A Disturbance-Rejection Problem with Dynamic Compensator for Linear ω-Periodic Discrete-Time Systems

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Abstract

In this paper the solvability conditions of the disturbance-rejection problem with dynamic compensator which was investigated by Grasselli and Longhi is investigated without assuming that the order of dynamic compensator is equal to that of system plant.

Keywords : Disturbance-rejection, dynamic compensator, ω-periodic discrete-time systems, geometric approach

1 Introduction

In the framework of the so-called geometric approach Grasselli and Longhi[?, ?] studied the notions of $(A(\cdot), B(\cdot))$-invariance and $(C(\cdot), A(\cdot))$-invariance for linear ω-periodic discrete-time systems, and the disturbance-rejection problem with state feedback was studied. Further, they studied the notion of $(C(\cdot), A(\cdot), B(\cdot))$-pair for linear ω-periodic discrete-time systems and then necessary and sufficient conditions for the solvability of the disturbance-rejection problem with dynamic compensator (DRPDC) were given under the assumption that the order (n) of system plant is equal to the order (μ) of dynamic compensator[?]. However, if the order of system plant is large, then the order of dynamic compensator which solves the DRPDC is also large. Therefore, it is important to investigate about the low order dynamic compensator which solves the DRPDC.

The objective of this paper is to give the solvability conditions of the DRPDC without assuming $n = \mu$.

The present investigation is organized as follows. Section 2 gives the notions of some invariances. In Section 3 the notion of $(C(\cdot), A(\cdot), B(\cdot))$-pair and its useful results are given. In Section 4 the main results are given. Finally, we make some concluding remarks in Section 5.

2 Preliminaries

The following notations are used throughout this investigation. $\boldsymbol{N}$:= the set of all natural numbers, $\boldsymbol{Z}$:= the set of all integers, $\boldsymbol{Z}_{k_0}^{\omega} := \{k_0+1, k_0+2, \ldots, k_0+\omega\}$ for $k_0 \in \boldsymbol{Z}$ and $\omega \in \boldsymbol{N}$, $\boldsymbol{R}^s$:= s dimensional Euclidean space and $\boldsymbol{R}^{p\times q}$:= the set of all linear maps from $\boldsymbol{R}^q$ to $\boldsymbol{R}^p$. For a linear map A from a vector space X to a vector space Y $\mathrm{Im}A$:= the image of A, $\mathrm{Ker}A$:= the kernel of A, $A|_V$:= the restriction map of A on a subspace V of X and $A^{-1}\Omega := \{x \in X \mid Ax \in \Omega\}$ for a subspace Ω of Y. The notations $\oplus$ and $\cong$ indicate the direct sum and the isomorphic, respectively. For a linear map-valued function $A(\cdot)(A(k) \in \boldsymbol{R}^{p\times q}, k \in \boldsymbol{Z})$, $A(\cdot)$ is said to be ω-periodic for a given $\omega \in \boldsymbol{N}$ if $A(k) = A(k+\omega)$ for all $k \in \boldsymbol{Z}$. Further, for a subspace-valued function $V(\cdot)$ $(V(k) \subset \boldsymbol{R}^s, k \in \boldsymbol{Z})$, $V(\cdot)$ is said to be ω-periodic for a given $\omega \in \boldsymbol{N}$ if $V(k) = V(k+\omega)$ for all $k \in \boldsymbol{Z}$.

Now, consider a linear ω-periodic discrete-time system given by

$$S : \begin{cases} x(k+1) &= A(k)x(k) + B(k)u(k), \\ y(k) &= C(k)x(k), \ k \in \boldsymbol{Z} \end{cases}$$

where $x(k) \in X := \boldsymbol{R}^n$ is the state, $u(k) \in U := \boldsymbol{R}^m$ is the input, $y(k) \in Y := \boldsymbol{R}^p$ is the measurement output and $A(\cdot)$ $(A(k) \in \boldsymbol{R}^{n\times n})$, $B(\cdot)$ $(B(k) \in \boldsymbol{R}^{n\times m})$ and $C(\cdot)$ $(C(k) \in \boldsymbol{R}^{p\times n})$ are ω-periodic.

Definition 2.1 Let $V(\cdot)$ $(V(k) \subset X)$ be ω-periodic subspace-valued function.

(i) $V(\cdot)$ is said to be $(A(\cdot), B(\cdot))$-invariant if $A(k)V(k) \subset V(k+1) + \mathrm{Im}B(k)$ for all $k \in \boldsymbol{Z}$.

(ii) $V(\cdot)$ is said to be $(C(\cdot), A(\cdot))$-invariant if $A(k)(V(k) \cap \mathrm{Ker}C(k)) \subset V(k+1)$ for all $k \in \boldsymbol{Z}$. □

Lemma 2.1 [?, ?] Let $V(\cdot)$ ($V(k) \subset X$) be ω-periodic subspace-valued function.

(i) $V(\cdot)$ is $(A(\cdot), B(\cdot))$-invariant if and only if there exists an ω-periodic feedback $F(\cdot)$ ($F(k) \in \boldsymbol{R}^{m\times n}$) such that

$$(A(k) + B(k)F(k))V(k) \subset V(k+1) \ \text{ for all } k \in \boldsymbol{Z}.$$

(ii) $V(\cdot)$ is $(C(\cdot), A(\cdot))$-invariant if and only if there exists an ω-periodic $G(\cdot)$ ($G(k) \in \boldsymbol{R}^{n\times p}$) such that

$$(A(k) + G(k)C(k))V(k) \subset V(k+1) \ \text{ for all } k \in \boldsymbol{Z}. \quad \Box$$

For a given ω-periodic subspace-valued function $\Lambda(\cdot)$ ($\Lambda(k) \subset X$), define the following two classes of ω-periodic subspace-valued functions.

$$\boldsymbol{V}(A(\cdot), B(\cdot); \Lambda(\cdot)) := \Big\{V(\cdot) \mid V(\cdot) \text{ is } (A(\cdot), B(\cdot))\text{-invariant and } V(k) \subset \Lambda(k) \text{ for all } k \in \boldsymbol{Z}\Big\}.$$

$$\boldsymbol{V}(\Lambda(\cdot); C(\cdot), A(\cdot)) := \Big\{V(\cdot) \mid V(\cdot) \text{ is } (C(\cdot), A(\cdot))\text{-invariant and } \Lambda(k) \subset V(k) \text{ for all } k \in \boldsymbol{Z}\Big\}.$$

Further, the following definitions are introduced.

$$\boldsymbol{F}(V(\cdot)) := \{F(\cdot) : \omega\text{-periodic} \mid (A(k)+B(k)F(k))V(k) \subset V(k+1) \text{ for all } k \in \boldsymbol{Z}\}.$$

$$\boldsymbol{G}(V(\cdot)) := \{G(\cdot) : \omega\text{-periodic} \mid (A(k)+G(k)C(k))V(k) \subset V(k+1) \text{ for all } k \in \boldsymbol{Z}\}.$$

Definition 2.2

(i) $V^*(\cdot)$ is said to be the maximal element of $\boldsymbol{V}(A(\cdot), B(\cdot); \Lambda(\cdot))$ if $V^*(\cdot) \in \boldsymbol{V}(A(\cdot), B(\cdot); \Lambda(\cdot))$ and $V(k) \subset V^*(k)$ ($k \in \boldsymbol{Z}$) for all $V(\cdot)$ of $\boldsymbol{V}(A(\cdot), B(\cdot); \Lambda(\cdot))$.

(ii) $V_*(\cdot)$ is said to be the minimal element of $\boldsymbol{V}(\Lambda(\cdot); C(\cdot), A(\cdot))$ if $V_*(\cdot) \in \boldsymbol{V}(\Lambda(\cdot); C(\cdot), A(\cdot))$ and $V_*(k) \subset V(k)$ ($k \in \boldsymbol{Z}$) for all $V(\cdot)$ of $\boldsymbol{V}(\Lambda(\cdot); C(\cdot), A(\cdot))$. $\Box$

Then, the following lemma gives the computational algorithms of $V^*(\cdot)$ and $V_*(\cdot)$.

Lemma 2.2 [?, ?]

(i) $\boldsymbol{V}(A(\cdot), B(\cdot); \Lambda(\cdot))$ has the maximal element $V^*(\cdot)$ in the sence of Definition 2.2 and it can be computed in a finite number of steps through the sequence:

$V^0 := \Lambda(k) \quad (k \in \boldsymbol{Z})$,

$V^i(k) := \Lambda(k) \cap A^{-1}(k)(V^{i-1}(k+1) + \mathrm{Im}B(k)) \quad (k \in \boldsymbol{Z}), \quad i = 1, 2, \cdots$.

(ii) $\boldsymbol{V}(\Lambda(\cdot); C(\cdot), A(\cdot))$ has the minimal element $V_*(\cdot)$ in the sence of Definition 2.2 and it can be computed in a finite number of steps through the sequence:

$V^0 := \Lambda(k) \quad (k \in \boldsymbol{Z})$,

$V^i(k) := \Lambda(k+1) + A(k)(V^{i-1}(k) \cap \mathrm{Ker}C(k)) \quad (k \in \boldsymbol{Z}), \quad i = 1, 2, \cdots$. $\Box$

3 $(C(\cdot), A(\cdot), B(\cdot))$-pairs

In this section the properties of $(C(\cdot), A(\cdot), B(\cdot))$-pairs which will be needed in Section 4 are investigated. Consider the following linear ω-periodic discrete-time system :

$$\Sigma : \begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) + E(k)\xi(k), \\ y(k) = C(k)x(k), \\ z(k) = D(k)x(k), \ k \in \boldsymbol{Z}, \end{cases}$$

where $x(k) \in X := \boldsymbol{R}^n$, $u(k) \in U := \boldsymbol{R}^m$, $\xi(k) \in Q := \boldsymbol{R}^s$, $y(k) \in Y := \boldsymbol{R}^p$ and $z(k) \in Z := \boldsymbol{R}^q$ are the state, the input, the disturbance, the measurement output and the controlled output, respectively. And $A(\cdot)$ ($A(k) \in \boldsymbol{R}^{n\times n}$), $B(\cdot)$ ($B(k) \in \boldsymbol{R}^{n\times m}$), $C(\cdot)$ ($C(k) \in \boldsymbol{R}^{p\times n}$), $D(\cdot)$ ($D(k) \in \boldsymbol{R}^{q\times n}$) and $E(\cdot)$ ($E(k) \in \boldsymbol{R}^{n\times s}$) are ω-periodic.

Now, introduce an ω-periodic dynamic compensator ($K(\cdot)$, $L(\cdot)$, $M(\cdot)$, $N(\cdot)$) defined in $W := \boldsymbol{R}^\mu$ of the following form :

$$\Sigma_c : \begin{cases} w(k+1) = N(k)w(k) + M(k)y(k), \\ u(k) = L(k)w(k) + K(k)y(k), \end{cases}$$

where $N(k) \in \boldsymbol{R}^{\mu\times\mu}$, $M(k) \in \boldsymbol{R}^{\mu\times p}$, $L(k) \in \boldsymbol{R}^{m\times\mu}$ and $K(k) \in \boldsymbol{R}^{m\times p}$ are ω-periodic.

If the compensator Σ_c is applied to system Σ, the resulting ω-periodic extended system on the extended state space $X^e := X \oplus W$ is easily seen to be

$$\Sigma^e : \begin{cases} \begin{bmatrix} x(k+1) \\ w(k+1) \end{bmatrix} = \begin{bmatrix} A(k) + B(k)K(k)C(k) & B(k)L(k) \\ M(k)C(k) & N(k) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} + \begin{bmatrix} E(k) \\ 0 \end{bmatrix} \xi(k), \\ z(k) = \begin{bmatrix} D(k) & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}. \end{cases}$$

For the system Σ^e, define

$$x^e(k) := \begin{bmatrix} x(k) \\ w(k) \end{bmatrix},$$

$$A^e(k) := \begin{bmatrix} A(k)+B(k)K(k)C(k) & B(k)L(k) \\ M(k)C(k) & N(k) \end{bmatrix},$$

$$E^e(k) := \begin{bmatrix} E(k) \\ 0 \end{bmatrix},$$

$$D^e(k) := \begin{bmatrix} D(k) & 0 \end{bmatrix},$$

$\Phi^e(k,k_0) := A^e(k-1)A^e(k-2)\cdots A^e(k_0)$ for $k > k_0$ $(k, k_0 \in \boldsymbol{Z})$ and $\Phi^e(k,k) := I_{n+\mu}$ $(k \in \boldsymbol{Z})$, where $I_{n+\mu}$ is the identity map of dimension $(n+\mu)$.

Now, the difinition of $(C(\cdot), A(\cdot), B(\cdot))$-pair is given.

Definition 3.1 Let $V_1(\cdot)$, $V_2(\cdot)$ $(V_1(k), V_2(k) \subset X)$ be ω-periodic subspace-valued functions. A pair $(V_1(\cdot), V_2(\cdot))$ is said to be $(C(\cdot), A(\cdot), B(\cdot))$-pair if the following three conditions are satisfied.

(i) $V_1(\cdot)$ is $(C(\cdot), A(\cdot))$-invariant.

(ii) $V_2(\cdot)$ is $(A(\cdot), B(\cdot))$-invariant.

(iii) $V_1(k) \subset V_2(k)$ for all $k \in \boldsymbol{Z}$. □

For an extended system Σ^e, we give the following definition.

Definition 3.2 Let $V^e(\cdot)$ $(V^e(k) \subset X^e)$ be an ω-periodic subspace-valued function. $V^e(\cdot)$ is said to be an $A^e(\cdot)$-invariant if $A^e(k)V^e(k) \subset V^e(k+1)$ for all $k \in \boldsymbol{Z}$. □

Definition 3.3 Let $V^e(\cdot)$ $(V^e(k) \subset X^e)$ be an ω-periodic subspace-valued function. Then, the following two subspace-valued functions $V_s(\cdot)$ and $V_p(\cdot)$ are defined:

$$V_s(k) := \left\{ x \in X \,\middle|\, \begin{bmatrix} x \\ 0 \end{bmatrix} \in V^e(k) \right\} \text{ and}$$

$$V_p(k) := \left\{ x \in X \,\middle|\, \begin{bmatrix} x \\ w \end{bmatrix} \in V^e(k) \text{ for some } w \in W \right\}$$

$$= P_X(V^e(k)),$$

where P_X is the projection map from X^e onto X along W. □

The following lemma was given by Grasselli and Longhi.

Lemma 3.1 [?] Let $V^e(\cdot)$ $(V^e(k) \subset X^e)$ be an ω-periodic subspace-valued function. If $V^e(\cdot)$ is an $A^e(\cdot)$-invariant, then the pair $(V_s(\cdot), V_p(\cdot))$ is $(C(\cdot), A(\cdot), B(\cdot))$-pair. □

The following two lemmas are used to prove Lemma 3.4.

Lemma 3.2 If a pair $(V_1(\cdot), V_2(\cdot))$ is $(C(\cdot), A(\cdot), B(\cdot))$-pair, then there exist $F(\cdot) \in \boldsymbol{F}(V_2(\cdot))$, $G(\cdot) \in \boldsymbol{G}(V_1(\cdot))$, $G_0(k) \in \boldsymbol{R}^{n\times p}$, $F_0(k) \in \boldsymbol{R}^{m\times n}$ and $K(k) \in \boldsymbol{R}^{m\times p}$ such that

$F(k) = K(k)C(k) + F_0(k)$, $G(k) = B(k)K(k) + G_0(k)$, $\mathrm{Ker}F_0(k) \supset V_1(k)$ and $\mathrm{Im}G_0(k) \subset V_2(k+1)$ for all $k \in \boldsymbol{Z}$.

Proof. Suppose that a pair $(V_1(\cdot), V_2(\cdot))$ is $(C(\cdot), A(\cdot), B(\cdot))$-pair.

Claim 1: There exists a $G(\cdot) \in \boldsymbol{G}(V_1(\cdot))$ such that $\mathrm{Im}G(k) \subset V_2(k+1) + \mathrm{Im}B(k)$.
To prove Claim 1, choose an arbitrary element $G_1(\cdot) \in \boldsymbol{G}(V_1(\cdot))$. Define a subspace $\Gamma(k)$ such that $Y = C(k)V_1(k) \oplus \Gamma(k)$. Further, define a linear map $G(k)$ from Y to X such that

$$G(k) := \begin{cases} G_1(k) & \text{on } C(k)V_1(k), \\ 0 & \text{on } \Gamma(k). \end{cases}$$

Then, it can be easily obtained that $G(\cdot) \in \boldsymbol{G}(V_1(\cdot))$ and $\mathrm{Im}G(k) \subset V_2(k+1) + \mathrm{Im}B(k)$.

Claim 2: There exists maps $K(k) \in \boldsymbol{R}^{m\times p}$ and $G_0(k) \in \boldsymbol{R}^{n\times p}$ such that $G(k) = B(k)K(k) + G_0(k)$ and $\mathrm{Im}G_0(k) \subset V_2(k+1)$.
To prove Claim 2 let $\{y_1, \cdots, y_p\}$ be a basis of Y. Then, it follows from Claim 1 that there exist an $x_i(k) \in V_2(k+1)$ and a $u_i(k) \in U$ such that $G(k)y_i = x_i(k) + B(k)u_i(k)$. Define linear maps $K(k)$ from Y to U and $G_0(k)$ from Y to X such that $K(k)y_i := u_i(k)$ $(i = 1, \cdots, p)$ and $G_0(k)y_i := x_i(k)$ $(i = 1, \cdots, p)$. Then, we have $G(k) = B(k)K(k) + G_0(k)$ and $\mathrm{Im}G_0(k) \subset V_2(k+1)$ which proves Claim 2.

Claim 3: There exists a map $F_0(k) \in \boldsymbol{R}^{m\times n}$ such that $\mathrm{Ker}F_0(k) \supset V_1(k)$ and $(A(k) + G(k)C(k) + B(k)F_0(k))V_2(k) \subset V_2(k+1)$.
To prove Claim 3 define subspaces $\Omega(k)$ and $\Lambda(k)$ such that $X = \underbrace{V_1(k) \oplus \Omega(k)}_{V_2(k)} \oplus \Lambda(k)$. Further, the following relation can be easily obtained.

$$(A(k)+G(k)C(k))V_2(k) \subset V_2(k+1) + \mathrm{Im}B(k)$$

for all $k \in \boldsymbol{Z}$, which implies $V_2(\cdot)$ is $((A(\cdot)+G(\cdot)C(\cdot)), B(\cdot))$-invariant (e.g., see [?]). Thus, there exists an ω-periodic linear map $\tilde{F}(k)$ from X to U such that

$$(A(k)+G(k)C(k)+B(k)\tilde{F}(k))V_2(k) \subset V_2(k+1).$$

Further, define a linear map $F_0(k)$ from X to U such that

$$F_0(k) := \begin{cases} \tilde{F}(k) & \text{on } \Omega(k) \oplus \Lambda(k), \\ 0 & \text{on } V_1(k). \end{cases}$$

Then, $F_0(k)$ satisfies $\mathrm{Ker}F_0(k) \supset V_1(k)$ and

$$(A(k)+G(k)C(k)+B(k)F_0(k))V_2(k) \subset V_2(k+1)$$

which proves Claim 3.
Finally, define $F(k) := K(k)C(k)+F_0(k)$. Then, it can be easily obtained that $F(\cdot) \in \boldsymbol{F}(V_2(\cdot))$. This completes the proof. □

Lemma 3.3 Suppose that V_1 and V_2 are subspaces of X satisfying $V_1 \subset V_2$ and $W \cong \boldsymbol{R}^{\dim V_2 - \dim V_1}$. Then, a linear map R from V_2 to W can be defined such that $\mathrm{Ker}R = V_1$.
Proof. Since the proof follows easily, it is omitted. □

The following lemma plays an important role to prove the main theorem in the next section.

Lemma 3.4 If a pair $(V_1(\cdot), V_2(\cdot))$ is $(C(\cdot), A(\cdot), B(\cdot))$-pair, then there exist an ω-periodic dynamic compensator $(K(\cdot), L(\cdot), M(\cdot), N(\cdot))$ on W and a subspace-valued function $V^e(\cdot)$ ($V^e(k) \subset X^e = X \oplus W$) such that

$$\dim W = \dim\Big(\sum_{k \in \boldsymbol{Z}^{\omega}_{k_0}} V_2(k)\Big) - \dim\Big(\bigcap_{k \in \boldsymbol{Z}^{\omega}_{k_0}} V_1(k)\Big),$$

$\bigcap_{k \in \boldsymbol{Z}^{\omega}_{k_0}} V_1(k) = V_s(k)$, $V_2(k) = V_p(k)$ and $V^e(\cdot)$ is $A^e(\cdot)$-invariant.

Proof. Suppose that a pair $(V_1(\cdot), V_2(\cdot))$ is $(C(\cdot), A(\cdot), B(\cdot))$-pair. Define $W := \boldsymbol{R}^{\mu}$, where $\mu := \dim(\sum_{k \in \boldsymbol{Z}^{\omega}_{k_0}} V_2(k)) - \dim(\bigcap_{k \in \boldsymbol{Z}^{\omega}_{k_0}} V_1(k))$. Then, from Lemma 3.3 a linear map R from $(\sum_{k \in \boldsymbol{Z}^{\omega}_{k_0}} V_2(k))$ to W can be defined such that

$$\mathrm{Ker}R = \Big(\bigcap_{k \in \boldsymbol{Z}^{\omega}_{k_0}} V_1(k)\Big).$$

Further, define

$$V^e(k) := \left\{ \begin{bmatrix} x(k) \\ Rx(k) \end{bmatrix} \middle| x(k) \in V_2(k) \right\} \subset X \oplus W.$$

Then, it can be easily obtained $V_s(k) = \bigcap_{k \in \boldsymbol{Z}^{\omega}_{k_0}} V_1(k)$ and $V_p(k) = V_2(k)$ for all $k \in \boldsymbol{Z}$. Now, it follows from Lemma 3.2 that there exist $F(\cdot) \in \boldsymbol{F}(V_2(\cdot))$, $G(\cdot) \in \boldsymbol{G}(V_1(\cdot))$, $G_0(k) \in \boldsymbol{R}^{n \times p}$, $F_0(k) \in \boldsymbol{R}^{m \times n}$ and $K(k) \in \boldsymbol{R}^{m \times p}$ such that

$$F(k) = K(k)C(k) + F_0(k),\ G(k) = B(k)K(k) + G_0(k),$$

$\mathrm{Ker}F_0(k) \supset V_1(k)$ and $\mathrm{Im}G_0(k) \subset V_2(k)$ for all $k \in \boldsymbol{Z}$. Then, since $\mathrm{Ker}R (= \bigcap_{k \in \boldsymbol{Z}^{\omega}_{k_0}} V_1(k)) \subset \mathrm{Ker}F_0(k)$, there exists a linear map $L(k)$ from W to X satisfying

$$L(k)R = F_0(k)|_{(\sum_{k \in \boldsymbol{Z}^{\omega}_{k_0}} V_2(k))}.$$

Further, since $\mathrm{Ker}R|_{V_2(k)} \subset \mathrm{Ker}R(A(k)+B(k)F(k)+G_0(k)C(k))|_{V_2(k)}$, there exists a linear map $N(k)$ on W satisfying

$$N(k)R|_{V_2(k)} = R(A(k)+B(k)F(k)+G_0(k)C(k))|_{V_2(k)}.$$

Define a linear map $M(k) := -RG_0(k)$ from Y to W. Then, it can be proved that $V^e(\cdot)$ is $A^e(\cdot)$-invariant. Further, it remarks the dynamic compensator $(K(\cdot), L(\cdot), M(\cdot), N(\cdot))$ is ω-periodic. This completes the proof. □

4 Disturbance-Rejection

In this section a disturbance-rejection problem with dynamic compensator is formulated and then the solvability conditions of the problem are given without assuming that the order of dynamic compensator is equal to that of system plant.

Consider an extended system Σ^e in Section 3. The disturbance-rejection problem can be stated as follows. Given ω-periodic linear map-valued functions $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ and $E(\cdot)$ of the system Σ, find (if possible) an ω-periodic dynamic compensator $(K(\cdot), L(\cdot), M(\cdot), N(\cdot))$ of Σ_c such that

$$D^e(k) \sum_{h=k_0}^{k-1} \Phi^e(k, h+1) E^e(h) \xi(h) = 0$$

for all $\xi(\cdot)$ and $k_0 \in \boldsymbol{Z}, k \geq k_0$ where k_0 is an initial time.

Noticing that a subspace generated by the disturbances $\xi(\cdot)$ is $\sum_{h=k_0}^{k-1}\Phi^e(k,h+1)\mathrm{Im}E^e(h)$, a disturbance rejection problem with dynamic compensator can be stated as follows.

Disturbance-Rejection Problem with Dynamic Compensator (DRPDC) Given ω-periodic linear map-valued functions $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ and $E(\cdot)$ of the system Σ, find (if possible) an ω-periodic dynamic compensator ($K(\cdot)$, $L(\cdot)$, $M(\cdot)$, $N(\cdot)$) of Σ_c such that

$$\sum_{h=k_0}^{k-1}\Phi^e(k,h+1)\mathrm{Im}E^e(h)\subset \mathrm{Ker}D^e(k)$$

for all $k_0\in \boldsymbol{Z}, k\geq k_0$. □

The following theorem gives necessary conditions for the solvability of the problem.

Theorem 4.1 If the DRPDC is solvable, then there exists a $(C(\cdot),A(\cdot),B(\cdot))$-pair $(V_1(\cdot),V_2(\cdot))$ such that

$\mathrm{Im}E(k-1)\subset V_1(k)\subset V_2(k)\subset \mathrm{Ker}D(k)$ for all $k\in\boldsymbol{Z}$.

Proof. Suppose that the DRPDC is solvable. Then, there exists an ω-periodic dynamic compensator ($K(\cdot)$, $L(\cdot)$, $M(\cdot)$, $N(\cdot)$) such that

$$\sum_{h=k_0}^{k-1}\Phi^e(k,h+1)\mathrm{Im}E^e(h)\subset \mathrm{Ker}D^e(k)$$

for all $k_0\in \boldsymbol{Z}, k\geq k_0$. Define, a subspace-valued function $V^e(\cdot)$ such that

$$V^e(k):=\sum_{h=k_0}^{k-1}\Phi^e(k,h+1)\mathrm{Im}E^e(h).$$

Then, since $V^e(\cdot)$ is ω-periodic and $A^e(\cdot)$-invariant, it follows from Lemma 3.1 that $(V_s(\cdot),V_p(\cdot))$ is a ($C(\cdot)$, $A(\cdot)$, $B(\cdot)$)-pair. Further, it can be easily obtained that

$$\mathrm{Im}E(k-1)\subset V_s(k) \text{ and } V_p(k)\subset \mathrm{Ker}D(k).$$

Thus, we have

$\mathrm{Im}E(k-1)\subset V_s(k)\subset V_p(k)\subset \mathrm{Ker}D(k)$ for all $k\in\boldsymbol{Z}$.□

Corollary 4.1 Suppose that $V^*(\cdot)$ is the maximal element of $\boldsymbol{V}(A(\cdot),B(\cdot);\mathrm{Ker}D(\cdot))$ and $V_*(\cdot)$ is the minimal element of $\boldsymbol{V}(\mathrm{Im}E(\cdot-1);C(\cdot),A(\cdot))$. If the DRPDC is solvable, then $V_*(k)\subset V^*(k)$ for all $k\in\boldsymbol{Z}$.

Proof. The proof follows from Theorem 4.1. □

The following theorem gives sufficient conditions for the solvability of the problem.

Theorem 4.2 If there exists a $(C(\cdot),A(\cdot),B(\cdot))$-pair $(V_1(\cdot),V_2(\cdot))$ such that

$\mathrm{Im}E(k-1)\subset \bigcap_{k\in\boldsymbol{Z}_{k_0}^{\omega}} V_1(k), V_2(k)\subset\mathrm{Ker}D(k)$ for all $k\in\boldsymbol{Z}$,

then the DRPDC is solvable. In this case, the order of dynamic compensator which is necessary for the solution of the DRPDC is given by

$$\mu:=\dim(\sum_{k\in\boldsymbol{Z}_{k_0}^{\omega}}V_2(k))-\dim(\bigcap_{k\in\boldsymbol{Z}_{k_0}^{\omega}}V_1(k)).$$

Proof. Suppose that there exists a ($C(\cdot)$, $A(\cdot)$, $B(\cdot)$)-pair ($V_1(\cdot)$, $V_2(\cdot)$) such that the stated conditions are satisfied. Then, it follows from Lemma 3.4 that there exist an ω-periodic dynamic compensator $(K(\cdot),L(\cdot),M(\cdot),N(\cdot))$ on W and a subspace-valued function $V^e(\cdot)$ ($V^e(k)\subset X^e=X\oplus W$) such that

$$\dim W=\dim(\sum_{k\in\boldsymbol{Z}_{k_0}^{\omega}}V_2(k))-\dim(\bigcap_{k\in\boldsymbol{Z}_{k_0}^{\omega}}V_1(k)),$$

$\bigcap_{k\in\boldsymbol{Z}_{k_0}^{\omega}}V_1(k)=V_s(k)$, $V_2(k)=V_p(k)$ and $V^e(\cdot)$ is $A^e(\cdot)$-invariant, where $V^e(k):=\left\{\begin{bmatrix}x(k)\\Rx(k)\end{bmatrix}|x(k)\in V_2(k)\right\}$ and $\mathrm{Ker}R=\bigcap_{k\in\boldsymbol{Z}_{k_0}^{\omega}}V_1(k)$.

Then, we have

$$\mathrm{Im}E^e(k-1)\subset V^e(k)\subset \mathrm{Ker}D^e(k).$$

Thus,

$$\begin{aligned}\sum_{h=k_0}^{k-1}\Phi^e(k,h+1)\mathrm{Im}E^e(h)&\subset\sum_{h=k_0}^{k-1}V^e(k)\\&=V^e(k)\\&\subset\mathrm{Ker}D^e(k)\end{aligned}$$

for all $k_0\in\boldsymbol{Z}, k\geq k_0$, which implies the DRPDC is solvable. □

Corollary 4.2 Suppose that $V^*(\cdot)$ is the maximal element of $\boldsymbol{V}(A(\cdot),B(\cdot);\mathrm{Ker}D(\cdot))$ and $V_*(\cdot)$ is the minimal element of $\boldsymbol{V}(\mathrm{Im}E(\cdot-1);C(\cdot),A(\cdot))$. If $\mathrm{Im}E(k-1)\subset\bigcap_{k\in\boldsymbol{Z}_{k_0}^{\omega}}V_*(k)$ and $V_*(k)\subset V^*(k)$ for all $k\in\boldsymbol{Z}$, then the DRPDC is solvable. In this case, the order of dynamic compensator which is necessary for the solution of the DRPDC is given by

$$\mu_* := \dim(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V^*(k)) - \dim(\bigcap_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V_*(k)).$$

Proof. The proof follows from Theorem 4.2. □

We remark that $1 \leq \mu \leq \mu_* \leq \dim(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} \mathrm{Ker}D(k)) - \dim(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} \mathrm{Im}E(k-1)) < n$:=the order of system plant Σ.

Finally, we consider the case that the disturbance map $E(k)$ does not depend on time k. Namely, assume that $E := E(k_0+1) = \cdots = E(k_0+\omega)$.

Theorem 4.3 Assume that $E := E(k_0+1) = \cdots = E(k_0+\omega)$. Then, the DRPDC is solvable if and only if there exists a $(C(\cdot), A(\cdot), B(\cdot))$-pair $(V_1(\cdot), V_2(\cdot))$ such that

$$\mathrm{Im}E \subset V_1(k) \subset V_2(k) \subset \mathrm{Ker}D(k) \quad \text{for all } k \in \boldsymbol{Z}.$$

In this case, the minimal order of dynamic compensator which is necessary for the solution of the DRPDC is given by

$$\mu^E_{min} := \min\{\dim(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V_2(k)) - \dim(\bigcap_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V_1(k)) \mid (V_1(\cdot), V_2(\cdot)) \text{ is } (C(\cdot), A(\cdot), B(\cdot))\text{-pair and } \mathrm{Im}E \subset V_1(k) \subset V_2(k) \subset \mathrm{Ker}D(k) \text{ for all } k \in \boldsymbol{Z}\}.$$

Proof. The proof follows from Theorems 4.1 and 4.2. □

Corollary 4.3 Assume that $E := E(k_0+1) = \cdots = E(k_0+\omega)$. And suppose that $V^*(\cdot)$ is the maximal element of $\boldsymbol{V}(A(\cdot), B(\cdot); \mathrm{Ker}D(\cdot))$ and $V^E_*(\cdot)$ is the minimal element of $\boldsymbol{V}(\mathrm{Im}E; C(\cdot), A(\cdot))$. Then, the DRPDC is solvable if and only if $V^E_*(k) \subset V^*(k)$ for all $k \in \boldsymbol{Z}$. In this case, the order of dynamic compensator which is necessary for the solution of the DRPDC is given by

$$\mu^E_* := \dim(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V^*(k)) - \dim(\bigcap_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V^E_*(k)).$$

Proof. The proof follows from Corollaries 4.1 and 4.2. □

We remark that $1 \leq \mu^E_{min} \leq \mu^E_* \leq \dim(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} \mathrm{Ker}D(k)) - \dim(\mathrm{Im}E) < n$:= the order of system plant Σ.

5 Concluding Remarks

In this paper necessary conditions and / or sufficient conditions for the solvability of the disturbance-rejection problem with dynamic compensator were given for linear ω-periodic discrete-time systems without assuming that the order of dynamic compensator is equal to that of system plant. Further, the minimal order of dynamic compensator which solves the DRPDC was given under the assumption that the disturbance map $E(k)$ does not depent on time k. As a future study we need to investigate the stability problem of disturbance-rejected systems.

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