Direct Adaptive Control for Nonlinear Uncertain Systems with Bounded Energy L_2 Disturbances

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Abstract

A direct adaptive nonlinear control framework for multivariable nonlinear uncertain systems with exogenous L_2 disturbances is developed. The proposed framework is Lyapunov-based and guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant, as well as a nonexpansivity constraint on the closed-loop input-output map. Finally, an illustrative numerical example is provided to demonstrate the efficacy of the proposed approach.

1. Introduction

In a recent paper [1], a direct adaptive control framework for adaptive stabilization, disturbance rejection, and command following of multivariable nonlinear uncertain systems with exogenous bounded amplitude disturbances was developed. In particular, a Lyapunov-based direct adaptive control framework was developed that requires a matching condition on the system disturbance and guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Furthermore, the remainder of the state associated with the adaptive controller gains was shown to be Lyapunov stable. In the case where the nonlinear system was represented in normal form [2] with input-to-state stable zero dynamics [2, 3], the nonlinear adaptive controller was constructed *without* requiring knowledge of the system dynamics or the system disturbance.

In this paper, we generalize the results of [1] to uncertain nonlinear systems with exogenous L_2 disturbances. Specifically, we remove the matching condition on the system disturbance required in [1]. In addition, the proposed framework guarantees that the closed-loop nonlinear input-output map from uncertain exogenous L_2 disturbances to system performance variables is nonexpansive (gain bounded) and the zero trajectory of the closed-loop system is partially asymptotically stable.

We emphasize that the direct adaptive stabilization framework developed in this paper is distinct from the methods given in [4–7] predicated on model reference adaptive control. The work of [8, 9] on *linear* direct adaptive control is most closely related to the results presented herein. Specifically, specializing our result to single-input linear systems with no internal dynamics, we recover the result given in [9].

2. Adaptive Control for Nonlinear Systems with L_2 Disturbances

In this section we begin by considering the problem of characterizing adaptive feedback control laws for nonlinear uncertain systems with L_2 disturbances. Specifically, consider the controlled nonlinear uncertain system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t) + J(x(t))w(t), \quad x(0) = x_0, \quad t \geq 0, \tag{1}$$

with performance variables

$$z(t) = h(x(t)), \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $w(t) \in \mathbb{R}^d$, $t \geq 0$, is an unknown bounded energy L_2 disturbance, $z(t) \in \mathbb{R}^p$, $t \geq 0$, is a performance variable, $f : \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $J : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ and satisfies $h(0) = 0$. The control input $u(\cdot)$ in (1) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$. Furthermore, for the nonlinear system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $f(\cdot)$, $G(\cdot)$, $u(\cdot)$, and $w(\cdot)$ satisfy sufficient regularity conditions such that (1) has a unique solution forward in time.

Theorem 2.1. Consider the nonlinear system $\mathcal{G}$ given by (1) and (2). Assume there exists a matrix $K_{\rm g} \in \mathbb{R}^{m\times s}$ and functions $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, such that the zero solution $x(t) \equiv 0$ to

$$\dot{x}(t) = f(x(t)) + G(x(t))\hat{G}(x(t))K_{\rm g}F(x(t)) \triangleq f_{\rm c}(x(t)), \quad x(0) = x_0, \ t \geq 0, \tag{3}$$

is globally asymptotically stable. Furthermore, assume there exists a C^1 function $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$ such that $V_{\rm s}(\cdot)$ is positive definite, radially unbounded, $V_{\rm s}(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 = V_{\rm s}'(x)f_{\rm c}(x) + \Gamma(x), \tag{4}$$

where

$$\Gamma(x) \triangleq \tfrac{1}{4\gamma^2}V_{\rm s}'(x)J(x)J^{\rm T}(x)V_{\rm s}'^{\rm T}(x) + h^{\rm T}(x)h(x). \tag{5}$$

Finally, let $Q \in \mathbb{R}^{m\times m}$ and $Y \in \mathbb{R}^{s\times s}$, be positive definite. Then the adaptive feedback control law

$$u(t) = \hat{G}(x(t))K(t)F(x(t)), \tag{6}$$

where $K(t) \in \mathbb{R}^{m\times s}$, $t \geq 0$, with update law

$$\dot{K}(t) = -\tfrac{1}{2}Q\hat{G}^{\rm T}(x(t))G^{\rm T}(x(t))V_{\rm s}'^{\rm T}(x(t))F^{\rm T}(x(t))Y, \tag{7}$$

guarantees that the zero solution $(x(t), K(t)) \equiv (0, 0)$ of the closed-loop system given by (1), (6), and (7) is Lyapunov stable and $h(x(t)) \to 0$ as $t \to \infty$. If, in addition, $h^{\rm T}(x)h(x) > 0$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$. Furthermore, the solution $x(t)$, $t \geq 0$, to the closed-loop system

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given by (1), (6), and (7) satisfies the nonexpansivity constraint

$$\int_0^T z^{\mathrm{T}}(t)z(t)\,\mathrm{d}t \;\le\; \gamma^2\int_0^T w^{\mathrm{T}}(t)w(t)\,\mathrm{d}t + V(x(0),K(0)),\qquad T\ge 0,\quad w(\cdot)\in \mathrm{L}_2,\quad (8)$$

where

$$V(x,K) \triangleq V_{\mathrm{s}}(x) + \mathrm{tr}\,Q^{-1}(K-K_{\mathrm{g}})Y^{-1}(K-K_{\mathrm{g}})^{\mathrm{T}}. \quad (9)$$

Proof. Note that with $u(t)$, $t\ge 0$, given by (6) it follows from (1) that

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + G(x(t))\hat{G}(x(t))K(t)F(x(t))\\ &\quad +J(x(t))w(t),\ x(0)=x_0,\ w(\cdot)\in\mathrm{L}_2,\ t\ge 0,\end{aligned}\quad (10)$$

or, equivalently, using the definition for $f_{\mathrm{c}}(x)$ given in (3),

$$\begin{aligned}\dot{x}(t) &= f_{\mathrm{c}}(x(t)) + G(x(t))\hat{G}(x(t))(K(t)-K_{\mathrm{g}})F(x(t))\\ &\quad +J(x(t))w(t),\ x(0)=x_0,\ w(\cdot)\in\mathrm{L}_2,\ t\ge 0.\end{aligned}\quad (11)$$

To show Lyapunov stability of the closed-loop system (7) and (11) consider the Lyapunov function candidate given by (9). Note that $V(0,K_{\mathrm{g}})=0$ and, since $V_{\mathrm{s}}(\cdot)$, Q, and Y are positive definite, $V(x,K)>0$ for all $(x,K)\ne(0,K_{\mathrm{g}})$. Furthermore, $V(x,K)$ is radially unbounded. Now, letting $x(t)$, $t\ge 0$, denote the solution to (11) and using (4) and (7), it follows that the Lyapunov derivative along the undisturbed ($w(t)\equiv 0$) closed-loop system trajectories is given by

$$\begin{aligned}&\dot{V}(x(t),K(t))\\ &\quad= V_{\mathrm{s}}'(x(t))[f_{\mathrm{c}}(x(t))\\ &\qquad +G(x(t))\hat{G}(x(t))(K(t)-K_{\mathrm{g}})F(x(t))]\\ &\qquad +2\mathrm{tr}\,Q^{-1}(K(t)-K_{\mathrm{g}})Y^{-1}\dot{K}^{\mathrm{T}}(t)\\ &\quad= -\Gamma(x(t))\\ &\qquad +\mathrm{tr}\Big[(K(t)-K_{\mathrm{g}})F(x(t))V_{\mathrm{s}}'(x(t))G(x(t))\hat{G}(x(t))\Big]\\ &\qquad -\mathrm{tr}\Big[(K(t)-K_{\mathrm{g}})F(x(t))V_{\mathrm{s}}'(x(t))G(x(t))\hat{G}(x(t))\Big]\\ &\quad= -\Gamma(x(t))\\ &\quad\le 0,\end{aligned}\quad (12)$$

which proves that the solution $(x(t),K(t))\equiv(0,K_{\mathrm{g}})$ to (7) and (11) with $w(t)\equiv 0$ is Lyapunov stable. Furthermore, it follows from Theorem 4.4 of [5] that $h(x(t))\to 0$ as $t\to\infty$ for all $x_0\in\mathbb{R}^n$. In addition, if $h^{\mathrm{T}}(x)h(x)>0$, $x\ne 0$, then $x(t)\to 0$ as $t\to\infty$ for all $x_0\in\mathbb{R}^n$.

Finally, to show that the nonexpansivity constraint (8) holds, note that, for all $w\in\mathbb{R}^d$,

$$\begin{aligned}0 &\le \left[\tfrac{1}{2\gamma}J^{\mathrm{T}}(x)V_{\mathrm{s}}'^{\mathrm{T}}(x)-\gamma w\right]^{\mathrm{T}}\left[\tfrac{1}{2\gamma}J^{\mathrm{T}}(x)V_{\mathrm{s}}'^{\mathrm{T}}(x)-\gamma w\right]\\ &= \Gamma(x)+\gamma^2 w^{\mathrm{T}}w - z^{\mathrm{T}}z - V_{\mathrm{s}}'^{\mathrm{T}}(x)J(x)w.\end{aligned}\quad (13)$$

Now, let $w(\cdot)\in\mathrm{L}_2$ and let $x(t)$, $t\ge 0$, denote the solution of the closed-loop system (7), (11). Then the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}&\dot{V}(x(t),K(t))\\ &\quad= V_{\mathrm{s}}'(x(t))[f_{\mathrm{c}}(x(t))\\ &\qquad +G(x(t))\hat{G}(x(t))(K(t)-K_{\mathrm{g}})F(x(t))+J(x(t))w(t)]\\ &\qquad +2\mathrm{tr}\,Q^{-1}(K(t)-K_{\mathrm{g}})Y^{-1}\dot{K}^{\mathrm{T}}(t)\\ &\quad= -\Gamma(x(t))\\ &\qquad +\mathrm{tr}\,(K(t)-K_{\mathrm{g}})F(x(t))V_{\mathrm{s}}'(x(t))G(x(t))\hat{G}(x(t))\\ &\qquad +V_{\mathrm{s}}'(x(t))J(x(t))w(t)\\ &\qquad -\mathrm{tr}\,(K(t)-K_{\mathrm{g}})F(x(t))V_{\mathrm{s}}'(x(t))G(x(t))\hat{G}(x(t))\\ &\quad= -\Gamma(x(t))+V_{\mathrm{s}}'(x(t))J(x(t))w(t)\\ &\quad\le \gamma^2 w^{\mathrm{T}}(t)w(t)-z^{\mathrm{T}}(t)z(t).\end{aligned}\quad (14)$$

Now, integrating (14) over $[0,\,T]$ yields

$$\begin{aligned}V(x(T),K(T)) &\le \int_0^T\big[\gamma^2 w^{\mathrm{T}}(t)w(t)-z^{\mathrm{T}}(t)z(t)\big]\mathrm{d}t\\ &\quad +V(x(0),K(0)),\ T>0,\ w(\cdot)\in\mathrm{L}_2,\end{aligned}\quad (15)$$

which, by noting that $V(x(T))\ge 0$, $T\ge 0$, yields (8). □

Remark 2.1. Note that the conditions in Theorem 2.1 imply that $x(t)\to 0$ as $t\to\infty$ and hence it follows from (7) that $(x(t),K(t))\to\mathcal{M}\triangleq\{(x,K)\in\mathbb{R}^n\times\mathbb{R}^{m\times s}: x=0,\ \dot{K}=0\}$ as $t\to\infty$.

It is important to note that the adaptive control law (6) and (7) does *not* require explicit knowledge of the gain matrix K_{g}; even though Theorem 2.1 requires the existence of K_{g}, $F(x)$, and $\hat{G}(x)$ such that the zero solution $x(t)\equiv 0$ to (3) is globally asymptotically stable. Furthermore, if (1) is in controllable normal form with asymptotically stable internal dynamics [2], then we can always construct functions $\hat{G}:\mathbb{R}^n\to\mathbb{R}^{m\times m}$ and $F:\mathbb{R}^n\to\mathbb{R}^s$, with $F(0)=0$, such that the zero solution $x(t)\equiv 0$ to (3) is globally asymptotically stable *without* requiring knowledge of the system dynamics. To see this assume that the nonlinear uncertain system $\mathcal{G}$ is generated by

$$q_i^{(r_i)}(t) = f_{\mathrm{u}\,i}(q(t)) + \sum_{j=1}^m G_{\mathrm{s}\,(i,j)}(q(t))u_j(t) + \sum_{k=1}^d \hat{D}_{(i,k)}w_k(t),\ q(0)=q_0,\ i=1,\cdots,m,\quad (16)$$

where $q_i^{(r_i)}$ denotes the r_i^{th} derivative of q_i, r_i denotes the relative degree with respect to the output q_i, $f_{\mathrm{u}\,i}(q)=f_{\mathrm{u}\,i}(q_1,\cdots,q_1^{(r_1-1)},\cdots,q_m,\cdots,q_m^{(r_m-1)})$, $G_{\mathrm{s}\,(i,j)}(q)=G_{\mathrm{s}\,(i,j)}(q_1,\cdots,q_1^{(r_1-1)},\cdots,q_m,\cdots,q_m^{(r_m-1)})$, $\hat{D}_{(i,k)}\in\mathbb{R}$, $i=1,\cdots,m, k=1,\cdots,d$, and $w_k(t)\in\mathbb{R}$, $t\ge 0$, $k=1,\cdots,d$. Here we assume that the square matrix function $G_{\mathrm{s}}(q)$ composed of the entries $G_{\mathrm{s}\,(i,j)}(q)$, $i,j=1,\cdots,m$, is such that $\det G_{\mathrm{s}}(q)\ne 0$, $q\in\mathbb{R}^{\hat{r}}$, where $\hat{r}=r_1+\cdots+r_m$ is the (vector) relative degree of (16). Furthermore, since (16) is in a form where it does not possess internal dynamics, it follows that $\hat{r} = n$. The case where (16) possesses input-to-state stable internal dynamics can be handled as shown in [1].

Next, define $x_i \triangleq \left[q_i, \cdots, q_i^{(r_i-2)} \right]^{\mathrm{T}}$, $i = 1, \cdots, m$, $x_{m+1} \triangleq \left[q_1^{(r_1-1)}, \cdots, q_m^{(r_m-1)} \right]^{\mathrm{T}}$, and $x \triangleq \left[x_1^{\mathrm{T}}, \cdots, x_{m+1}^{\mathrm{T}} \right]^{\mathrm{T}}$, so that (16) can be described as (1) with

$$f(x) = \tilde{A}x + \tilde{f}_{\mathrm{u}}(x), \quad G(x) = \left[\begin{array}{c} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{array} \right], \quad J(x) = D = \left[\begin{array}{c} 0_{(n-m)\times d} \\ \hat{D} \end{array} \right], \tag{17}$$

where

$$\tilde{A} = \left[\begin{array}{c} A_0 \\ 0_{m\times n} \end{array} \right], \quad \tilde{f}_{\mathrm{u}}(x) = \left[\begin{array}{c} 0_{(n-m)\times 1} \\ f_{\mathrm{u}}(x) \end{array} \right],$$

$A_0 \in \mathbb{R}^{(n-m)\times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [10], $f_{\mathrm{u}} : \mathbb{R}^n \to \mathbb{R}^m$ is an unknown function and satisfies $f_{\mathrm{u}}(0) = 0$, $G_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and $\hat{D} \in \mathbb{R}^{m\times d}$. Here, we assume that $f_{\mathrm{u}}(x)$ is unknown and is parameterized as $f_{\mathrm{u}}(x) = \Theta f_{\mathrm{n}}(x)$, where $f_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^q$ and satisfies $f_{\mathrm{n}}(0) = 0$, and $\Theta \in \mathbb{R}^{m\times q}$ is a matrix of uncertain constant parameters.

Next, to apply Theorem 2.1 to the uncertain system (1) with $f(x)$, $G(x)$, and $J(x) = D$ given by (17), let $K_{\mathrm{g}} \in \mathbb{R}^{m\times s}$, where $s = q + r$, be given by

$$K_{\mathrm{g}} = [\, \Theta_{\mathrm{n}} - \Theta,\ \Phi_{\mathrm{n}} \,], \tag{18}$$

where $\Theta_{\mathrm{n}} \in \mathbb{R}^{m\times q}$ and $\Phi_{\mathrm{n}} \in \mathbb{R}^{m\times r}$ are known matrices, and let $\hat{G}(x) = G_{\mathrm{s}}^{-1}(x)$ and

$$F(x) = \left[\begin{array}{c} f_{\mathrm{n}}(x) \\ \hat{f}_{\mathrm{n}}(x) \end{array} \right], \tag{19}$$

where $\hat{f}_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^r$ and satisfies $\hat{f}_{\mathrm{n}}(0) = 0$ is an arbitrary function. In this case, it follows that

$$\begin{aligned} f_{\mathrm{c}}(x) &= f(x) + G(x)\hat{G}(x)K_{\mathrm{g}}F(x) \\ &= \tilde{A}x + \tilde{f}_{\mathrm{u}}(x) + \left[\begin{array}{c} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{array} \right] G_{\mathrm{s}}^{-1}(x) \\ &\quad \cdot \left[\ \Theta_{\mathrm{n}} f_{\mathrm{n}}(x) - \Theta f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}} \hat{f}_{\mathrm{n}}(x)\ \right] \\ &= \tilde{A}x + \left[\begin{array}{c} 0_{(n-m)\times 1} \\ \Theta_{\mathrm{n}} f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}} \hat{f}_{\mathrm{n}}(x) \end{array} \right]. \end{aligned} \tag{20}$$

Now, since $\Theta_{\mathrm{n}} \in \mathbb{R}^{m\times q}$ and $\Phi_{\mathrm{n}} \in \mathbb{R}^{m\times r}$ are arbitrary constant matrices and $\hat{f}_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^r$ is an arbitrary function we can always construct K_{g} and $F(x)$ without knowledge of $f(x)$ such that the zero solution $x(t) \equiv 0$ to (3) can be made globally asymptotically stable. In particular, choosing $\Theta_{\mathrm{n}} f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}} \hat{f}_{\mathrm{n}}(x) = \hat{A}x$, where $\hat{A} \in \mathbb{R}^{m\times n}$, it follows that (20) has the form $f_{\mathrm{c}}(x) = A_{\mathrm{c}}x$, where $A_{\mathrm{c}} = [A_0^{\mathrm{T}}, \hat{A}^{\mathrm{T}}]^{\mathrm{T}}$ is in multivariable controllable canonical form. In addition, in the case where $J(x) = D$ and $h(x) = Ex$, the adaptive controller (7) can be constructed to guarantee the nonexpansivity constraint (8) using standard *linear* H_∞ methods. Specifically, choosing $f_{\mathrm{c}}(x) = A_{\mathrm{c}}x$, where A_{c} is asymptotically stable and in multivariable controllable canonical form, it follows from standard H_∞ theory [11] that if (A_{c}, E) is observable and $\|G(s)\|_\infty < \gamma$, where $G(s) = E(sI_n - A_{\mathrm{c}})^{-1}D$, then there exists a positive-definite matrix P satisfying the bounded real Riccati equation

$$0 = A_{\mathrm{c}}^{\mathrm{T}}P + PA_{\mathrm{c}} + \gamma^{-2}PDD^{\mathrm{T}}P + E^{\mathrm{T}}E. \tag{21}$$

It is well known that (21) has a nonnegative-definite solution if and only if the Hamiltonian matrix

$$\mathcal{H} = \left[\begin{array}{cc} A_{\mathrm{c}} & \gamma^{-2}DD^{\mathrm{T}} \\ -E^{\mathrm{T}}E & -A_{\mathrm{c}}^{\mathrm{T}} \end{array} \right], \tag{22}$$

has no purely imaginary eigenvalues. If, in addition, $E^{\mathrm{T}}E > 0$, then $P > 0$. In this case, with Lyapunov function $V_{\mathrm{s}}(x) = x^{\mathrm{T}}Px$, the adaptive feedback controller (6) with update law (7), or, equivalently,

$$\dot{K}(t) = -Q\hat{G}^{\mathrm{T}}(x(t))G^{\mathrm{T}}(x(t))Px(t)F^{\mathrm{T}}(x(t))Y, \tag{23}$$

guarantees global asymptotic stability of the nonlinear undisturbed ($w(t) \equiv 0$) dynamical system (1), where $f(x)$ and $G(x)$ are given by (17). Furthermore, the solution $x(t)$, $t \geq 0$, of the closed-loop *nonlinear* dynamical system (1) is guaranteed to satisfy the nonexpansivity constraint (8).

Next, we consider the case where $f(x)$ and $G(x)$ are uncertain. Specifically, we assume that $G_{\mathrm{s}}(x)$ is unknown and is parameterized as $G_{\mathrm{s}}(x) = B_{\mathrm{u}}G_{\mathrm{n}}(x)$, where $G_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is known and satisfies $\det G_{\mathrm{n}}(x) \neq 0$, $x \in \mathbb{R}^n$, and $B_{\mathrm{u}} \in \mathbb{R}^{m\times m}$, with $\det B_{\mathrm{u}} \neq 0$, is a symmetric sign definite matrix but the sign definiteness of B_{u} is known; that is, $B_{\mathrm{u}} > 0$ or $B_{\mathrm{u}} < 0$. For the statement of the next result define $B_0 \triangleq \left[0_{m\times(n-m)},\ I_m\right]^{\mathrm{T}}$ for $B_{\mathrm{u}} > 0$, and $B_0 \triangleq \left[0_{m\times(n-m)},\ -I_m\right]^{\mathrm{T}}$ for $B_{\mathrm{u}} < 0$.

Corollary 2.1. Consider the nonlinear system $\mathcal{G}$ given by (1) with $f(x)$, $G(x)$, and $J(x)$ given by (17) and $G_{\mathrm{s}}(x) = B_{\mathrm{u}}G_{\mathrm{n}}(x)$, where B_{u} is an unknown symmetric matrix and the sign definiteness of B_{u} is known. Assume there exists a matrix $K_{\mathrm{g}} \in \mathbb{R}^{m\times s}$ and a function $F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, such that the zero solution $x(t) \equiv 0$ to (3) is globally asymptotically stable. Furthermore, assume there exists a C^1 function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$ such that $V_{\mathrm{s}}(\cdot)$ is positive definite, radially unbounded, $V_{\mathrm{s}}(0) = 0$, and, for all $x \in \mathbb{R}^n$, and (4) holds. Finally, let $Y \in \mathbb{R}^{s\times s}$ be positive definite. Then the adaptive feedback control law

$$u(t) = G_{\mathrm{n}}^{-1}(x(t))K(t)F(x(t)), \tag{24}$$

where $K(t) \in \mathbb{R}^{m\times s}$, $t \geq 0$, with update law

$$\dot{K}(t) = -\tfrac{1}{2}B_0 V_{\mathrm{s}}'^{\mathrm{T}}(x(t))F^{\mathrm{T}}(x(t))Y, \tag{25}$$

guarantees that the zero solution $(x(t), K(t)) \equiv (0, 0)$ of the closed-loop system given by (1), (6), and (7) is Lyapunov stable and $h(x(t)) \to 0$ as $t \to \infty$. If, in addition, $h^{\mathrm{T}}(x)h(x) > 0$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$. Furthermore, the solution $x(t)$, $t \geq 0$, to the closed-loop system given by (1), (6), and (7) satisfies the nonexpansivity constraint (8).

Proof. The result is a direct consequence of Theorem 2.1. First, let $\hat{G}(x) = G_{\rm n}^{-1}(x)$ so that $G(x)\hat{G}(x) = [0_{m\times(n-m)}, B_{\rm u}]^{\rm T}$. Next, since Q is an arbitrary positive-definite matrix, Q in (7) can be replaced by $q|B_{\rm u}|^{-1}$, where q is a positive constant and $|B_{\rm u}| = (B_{\rm u}^2)^{\frac{1}{2}}$, where $(\cdot)^{\frac{1}{2}}$ denotes the (unique) positive-definite square root. Now, since $B_{\rm u}$ is symmetric and sign definite it follows from the Schur decomposition that $B_{\rm u} = UD_{B_{\rm u}}U^{\rm T}$, where U is orthogonal and $D_{B_{\rm u}}$ is real diagonal. Hence, $|B_{\rm u}|^{-1}B^{\rm T} = [0_{m\times(n-m)}, \mathcal{I}_m] = B_0^{\rm T}$, where $\mathcal{I}_m = I_m$ for $B_{\rm u} > 0$ and $\mathcal{I}_m = -I_m$ for $B_{\rm u} < 0$. Now, (7), with qY replaced by Y, implies (25). □

It is important to note that if, as discussed above, $K_{\rm g}$ and $F(x)$ are constructed to give $f_{\rm c}(x) = A_{\rm c}x$ in (3) and in the case where $J(x) = D$ and $h(x) = Ex$, where $A_{\rm c}$ is an asymptotically stable matrix in multivariable controllable canonical form, then considerable simplification occurs in Corollary 2.1. Specifically, in this case $V_{\rm s}(x) = x^{\rm T}Px$, where $P > 0$ satisfies (21), and hence (25) becomes

$$\dot{K}(t) = -B_0^{\rm T}Px(t)F^{\rm T}(x(t))Y. \qquad (26)$$

3. Illustrative Numerical Example

In this section we present a numerical example to demonstrate the utility of the proposed direct adaptive control framework for adaptive stabilization and L_2 disturbance rejection.

Consider the nonlinear dynamical system representing a controlled rigid spacecraft given by

$$\dot{x}(t) = -I_{\rm b}^{-1}XI_{\rm b}x(t) + I_{\rm b}^{-1}u(t) + Dw(t), \quad x(0) = x_0,\ w(\cdot) \in \mathrm{L}_2,\ t \geq 0, \qquad (27)$$

where $x = [x_1, x_2, x_3]^{\rm T}$ represents the angular velocities of the spacecraft with respect to the body-fixed frame, $I_{\rm b} \in \mathbb{R}^{3\times3}$ is an unknown positive-definite inertia matrix of the spacecraft, $u(t) = [u_1, u_2, u_3]^{\rm T}$ is a control vector with control inputs providing body-fixed torques about three mutually perpendicular axes defining the body-fixed frame of the spacecraft, $D \in \mathbb{R}^{3\times1}$, and X denotes the skew-symmetric matrix

$$X \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

Note that (27) can be written in state space form (1) with $f(x) = -I_{\rm b}^{-1}XI_{\rm b}x$, $G(x) = I_{\rm b}^{-1}$, and $J(x) = D$. Here, we assume that the inertia matrix $I_{\rm b}$ of the spacecraft is symmetric and positive definite but unknown. Since $f(x)$ is a quadratic function, we parameterize $f(x)$ as $f(x) = \Theta f_{\rm n}(x)$, where $\Theta \in \mathbb{R}^{3\times6}$ is an unknown matrix and $f_{\rm n}(x) = [x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_3x_1]^{\rm T}$. Next, let $F(x) = \left[f_{\rm n}^{\rm T}(x), x^{\rm T}\right]^{\rm T}$, $\hat{G}(x) \equiv I_3$, and $K_{\rm g} = I_{\rm b}\left[-\Theta, \Phi_{\rm n}\right]$, where $\Phi_{\rm n} \in \mathbb{R}^{3\times3}$, is an arbitrary matrix, so that

$$f_{\rm c}(x) = \Phi_{\rm n}x = A_{\rm c}x.$$

Now, with the proper choice of $\Phi_{\rm n}$, it follows from Corollary 2.1 that the adaptive feedback controller (24) with update law (25) guarantees that $x(t) \to 0$ as $t \to \infty$ with $w(t) \equiv 0$. Furthermore, the closed-loop nonlinear input-output map from L_2 disturbances $Dw(t)$ to performance variable $z(t) = Ex(t)$ satisfies the nonexpansivity constraint (8). Here, we choose $A_{\rm c} = -10I_3$, $E^{\rm T}E = 2I_3$, and $\gamma = 1.4$, so that P satisfying (21) is given by

$$P = \begin{bmatrix} 0.1653 & 0.0408 & 0.0245 \\ 0.0408 & 0.1255 & 0.0153 \\ 0.0245 & 0.0153 & 0.1092 \end{bmatrix}.$$

With

$$I_{\rm b} = \begin{bmatrix} 20 & 0 & 0.9 \\ 0 & 17 & 0 \\ 0.9 & 0 & 15 \end{bmatrix}, \quad Y = 10I_9, \quad D = \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix},$$

$$w(t) = e^{-0.2t}\sin 1.8t,$$

and initial conditions $x(0) = [0.4, 0.2, -0.2]$ and $K(0) = 0_{3\times9}$, Figure 3.1 shows the angular velocities versus time. Figure 3.2 shows the control signal versus time. An alternative adaptive feedback controller that also does not require knowledge of the inertia of the spacecraft is presented in [12]. However, unlike the proposed controller, the adaptive controller presented in [12] is tailored to the spacecraft attitude control problem and does not address L_2 disturbance rejection.

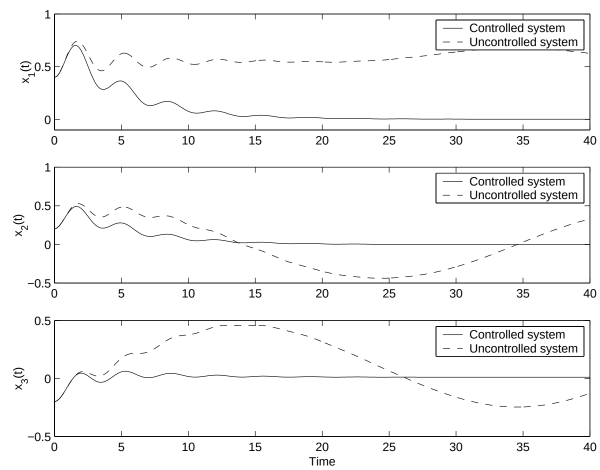

Figure 3.1: Angular velocities versus time

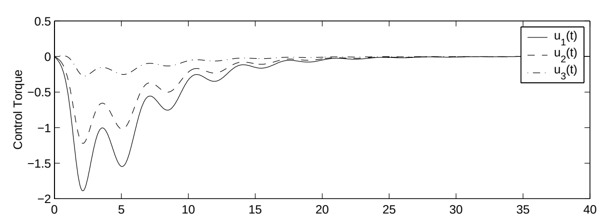

Figure 3.2: Control signal versus time

4. Conclusion

A direct adaptive nonlinear control framework for adaptive stabilization of multivariable nonlinear uncertain systems with exogenous L_2 disturbances was developed.

Using Lyapunov methods the proposed framework was shown to guarantee partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant, as well as a nonexpansivity constraint on the closed-loop input-output map.

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