d}0}(\cdot,\cdot))$. Then it follows that

$$0 = -\dot{V}(x(t)) + V'(x(t))F_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t)), \qquad x(t) \notin \mathcal{Z}_x, \quad t_k < t \leq t_{k+1}, \tag{70}$$

$$0 = -\Delta V(x(t_k)) + V(x(t_k) + F_{\mathrm{d}}(x(t_k), u_{\mathrm{d}}(t_k))) - V(x(t_k)), \quad x(t_k) \in \mathcal{Z}_x. \tag{71}$$

Hence,

$$\begin{aligned} L_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t)) + \Gamma_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t)) &= -\dot{V}(x(t)) + L_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t)) \\ &\quad + V'(x(t))F_{\mathrm{c}0}(x(t), u_{\mathrm{c}}(t)) + \Gamma_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t)) \\ &= -\dot{V}(x(t)) + H_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t)), \; x(t) \notin \mathcal{Z}_x, \; t_k < t \leq t_{k+1}. \end{aligned} \tag{72}$$

Similarly,

$$\begin{aligned} L_{\mathrm{d}}(x(t_k), u_{\mathrm{d}}(t_k)) + \Gamma_{\mathrm{d}}(x(t_k), u_{\mathrm{d}}(t_k)) &= -\Delta V(x(t_k)) \\ &\quad + L_{\mathrm{d}}(x(t_k), u_{\mathrm{d}}(t_k)) + \Delta V(x(t_k)) + \Gamma_{\mathrm{d}}(x(t_k), u_{\mathrm{d}}(t_k)) \\ &= -\Delta V(x(t_k)) + H_{\mathrm{d}}(x(t_k), u_{\mathrm{d}}(t_k)), \qquad x(t_k) \in \mathcal{Z}_x. \end{aligned} \tag{73}$$

Now, using (65), (66), and (68), and a fact that $(u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot)) \in \mathcal{C}(x_0)$, it follows that

$$\begin{aligned} \mathcal{J}(x_0, u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot)) &= \int_0^\infty [-\dot{V}(x(t)) + H_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t))]\mathrm{d}t \\ &\quad + \sum_{k \in \mathcal{N}_{[0,\infty)}} [-\Delta V(x(t_k)) + H_{\mathrm{d}}(x(t_k), u_{\mathrm{d}}(t_k))] \\ &= -\lim_{t \to \infty} V(x(t)) + V(x_0) + \int_0^\infty H_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t))\mathrm{d}t \\ &\quad + \sum_{k \in \mathcal{N}_{[0,\infty)}} H_{\mathrm{d}}(x(t_k), u_{\mathrm{d}}(t_k)) \\ &\geq V(x_0) \\ &= \mathcal{J}(x_0, \phi_{\mathrm{c}}(x(\cdot)), \phi_{\mathrm{d}}(x(\cdot))), \end{aligned} \tag{74}$$

which yields (69). □

Next, we specialize Theorem 3.1 to linear uncertain impulsive dynamical systems. Specifically, in this case we consider $\mathcal{F} = \mathcal{F}_{\mathrm{c}} \times \mathcal{F}_{\mathrm{d}}$ to be the set of uncertain linear impulsive dynamical systems given by

$$\mathcal{F}_{\mathrm{c}} = \{(A_{\mathrm{c}} + \Delta A_{\mathrm{c}})x + B_{\mathrm{c}}u_{\mathrm{c}} : x \in \mathbb{R}^n, \, u_{\mathrm{c}} \in \mathbb{R}^{m_{\mathrm{c}}}, \, A_{\mathrm{c}} \in \mathbb{R}^{n \times n}, \, B_{\mathrm{c}} \in \mathbb{R}^{n \times m_{\mathrm{c}}}, \, \Delta A_{\mathrm{c}} \in \mathbf{\Delta}_{A_{\mathrm{c}}}\},$$

$$\mathcal{F}_{\mathrm{d}} = \{(A_{\mathrm{d}} + \Delta A_{\mathrm{d}})x + B_{\mathrm{d}}u_{\mathrm{d}} : x \in \mathbb{R}^n, \, u_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}}}, \, A_{\mathrm{d}} \in \mathbb{R}^{n \times n}, \, B_{\mathrm{d}} \in \mathbb{R}^{n \times m_{\mathrm{d}}}, \, \Delta A_{\mathrm{d}} \in \mathbf{\Delta}_{A_{\mathrm{d}}}\}.$$

For the following result let $R_{1\mathrm{c}} \in \mathbb{P}^n$, $R_{2\mathrm{c}} \in \mathbb{P}^{m_{\mathrm{c}}}$, $R_{1\mathrm{d}} \in \mathbb{N}^n$, and $R_{2\mathrm{d}} \in \mathbb{N}^{m_{\mathrm{d}}}$ be given and define $R_{2a\mathrm{d}} \triangleq R_{2\mathrm{d}} + B_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}} + \Omega_{\mathrm{d}_{u_{\mathrm{d}}u_{\mathrm{d}}}}(P)$ for arbitrary $P \in \mathbb{P}^n$ and $\Omega_{\mathrm{d}_{u_{\mathrm{d}}u_{\mathrm{d}}}} : \mathbb{N}^n \to \mathbb{N}^{m_{\mathrm{d}}}$.

Corollary 3.1. Consider the linear state-dependent uncertain impulsive controlled dynamical system

$$\dot{x}(t) = (A_{\mathrm{c}} + \Delta A_{\mathrm{c}})x(t) + B_{\mathrm{c}}u_{\mathrm{c}}(t), \quad x(0) = x_0, \; t \geq 0, \quad x(t) \notin \mathcal{Z}, \tag{75}$$

$$\Delta x(t) = (A_{\mathrm{d}} + \Delta A_{\mathrm{d}} - I_n)x(t) + B_{\mathrm{d}}u_{\mathrm{d}}(t), \quad x(t) \in \mathcal{Z}, \tag{76}$$

with the hybrid quadratic performance functional

$$\begin{aligned} J_{(\Delta A_{\mathrm{c}}, \Delta A_{\mathrm{d}})}(x_0, u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot)) &\triangleq \int_0^\infty [L_{1\mathrm{c}}(x(t)) + u_{\mathrm{c}}^{\mathrm{T}}(t)R_{2\mathrm{c}}u_{\mathrm{c}}(t)]\mathrm{d}t \\ &\quad + \sum_{k \in \mathcal{N}_{[0,\infty)}} [L_{1\mathrm{d}}(x(t_k)) + u_{\mathrm{d}}^{\mathrm{T}}(t_k)R_{2\mathrm{d}}u_{\mathrm{d}}(t_k)], \end{aligned} \tag{77}$$

where $(u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot))$ is admissible, $(\Delta A_{\mathrm{c}}, \Delta A_{\mathrm{d}}) \in \mathbf{\Delta}$, $L_{1\mathrm{c}}(x(t)) = x^{\mathrm{T}}(t)R_{1\mathrm{c}}x(t)$, and $L_{1\mathrm{d}}(x(t_k)) = x^{\mathrm{T}}(t_k)R_{1\mathrm{d}}x(t_k)$. Furthermore, assume there exist $P \in \mathbb{P}^n$, $\Omega_{\mathrm{c}} : \mathbb{P}^n \to \mathbb{N}^n$, $\Omega_{\mathrm{d}_{xx}} : \mathbb{P}^n \to \mathbb{N}^n$, $\Omega_{\mathrm{d}_{xu_{\mathrm{d}}}} : \mathbb{N}^n \to \mathbb{R}^{n \times m_{\mathrm{d}}}$, and $\Omega_{\mathrm{d}_{u_{\mathrm{d}}u_{\mathrm{d}}}} : \mathbb{N}^n \to \mathbb{N}^{m_{\mathrm{d}}}$, such that

$$x^{\mathrm{T}}(\Delta A_{\mathrm{c}}^{\mathrm{T}}P + P\Delta A_{\mathrm{c}})x \leq x^{\mathrm{T}}\Omega_{\mathrm{c}}(P)x, \; x \notin \mathcal{Z}, \Delta A_{\mathrm{c}} \in \mathbf{\Delta}_{A_{\mathrm{c}}}, \tag{78}$$

$$\begin{aligned} & x^{\mathrm{T}}(\Delta A_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} + A_{\mathrm{d}}^{\mathrm{T}}P\Delta A_{\mathrm{d}} - \Delta A_{\mathrm{d}}PB_{\mathrm{d}}R_{2a\mathrm{d}}^{-1}(B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} \\ & + \Omega_{\mathrm{d}_{xu_{\mathrm{d}}}}^{\mathrm{T}}(P)) - (B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} + \Omega_{\mathrm{d}_{xu_{\mathrm{d}}}}^{\mathrm{T}}(P))^{\mathrm{T}}R_{2a\mathrm{d}}^{-1}B_{\mathrm{d}}^{\mathrm{T}}P\Delta A_{\mathrm{d}} \\ & + \Delta A_{\mathrm{d}}^{\mathrm{T}}P\Delta A_{\mathrm{d}})x \leq x^{\mathrm{T}}(\Omega_{\mathrm{d}_{xx}}(P) - \Omega_{\mathrm{d}_{xu_{\mathrm{d}}}}(P)R_{2a\mathrm{d}}^{-1}(B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} \\ & + \Omega_{\mathrm{d}_{xu_{\mathrm{d}}}}^{\mathrm{T}}(P)) - (B_{\mathrm{d}}^{\mathrm{T}}P + \Omega_{\mathrm{d}_{xu_{\mathrm{d}}}}^{\mathrm{T}}(P))^{\mathrm{T}}R_{2a\mathrm{d}}^{-1}\Omega_{\mathrm{d}_{xu_{\mathrm{d}}}}^{\mathrm{T}}(P) \\ & + (B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} + \Omega_{\mathrm{d}_{xu_{\mathrm{d}}}}^{\mathrm{T}}(P))^{\mathrm{T}}R_{2a\mathrm{d}}^{-1}\Omega_{\mathrm{d}_{u_{\mathrm{d}}u_{\mathrm{d}}}}(P)R_{2a\mathrm{d}}^{-1}(B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} \\ & + \Omega_{\mathrm{d}_{xu_{\mathrm{d}}}}^{\mathrm{T}}(P)))x, \qquad x \in \mathcal{Z}, \quad \Delta A_{\mathrm{d}} \in \mathbf{\Delta}_{A_{\mathrm{d}}}, \end{aligned} \tag{79}$$

and

$$0 = x^{\mathrm{T}}(A_{\mathrm{c}}^{\mathrm{T}}P + PA_{\mathrm{c}} + R_{1\mathrm{c}} + \Omega_{\mathrm{c}}(P) - PB_{\mathrm{c}}R_{2\mathrm{c}}^{-1}B_{\mathrm{c}}^{\mathrm{T}}P)x, \quad x \notin \mathcal{Z}, \tag{80}$$

$$0 < R_{2\mathrm{d}} + B_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}} + \Omega_{\mathrm{d}_{u_{\mathrm{d}}u_{\mathrm{d}}}}(P), \tag{81}$$

$$\begin{aligned} 0 = {} & x^{\mathrm{T}}(A_{\mathrm{d}}^{\mathrm{T}}PA - P + R_{1\mathrm{d}} + \Omega_{\mathrm{d}_{xx}}(P) - (B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} + \Omega_{\mathrm{d}_{xu_{\mathrm{d}}}}^{\mathrm{T}}(P))^{\mathrm{T}} \\ & \cdot R_{2a\mathrm{d}}^{-1}(B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} + \Omega_{\mathrm{d}_{xu_{\mathrm{d}}}}^{\mathrm{T}}(P)))x, \qquad x \in \mathcal{Z}. \end{aligned} \tag{82}$$

Then, with the hybrid feedback control law

$$\begin{aligned} u_{\mathrm{c}} &= \phi_{\mathrm{c}}(x) = -R_{2\mathrm{c}}^{-1}B_{\mathrm{c}}^{\mathrm{T}}Px, \quad x \notin \mathcal{Z}_x, \\ u_{\mathrm{d}} &= \phi_{\mathrm{d}}(x) = -R_{2a\mathrm{d}}^{-1}(B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} + \Omega_{\mathrm{d}_{xu_{\mathrm{d}}}}^{\mathrm{T}}(P))x, \quad x \in \mathcal{Z}_x, \end{aligned}$$

the zero solution $x(t) \equiv 0$ to (75), (76) is globally asymptotically stable for all $x_0 \in \mathbb{R}^n$, $(\Delta A_{\mathrm{c}}, \Delta A_{\mathrm{d}}) \in \mathbf{\Delta}_{A_{\mathrm{c}}} \times \mathbf{\Delta}_{A_{\mathrm{d}}}$, and

$$\sup_{(\Delta A_{\mathrm{c}}, \Delta A_{\mathrm{d}}) \in \mathbf{\Delta}} J_{(\Delta A_{\mathrm{c}}, \Delta A_{\mathrm{d}})}(x_0) \leq \mathcal{J}(x_0, \phi_{\mathrm{c}}(\cdot), \phi_{\mathrm{d}}(\cdot)) = x_0^{\mathrm{T}}Px_0, \quad x_0 \in \mathbb{R}^n, \tag{83}$$

where

$$\begin{aligned} \mathcal{J}(x_0, u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot)) &\triangleq \int_0^\infty [x^{\mathrm{T}}(t)R_{1\mathrm{c}}x(t) + u_{\mathrm{c}}^{\mathrm{T}}(t)R_{2\mathrm{c}}u_{\mathrm{c}}(t) \\ &\quad + x^{\mathrm{T}}(t)\Omega_{\mathrm{c}}(P)x(t)]\mathrm{d}t + \sum_{k \in \mathcal{N}_{[0,\infty)}} [x^{\mathrm{T}}(t_k)R_{1\mathrm{d}}x(t_k) \\ &\quad + u_{\mathrm{d}}^{\mathrm{T}}(t_k)R_{2\mathrm{d}}u_{\mathrm{d}}(t_k) + x^{\mathrm{T}}(t_k)\Omega_{\mathrm{d}_{xx}}(P)x(t_k) \\ &\quad + 2x^{\mathrm{T}}(t_k)\Omega_{\mathrm{d}_{xu_{\mathrm{d}}}}(P)u_{\mathrm{d}}(t_k) + u_{\mathrm{d}}^{\mathrm{T}}(t_k)\Omega_{\mathrm{d}_{u_{\mathrm{d}}u_{\mathrm{d}}}}(P)u_{\mathrm{d}}(t_k)], \end{aligned} \tag{84}$$

and where $(u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot))$ is admissible and $x(t)$, $t \geq 0$, is a solution to (75), (76) with $(\Delta A_{\rm c}, \Delta A_{\rm d}) = (0,0)$. Furthermore,

$$\mathcal{J}(x_0, \phi_{\rm c}(x(\cdot)), \phi_{\rm d}(x(\cdot))) = \min_{(u_{\rm c}(\cdot), u_{\rm d}(\cdot)) \in \mathcal{C}(x_0)} \mathcal{J}(x_0, u_{\rm c}(\cdot), u_{\rm d}(\cdot)), \tag{85}$$

where $\mathcal{C}(x_0)$ is the set of asymptotically stabilizing hybrid controllers for the nominal impulsive dynamical system and $x_0 \in \mathbb{R}^n$.

Proof. The result is direct consequence of Theorem 3.1 with $F_{\rm c}(x, u_{\rm c}) = (A_{\rm c} + \Delta A_{\rm c})x + B_{\rm c}u_{\rm c}$, $F_{\rm c0}(x, u_{\rm c}) = A_{\rm c}x + B_{\rm c}u_{\rm c}$, $L_{\rm c}(x, u_{\rm c}) = x^{\rm T}R_{1{\rm c}}x + u_{\rm c}^{\rm T}R_{2{\rm c}}u_{\rm c}$, $\Gamma_{\rm c}(x, u_{\rm c}) = x^{\rm T}\Omega_{\rm c}(P)x$, $F_{\rm d}(x, u_{\rm d}) = (A_{\rm d} + \Delta A_{\rm d} - I_n)x + B_{\rm d}u_{\rm d}$, $F_{\rm d0}(x, u_{\rm d}) = (A_{\rm d} - I_n)x + B_{\rm d}u_{\rm d}$, $L_{\rm d}(x, u_{\rm d}) = x^{\rm T}R_{1{\rm d}}x + u_{\rm d}^{\rm T}R_{2{\rm d}}u_{\rm d}$, $\Gamma_{\rm d}(x, u_{\rm d}) = x^{\rm T}\Omega_{{\rm d}_{xx}}(P)x + 2x^{\rm T}\Omega_{{\rm d}_{xu_{\rm d}}}(P)u_{\rm d} + u_{\rm d}^{\rm T}\Omega_{{\rm d}_{u_{\rm d}u_{\rm d}}}(P)u_{\rm d}$, $V(x) = x^{\rm T}Px$, $\mathcal{D} = \mathbb{R}^n$, $\mathcal{U}_{\rm c} = \mathbb{R}^{m_{\rm c}}$, and $\mathcal{U}_{\rm d} = \mathbb{R}^{m_{\rm d}}$. □

4. Robust Nonlinear Hybrid Control with Polynomial Performance Functionals

In this section we specialize the results of Section 3 to linear uncertain impulsive dynamical systems controlled by inverse optimal nonlinear hybrid controllers that minimize a derived polynomial cost functional. Specifically, assume $\mathcal{F} = \mathcal{F}_{\rm c} \times \mathcal{F}_{\rm d}$ to be the set of uncertain impulsive dynamical systems given by

$$\mathcal{F}_{\rm c} = \{(A_{\rm c} + \Delta A_{\rm c})x + B_{\rm c}u_{\rm c} \colon x \in \mathbb{R}^n,\ u_{\rm c} \in \mathbb{R}^{m_{\rm c}},\ A_{\rm c} \in \mathbb{R}^{n \times n}\ B_{\rm c} \in \mathbb{R}^{n \times m_{\rm c}},\ \Delta A_{\rm c} \in \boldsymbol{\Delta}_{A_{\rm c}}\}, \tag{86}$$

$$\mathcal{F}_{\rm d} = \{(A_{\rm d} + \Delta A_{\rm d})x \colon x \in \mathbb{R}^n, A_{\rm d} \in \mathbb{R}^{n \times n}, \Delta A_{\rm d} \in \boldsymbol{\Delta}_{A_{\rm d}}\}. \tag{87}$$

For the results in this section we assume $u_{\rm d}(t_k) \equiv 0$. Furthermore, let $R_{1{\rm c}} \in \mathbb{P}^n$, $R_{1{\rm d}} \in \mathbb{N}^n$, $R_{2{\rm c}} \in \mathbb{P}^{m_{\rm c}}$, $\hat{R}_q, \hat{\hat{R}}_q \in \mathbb{N}^n$, $q = 2, \ldots, r$, be given, where r is a positive integer, and define $S_{\rm c} \triangleq B_{\rm c}R_{2{\rm c}}^{-1}B_{\rm c}^{\rm T}$.

Corollary 4.1. Consider the linear uncertain controlled impulsive dynamical system

$$\dot{x}(t) = (A_{\rm c} + \Delta A_{\rm c})x(t) + B_{\rm c}u_{\rm c}(t), \quad x(0) = x_0,\ x(t) \notin \mathcal{Z}_x, \tag{88}$$

$$\Delta x(t) = (A_{\rm d} + \Delta A_{\rm d} - I_n)x(t), \quad x(t) \in \mathcal{Z}_x, \tag{89}$$

where $u_{\rm c}(\cdot)$ is admissible and $(\Delta A_{\rm c}, \Delta A_{\rm d}) \in \boldsymbol{\Delta}$. Let $\Omega_{\rm c} : \mathcal{N}_{\rm P} \subseteq \mathbb{S}^n \to \mathbb{N}^n$ and $\Omega_{\rm d} : \mathcal{N}_{\rm P} \subseteq \mathbb{S}^n \to \mathbb{N}^n$ be such that

$$x^{\rm T}(\Delta A_{\rm c}^{\rm T}P + P\Delta A_{\rm c})x \le x^{\rm T}\Omega_{\rm c}(P)x,\ x \notin \mathcal{Z}_x,\ \Delta A_{\rm c} \in \boldsymbol{\Delta}_{A_{\rm c}}, \tag{90}$$

$$x^{\rm T}(\Delta A_{\rm d}^{\rm T}PA_{\rm d} + A_{\rm d}^{\rm T}P\Delta A_{\rm d} + \Delta A_{\rm d}^{\rm T}P\Delta A_{\rm d})x \le x^{\rm T}\Omega_{\rm d}(P)x, \quad x \in \mathcal{Z}_x, \quad \Delta A_{\rm d} \in \boldsymbol{\Delta}_{A_{\rm d}}, \tag{91}$$

where $P \in \mathcal{N}_{\rm P}$. Assume there exist $P \in \mathbb{P}^n$ and $M_q \in \mathbb{N}^n$, $q = 2, \ldots, r$, such that

$$0 = x^{\rm T}(A_{\rm c}^{\rm T}P + PA_{\rm c} + R_{1{\rm c}} + \Omega_{\rm c}(P) - PS_{\rm c}P)x,\ x \notin \mathcal{Z}_x, \tag{92}$$

$$0 = x^{\rm T}[(A_{\rm c} - S_{\rm c}P)^{\rm T}M_q + M_q(A_{\rm c} - S_{\rm c}P) + \hat{R}_q]x, \quad x \notin \mathcal{Z}_x, \quad q = 2, \ldots, r, \tag{93}$$

$$0 = x^{\rm T}(A_{\rm d}^{\rm T}PA_{\rm d} - P + R_{1{\rm d}} + \Omega_{\rm d}(P))x, \quad x \in \mathcal{Z}_x, \tag{94}$$

$$0 = x^{\rm T}(A_{\rm d}^{\rm T}M_qA_{\rm d} - M_q + \hat{\hat{R}}_q)x, \quad x \in \mathcal{Z}_x,\ q = 2, \ldots, r. \tag{95}$$

Then the zero solution $x(t) \equiv 0$ of the uncertain impulsive closed-loop system

$$\dot{x}(t) = (A_{\rm c} + \Delta A_{\rm c})x(t) + B_{\rm c}\phi_{\rm c}(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x, \tag{96}$$

$$\Delta x(t) = (A_{\rm d} + \Delta A_{\rm d} - I_n)x(t), \qquad x(t) \in \mathcal{Z}_x, \tag{97}$$

is globally asymptotically stable with the feedback control law

$$\phi_{\rm c}(x) = -R_{2{\rm c}}^{-1}B_{\rm c}^{\rm T}(P + \sum_{q=2}^{r}(x^{\rm T}M_qx)^{q-1}M_q)x,\ x \notin \mathcal{Z}_x, \tag{98}$$

and the hybrid performance functional (77) satisfies

$$\sup_{(\Delta A_{\rm c}, \Delta A_{\rm d}) \in \boldsymbol{\Delta}} J_{(\Delta A_{\rm c}, \Delta A_{\rm d})}(x_0, \phi_{\rm c}(x_0)) \le \mathcal{J}(x_0, \phi_{\rm c}(x_0)) = x_0^{\rm T}Px_0 + \sum_{q=2}^{r}\frac{1}{q}(x_0^{\rm T}M_qx_0)^q,\ x_0 \in \mathbb{R}^n, \tag{99}$$

where

$$\mathcal{J}(x_0, u_{\rm c}(\cdot)) \triangleq \int_0^\infty [L_{1{\rm c}}(x(t)) + u_{\rm c}^{\rm T}(t)R_{2{\rm c}}(x(t))u_{\rm c}(t) + \Gamma_{\rm c}(x(t))]{\rm d}t + \sum_{k \in \mathcal{N}_{[0,\infty)}}[L_{\rm d}(x(t_k)) + \Gamma_{\rm d}(x(t_k))], \tag{100}$$

and where $u_{\rm c}(\cdot)$ is admissible, and $x(t)$, $t \ge 0$, is a solution to (88), (89) with $(\Delta A_{\rm c}, \Delta A_{\rm d}) = (0,0)$, and

$$\Gamma_{\rm c}(x) = x^{\rm T}(\Omega_{\rm c}(P) + \sum_{q=2}^{r}(x^{\rm T}M_qx)^{q-1}\Omega_{\rm c}(M_q))x, \tag{101}$$

$$\Gamma_{\rm d}(x) = x^{\rm T}\Omega_{\rm d}(P)x + \sum_{q=2}^{r}\frac{1}{q}[(x^{\rm T}\hat{\hat{R}}_qx)\sum_{j=1}^{q}(x^{\rm T}M_qx)^{j-1} \cdot((x^{\rm T}(A_{\rm d}^{\rm T}M_qA_{\rm d} + \Omega_{\rm d}(M_q))x)^{q-j} - (x^{\rm T}A_{\rm d}^{\rm T}M_qA_{\rm d}x)^{q-j})], \tag{102}$$

where $u_{\rm c}(\cdot)$ is admissible and $(\Delta A_{\rm c}, \Delta A_{\rm d}) \in \boldsymbol{\Delta}$. In addition, the hybrid performance functional (77), with $R_{2{\rm c}}(x) = R_{2{\rm c}}$ and

$$L_{1{\rm c}}(x) = x^{\rm T}(R_{1{\rm c}} + \sum_{q=2}^{r}(x^{\rm T}M_qx)^{q-1}\hat{R}_q + [\sum_{q=2}^{r}(x^{\rm T}M_qx)^{q-1} \cdot M_q]^{\rm T}S_{\rm c}[\sum_{q=2}^{r}(x^{\rm T}M_qx)^{q-1}M_q])x, \tag{103}$$

$$L_{1{\rm d}}(x) = x^{\rm T}R_{1{\rm d}}x + \sum_{q=2}^{r}\frac{1}{q}[(x^{\rm T}\hat{\hat{R}}_qx)\sum_{j=1}^{q}(x^{\rm T}M_qx)^{j-1} \cdot(x^{\rm T}A_{\rm d}^{\rm T}M_qA_{\rm d}x)^{q-j}], \tag{104}$$

is minimized in the sense that

$$\mathcal{J}(x_0, \phi_{\rm c}(x(\cdot))) = \min_{u_{\rm c}(\cdot) \in \mathcal{C}(x_0)} \mathcal{J}(x_0, u_{\rm c}(\cdot)), \quad x_0 \in \mathbb{R}^n, \tag{105}$$

where $\mathcal{C}(x_0)$ is the set of asymptotically stabilizing controllers for the nominal impulsive dynamical system and $x_0 \in \mathbb{R}^n$.

Proof. The result is a direct consequence of Corollary 3.1. □

References

[1] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations.* Singapore, World Scientific, 1989.

[2] W. M. Haddad, V. Chellabonia, and N. A. Kablar, "Nonlinear Impulsive Dynamical Systems Part I: Stability and Dissipativity," *Proc. IEEE Conf. Dec. Contr.*, pp. 4404–4422, Phoenix, AZ, December 1999.

[3] W. M. Haddad, V. Chellabonia, and N. A. Kablar, "Nonlinear Impulsive Dynamical Systems Part II: Feedback Interconnections and Optimality," *Proc. IEEE Conf. Dec. Contr.*, pp. 5225–5234, Phoenix, AZ, December 1999.

[4] W. M. Haddad, V. Chellabonia, J. L. Fausz, and A. Leonessa, "Optimal Nonlinear Robust Control for Nonlinear Uncertain Systems," *Int. J. Contr.*, vol. 73, pp. 329–342, 2000.