A Unified Approach for Stability Robustness Computation of Quasipolynomials in a Convex Set

Guangdi Hu and Edward J. Davison
Systems Control Group
Dept. of Electrical and Computer Engineering
University of Toronto, 10 King's College Road
Toronto, Ontario, Canada M5S 3G4
E-mail: ted@control.toronto.edu, Fax: (416) 978 0804

Abstract: In this paper, the method of approach as used in Qiu and Davison (1992) and Hu and Davison (2000) is extended to the case of delay and neutral type quasipolynomials. We develop a unified approach to compute the stability robustness measure of quasipolynomials in a convex set using the framework of convex analysis. The procedure to compute the stability robustness measure which results from this approach is easy to implement. Finally, an example is given to demonstrate the effectiveness of this approach.

1 Problem Formulation and Computation

In this section, we refer to some results in convex analysis which will be used in the development to follow. The relevant notations and proofs such as the gauge, bounded, balanced, absorbing of S, ellipsoids, parallelotopes, crosspolytopes and polytopes can be found in Qiu and Davison (1992).

Let $\bar{\mathbb{C}}$ be the one point compactification of $\mathbb{C}$. Partition $\bar{\mathbb{C}}$ into two disjoint subsets $\mathbb{C}_g$ and $\mathbb{C}_b$, i.e. $\bar{\mathbb{C}} = \mathbb{C}_g \dot{\cup} \mathbb{C}_b$, such that $\mathbb{C}_g$ is open. Let $\mathcal{P}$ be the space of all real quasipolynomials. A quasipolynomial in $\mathcal{P}$ is said to be stable if its roots are contained in $\mathbb{C}_g$. We use $p(s,k)$ to denote the image of $k \in \mathbb{R}^r$ under a fixed affine map from $\mathbb{R}^r$ to $\mathcal{P}$ which is completely characterized by a matrix $F \in \mathbb{R}^{d\times r}$, delays τ_j $(j = 0, \cdots, m)$ and a vector $g \in \mathbb{R}^d$ as

$$\begin{aligned} p(s,k) &= \sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}(k) s^i e^{-\tau_j s} \\ &= [\ 1,\ s,\ \cdots,\ s^n,\ e^{-\tau_1 s},\ se^{-\tau_1 s},\ \cdots, \\ &\quad s^n e^{-\tau_1 s},\ \cdots,\ e^{-\tau_m s},\ se^{-\tau_m s},\ \cdots, \\ &\quad s^n e^{-\tau_m s}][Fk+g] \end{aligned} \tag{1}$$

For any bounded absorbing closed convex set $S \in \mathbb{R}^r$, let us define

$$\rho := \inf\{\mu_S(k) : k \in \mathbb{R}^r \text{ and } p(s,k) \text{ is unstable}\} \tag{2}$$

The purpose of this paper is to find a procedure to compute ρ when F, g, τ_j $(j = 0, \cdots, m)$, $\mathbb{C}_g$ and S are given. If $p(s,0)$ is unstable, we must have $\rho = 0$, so it is always assumed in the following that $p(s,0)$ is stable. Alternately we can write ρ as

$$\rho = \inf\{\mu_S(k) : k \in \mathbb{R}^r \text{ and } \exists s \in \mathbb{C}_b \text{ such that } p(s,k) = 0\}, \tag{3}$$

where $p(\infty,k) = 0$ means that $p(s,k)$ has a root at infinity.

Denote the boundary of $\mathbb{C}_g$ by $\partial\mathbb{C}_g$, i.e. $\partial\mathbb{C}_g = \mathbb{C}_b \cap \text{cl}(\mathbb{C}_g)$. This shows that

$$\rho = \inf_{s\in\partial\mathbb{C}_g}\{\inf\{\mu_S(k) : k \in \mathbb{R}^r \text{ and } p(s,k) = 0\}\}. \tag{4}$$

Define a function $\theta(s) : \partial\mathbb{C}_g \longrightarrow [0,\infty]$ by

$$\theta(s) = \inf\{\mu_S(k) : k \in \mathbb{R}^r \text{ and } p(s,k) = 0\}. \tag{5}$$

It is seen from (4) that the computation of ρ can be accomplished in two phases. The first phase is to find $\theta(s)$ for any fixed $s \in \partial\mathbb{C}_g$. The second phase is to carry out a search over all points in $\partial\mathbb{C}_g$, which is usually a one-dimensional curve in $\bar{\mathbb{C}}$, to find $\inf_{s\in\partial\mathbb{C}_g}\theta(s)$.

Now let $s \in \partial\mathbb{C}_g$ be fixed. Let

$$w := \frac{F'[1,\ s,\ \cdots,\ s^n,\ e^{-\tau_1 s},\ \cdots,\ s^n e^{-\tau_m s}]'}{-[1,\ s,\ \cdots,\ s^n,\ e^{-\tau_1 s},\ \cdots,\ s^n e^{-\tau_m s}]g} \tag{6}$$

and let $u := \Re(w)$, $v := \Im(w)$. Then $u, v \in \mathbb{R}^r$ and (6) is equivalent to

$$u'k = 1,\ v'k = 0. \tag{7}$$

Consequently, we have

$$\theta(s) = \inf\{\mu_S(k) : k \in \mathbb{R}^r \text{ and } u(s)'k = 1,\ v(s)'k = 0\} \tag{8}$$

The main theorem given in this section simplifies the quantity (8) to a form more readily computable.

Theorem If S is a balanced bounded absorbing closed

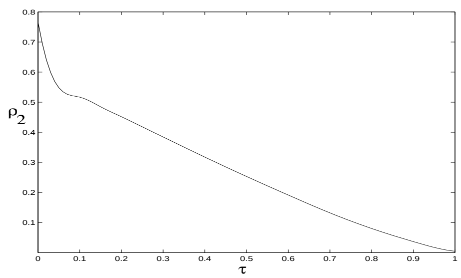

Figure 1: Robustness measure with respect to delay in the case 2-norm

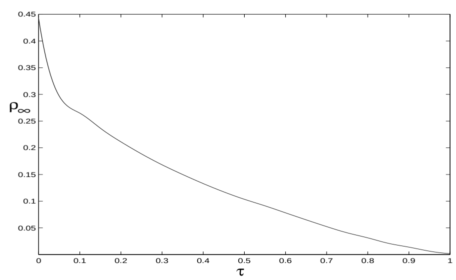

Figure 2: Robustness measure with respect to delay in the case ∞-norm

convex set in $\mathbb{R}^r$, then for any $u, v \in \mathbb{R}^r$,

$$\begin{aligned} & \inf\{\mu_S(k) : k \in \mathbb{R}^r \text{ and } u^{'} k = 1, v^{'} k = 0\} \\ = & \begin{cases} \infty & \text{if } \operatorname{rank}[u, v] \neq \operatorname{rank} \begin{bmatrix} u & v \\ 1 & 0 \end{bmatrix} \\ \frac{1}{\mu_{S^0}(u)} & \text{if } u \neq 0 \text{ and } v = 0 \\ \sup_{\alpha \in \mathbb{R}} \frac{1}{\mu_{S^0}(u+\alpha v)} & \text{if } \operatorname{rank}[u, v] = 2 \end{cases} \end{aligned} \tag{9}$$

Proof. The proof is identical to that given in Qiu and Davison (1992), and so it is omitted.

2 A numerical example

Example: The following quasipolynomial is stable for each $0 \leq \delta \leq (\pi/2-1)^2$ (see, Fu et al. (1989))

$$p(s, 0) = \delta + s^2 + 2se^{-s} + e^{-2s}.$$

Now we consider its coefficient perturbation form described by

$$\begin{aligned} p(s, k) \quad = \quad & [k_1 + (\pi/2-1)^2] + k_2 s + (k_3+1)s^2 \\ & + [k_4 + (k_5+2)s + k_6 s^2] e^{-\tau s} \\ & + [(k_7+1) + k_8 s + k_9 s^2] e^{-2\tau s}. \end{aligned}$$

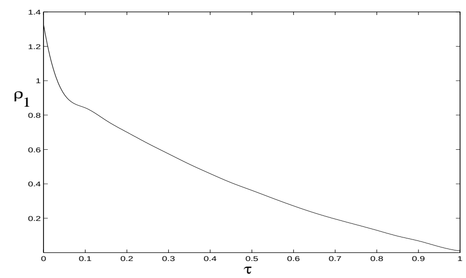

Figure 3: Robustness measure with respect to delay in the case 1-norm

which corresponds to the affine function given by $a(k) = Fk + g$, where $F = I_{9\times 9}$ (9×9 identity matrix), and k, g are given by

$$k' = [k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9]$$

and

$$g' = [(\pi/2-1)^2, 0, 1, 0, 2, 0, 1, 0, 0].$$

Stability radii with respect to delay are determined using Theorem and are given in Figures 1-3 for cases when the 1-norm, 2-norm and ∞-norm, respectively, are used.

3 Conclusions

An approach based on the framework of convex analysis is developed in this paper to study the stability robustness of quasipolynomials. This approach unifies and improves some recent results in the area of the stability robustness of quasipolynomials, motivates and solves new related problems.

References

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