An Asymptotic Expansion In A 3-Dimensional Degenerated Control Problem With Finite Horizon

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Abstract

We study a degenerate nonlinear optimal stochastic control problem with finite horizon ([2], [8] and [9]). Using the Hamilton–Jacobi–Bellman equation, we find an asymptotic expansion for this solution of this problem.

1 Introduction

The aim of this paper is the study and the approximation of a class of the control problem with finite horizon. We shall work on a particular problem which comes from a problem of optimization of controlled shock-absorber.

The control problem with finite horizon is to solve the *Hamilton–Bellman–Jacobi equation*.
And, we are interested in an asymptotic expansion of the solution of this equation.
The paper is organized as follows: In section 2, we present the control problem with finite horizon. We write this solution as an expansion in powers of ε. In sections 3 and 4, we get the equations of the terms of the expansion. The time-scale techniques are used in order to get the terms of the expansion. In the section 5, we present results on convergence.

2 Modelisation

This problem was studied in the 2-dimensional ergodic case ([7]). By modelling the road surface displacement by a 1-dimensional stationnary diffusion process $Z^\varepsilon(t) = Z(t/\varepsilon^2)$ where $Z(t)$ is defined by

$$\begin{aligned} dZ(t) &= b_3(Z(t))dt + \sqrt{2}\sigma_3(Z(t))dW_3(t) \\ Z(0) &= Z^0 \end{aligned}$$

we get a 3-dimensional problem ([6]).
Let us consider the stochastic problem

$$dX(t) = b(X(t), u(X(t)))dt + \sigma(X(t), u(X(t)))dW(t)$$

where $X(t)$ is a process with values in $\mathbb{R}^3$, $W(t)$ is a Wiener process in $\mathbb{R}^3$. b and σ are functions of $\mathbb{R} \times \mathbb{R}^2$ with values respectively in $\mathbb{R}^3$ and $\mathbb{R}^3 \times \mathbb{R}^3$ and defined by

$$b(X,u) = \begin{pmatrix} y \\ -(uy + \beta x + \gamma \text{signe}(y)) - \alpha z \\ \alpha^2 b_3(z) \end{pmatrix}$$

and,

$$\sigma(X,u) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}\alpha\sigma_3(z) \end{pmatrix}$$

where $\alpha = 1/\varepsilon$, β and γ are real strictly positive constants. In the previous equation, u is a feedback control which belongs to the class $\mathcal{U}$ of admissible controls defined by $0 < \underline{u} < u(X) < \overline{u} < \infty$.

We are interested in a control problem with finite horizon whose cost functional is

$$\phi^\varepsilon(t) = \inf_{u \in \mathcal{U}} J(u,t) = \inf_{u \in \mathcal{U}} E \int_0^t f(X(s), u(X(s)))ds$$

where the instantaneous cost f is given by

$$f(X,u) = (uy + \beta x + \gamma \text{signe}(y))^2.$$

The physical interpretation of this problem is the following: $r(t) = X_1(t)$ is solution of the equation

$$m\ddot{r} + v\dot{r} + Kr + F\text{signe}(\dot{r}) = m\ddot{e}$$

with $(m,\ K,\ F > 0)$, which discribes a one degree-of-freedom shock-absorber system with dry friction. r is the relative displacement, v the shock-absorber damping constant (the controlled parameter). $Kr + F\text{signe}(\dot{r})$ represents the restoring force including the dry friction term. $\ddot{e}$ is the random input of the system (i.e. the road surface displacement) which is supposed to be an stationnary process such that

$$\ddot{e} = -(1/\varepsilon)Z^\varepsilon$$

where $Z^\varepsilon(t) = Z(t/\varepsilon^2)$ with $b_3^{'}(z) \le -\mu < 0$ and $\underline{\sigma}^2 \le \sigma_3^2(z)$,$\forall z \in \mathbb{R}$.

Remarks We know that when ε goes towards to 0, $(1/\varepsilon)Z(t/\varepsilon^2)$ converges towards to the white noise ([4] and [11]). This white noise is the random input in the 2-dimensional problem ([6] and [7]).

Taking $u = v/m$, $\beta = K/m$, $\gamma = F/m$, the equation in $r(t)$ can be rewritten as the equation in $X(t)$.
The problem is to improve vehicle riding comfort by the choice of an adequate feedback v in order to maximize the confort, i.e. to minimize

$$J(u,t) = E\int_0^t |\ddot{r} - \ddot{e}|^2 ds.$$

The Hamilton–Jacobi–Bellman equation related to the control problem with finite horizon can be stated as

$$-\frac{\partial \phi^\varepsilon}{\partial t} + \inf_{u \in \mathcal{U}} \{\mathcal{L}_u \phi^\varepsilon + f(u)\} = 0$$

where f is the cost function and $\mathcal{L}_u$ is the infinitisimal generator of the diffusion process $X(t)$ defined by

$$\mathcal{L}_u \phi = \sum_{i=1}^{3} b_i(u)\frac{\partial \phi}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{3} a_{ij}\frac{\partial^2 \phi}{\partial x_i \partial x_j}.$$

3 The Solution of The Control Problem With Finite Horizon

We consider the control problem with finite horizon defined by

$$J(u,t) = E\left(\int_0^t f(X(s), u(X(s)))ds\right)$$

If we write

$$\phi^\varepsilon(t) = \inf_{u \in \mathcal{U}} J(u,t),$$

the equation of the control problem with finite horizon is defined by

$$-\frac{\partial \phi^\varepsilon}{\partial t} + \inf_{u \in \mathcal{U}} \left\{ y\frac{\partial \phi^\varepsilon}{\partial x} - (uy + \beta x + \gamma \text{signe}(y))\frac{\partial \phi^\varepsilon}{\partial y} - \frac{1}{\varepsilon} z \frac{\partial \phi^\varepsilon}{\partial y} + f(u) \right\} + \frac{1}{\varepsilon^2}\left\{ b(z)\frac{\partial \phi^\varepsilon}{\partial z} + \sigma^2(z)\frac{\partial^2 \phi^\varepsilon}{\partial z^2} \right\} = 0$$
$$\phi^\varepsilon(0) = 0$$

Then we look after expansions of the cost ϕ^ε and the feedback control u^ε in powers of ε. For that, we introduce the time scale $\tau = \frac{t}{\varepsilon^2}$.
And, we write the expressions of ϕ^ε and u^ε as

$$\begin{aligned} \phi^\varepsilon(t) &= \phi_0(t, \frac{t}{\varepsilon^2}) + \varepsilon \phi_1(t, \frac{t}{\varepsilon^2}) + \ldots \\ u^\varepsilon(t) &= u_0(t, \frac{t}{\varepsilon^2}) + \varepsilon u_1(t, \frac{t}{\varepsilon^2}) + \ldots \end{aligned}$$

We get the following equations

$$\begin{aligned} &- \frac{\partial}{\partial t}(\phi_0 + \ldots) - \frac{1}{\varepsilon^2}\frac{\partial}{\partial \tau}(\phi_0 + \ldots) + \left\{ y\frac{\partial}{\partial x}(\phi_0 + \ldots) \right. \\ &- ((u_0 + \ldots)y + \beta x + \gamma \text{signe}(y))\frac{\partial}{\partial y}(\phi_0 + \ldots) \\ &- \left. \frac{1}{\varepsilon} z \frac{\partial}{\partial y}(\phi_0 + \ldots) + f((u_0 + \ldots)) \right\} \\ &+ \frac{1}{\varepsilon^2}\left\{ b(z)\frac{\partial}{\partial z}(\phi_0 + \ldots) + \sigma^2(z)\frac{\partial^2}{\partial z^2}(\phi_0 + \ldots) \right\} = 0 \end{aligned}$$

and,

$$-y\frac{\partial}{\partial x}(\phi_0 + \ldots) - 2y\left(y(u_0 + \ldots) + \beta x + \gamma \text{signe}(y)\right) = 0$$

And, we introduce the following hamiltonians

$$\begin{aligned} H_0 &= y\frac{\partial \phi_0}{\partial x} - (uy + \beta x + \gamma \text{signe}(y))\frac{\partial \phi_0}{\partial y} - z\frac{\partial \phi_1}{\partial y} + f(u) \\ H_j &= y\frac{\partial \phi_j}{\partial x} - (uy + \beta x + \gamma \text{signe}(y))\frac{\partial \phi_j}{\partial y} - z\frac{\partial \phi_{j+1}}{\partial y} \end{aligned}$$

We note L the infinitesimal generator of $Z(t)$, defined by

$$L\phi = b(z)\frac{\partial \phi}{\partial z} + \sigma^2(z)\frac{\partial^2 \phi}{\partial z^2}$$

We define the following operators $D_u^j \phi = \dfrac{\partial^j \phi}{\partial u^j}$. Then we get the following equations

$$\begin{aligned} -\frac{\partial \phi_0}{\partial \tau} &+ Lu_0 = 0 \\ -\frac{\partial \phi_1}{\partial \tau} &+ L\phi_1 - z\frac{\partial \phi_0}{\partial y} = 0 \end{aligned}$$

and, for $l \geq 0$, we get

$$\begin{aligned}
- & \quad \frac{\partial \phi_{2(l+1)}}{\partial \tau} + L\phi_{2(l+1)} - \frac{\phi_{2l}}{\partial t} \\
+ & \quad \sum_{(l,\alpha)}^{2l} \frac{1}{\alpha_1! \ldots \alpha_l!} D_u^{(\alpha_1+\ldots+\alpha_l)} H_{\alpha_0}(u_0) u_1^{\alpha_1} \ldots u_l^{\alpha_l} = 0 \\
- & \quad \frac{\partial \phi_{2l+3}}{\partial \tau} + +L\phi_{2l+3} - \frac{\partial \phi_{2l+1}}{\partial t} \\
+ & \quad \sum_{(l,\alpha)}^{2l+1} \frac{1}{\alpha_1! \ldots \alpha_l!} D_u^{(\alpha_1+\ldots+\alpha_l)} H_{\alpha_0}(u_0) u_1^{\alpha_1} \ldots u_l^{\alpha_l} = 0
\end{aligned}$$

and,

$$\sum_{(l,\alpha)}^{l} \frac{1}{\alpha_1! \ldots \alpha_l!} D_u^{(1+\alpha_1+\ldots+\alpha_l)} H_{\alpha_0}(u_0) u_1^{\alpha_1} \ldots u_l^{\alpha_l} = 0$$

where we note

$$\sum_{(l,\alpha)}^{j} = \sum_{\alpha_0+\alpha_1+\ldots+l\alpha_l=j} \quad \text{with} \quad \alpha_0, \alpha_1, \ldots, \alpha_l \geq 0.$$

Then we get the following equations

$$\begin{aligned}
-\frac{\partial \phi_0}{\partial \tau} & + \quad L\phi_0 = 0 \\
-\frac{\partial \phi_1}{\partial \tau} & + \quad L\phi_1 - z\frac{\partial \phi_0}{\partial y} = 0 \\
-\frac{\partial \phi_2}{\partial \tau} & + \quad L\phi_2 - \frac{\partial \phi_0}{\partial t} - z\frac{\partial \phi_1}{\partial y} + y\frac{\partial \phi_0}{\partial x} \\
& - \quad (u_0 y + \beta x + \gamma \text{signe}(y))\frac{\partial \phi_0}{\partial y} \\
& + \quad (u_0 y + \beta x + \gamma \text{signe}(y))^2 = 0 \\
& \quad\quad -y\frac{\partial \phi_0}{\partial y} + 2y(u_0 y + \beta x + \gamma \text{signe}(y)) = 0 \\
-\frac{\partial \phi_3}{\partial \tau} & + \quad L\phi_3 - \frac{\partial \phi_1}{\partial t} - z\frac{\partial \phi_2}{\partial y} + y\frac{\partial \phi_1}{\partial x} \\
& - \quad (u_0 y + \beta x + \gamma \text{signe}(y))\frac{\partial \phi_1}{\partial y} = 0 \\
& \quad\quad -y\frac{\partial \phi_1}{\partial y} + 2y^2 u_1 = 0 \\
-\frac{\partial \phi_4}{\partial \tau} & + \quad L\phi_4 - \frac{\partial \phi_2}{\partial t} - z\frac{\partial \phi_3}{\partial y} + y\frac{\partial \phi_2}{\partial x} \\
& - \quad (u_0 y + \beta x + \gamma \text{signe}(y))\frac{\partial \phi_2}{\partial y} - y^2 u_1^2 = 0 \\
& \quad\quad -y\frac{\partial \phi_2}{\partial y} + 2y^2 u_2 = 0 \\
-\frac{\partial \phi_5}{\partial \tau} & + \quad L\phi_5 - \frac{\partial \phi_3}{\partial t} - z\frac{\partial \phi_4}{\partial y} + y\frac{\partial \phi_3}{\partial x}
\end{aligned}$$

$$\begin{aligned}
- & \quad (u_0 y + \beta x + \gamma \text{signe}(y))\frac{\partial \phi_3}{\partial y} - 2y^2 u_1 u_2 = 0 \\
& \quad -y\frac{\partial \phi_3}{\partial y} + 2y^2 u_3 = 0 \\
& \quad \vdots
\end{aligned}$$

4 Explicit Solutions of the Equations

We are going to give explicit solutions of the previous equations by introducing the invariant measure m (of the stationnary diffusion process associated with the infinitesimal generator L). First, we note

$$\phi_j(z) = \hat{w}_j + w_j(z) \text{ with } \int_{\mathbb{R}} w_j(z)m(z)dz = 0$$

And, we get equations for the quantities $\hat{w}_j$ and w_j respectively.
We introduce different first order operators depending or not on the quantity u.

$$\begin{aligned}
A\phi &= y\frac{\partial \phi}{\partial x} \\
B\phi &= z\frac{\partial \phi}{\partial y} \\
D(u)\phi &= (uy + \beta x + \gamma \text{signe}(y))\frac{\partial \phi}{\partial y}
\end{aligned}$$

By integration with respect of mdz, we get

$$\frac{\partial \hat{w}_0}{\partial \tau} = 0$$

So we get $\hat{w}_0(s,\tau) = \hat{w}_0(s)$, and by substraction, we get

$$-\frac{\partial w_0}{\partial \tau} + Lw_0 = 0$$

So we have $w_0(t,\tau,z) = 0$.
By integration with respect of mdz and as $\int_{\mathbb{R}} L(\phi_0)mdz = \int_{\mathbb{R}} \phi_0(L^* m)dz = 0$, we get

$$-\frac{\partial \hat{w}_1}{\partial \tau} - \int_{\mathbb{R}} B\phi_0 mdz = 0$$

As ϕ_0 is a function independant of z and Z is zero-mean process, we have $\int_{\mathbb{R}} B\phi_0 mdz = E(Z)\frac{\partial \phi_0}{\partial y} = 0$, we get $\hat{w}_1(t,\tau) = \hat{w}_1(t)$, and by substraction, we have

$$\begin{aligned}
-\frac{\partial w_1}{\partial \tau} + Lw_1 - B\hat{w}_0 &= 0 \\
w_1(t,0,z) &= 0
\end{aligned}$$

so we get

$$w_1(t,\tau,z) = -\int_0^\tau E_z(B(Z_r))dr\hat{w}_0(t).$$

By integration with respect of mdz, we get

$$\begin{aligned} -\frac{\partial \hat{w}_2}{\partial \tau} + \int_{\mathbb{R}} L\phi_2 mdz - \frac{\partial \hat{w}_0}{\partial t} + \int_{\mathbb{R}} A\phi_0 mdz & \\ -\int_{\mathbb{R}} B\phi_1 mdz - \int_{\mathbb{R}} D(u_0)\phi_0 mdz + \int_{\mathbb{R}} f(u_0) mdz &= 0 \end{aligned}$$

We have $\int_{\mathbb{R}} L(\phi_2)mdz = \int_{\mathbb{R}} \phi_2(L^* m)dz = 0$ and ϕ_0 and u_0 are independant of z; moreover, all quantities have limits as τ tends towards ∞, and,

$$\frac{\partial \hat{w}_2}{\partial \tau}(t,\infty) = 0$$

we have

$$\begin{aligned} -\frac{d\hat{w}_0}{dt} + [A - \int_0^\infty E(B(Z_s)B(Z_0))ds - D(u_0)]\hat{w}_0 & \\ +f(u_0) &= 0 \\ \hat{w}_0(0) &= 0 \end{aligned}$$

Remark This equation is an Hamilton–Bellman–Jacobi equation, and the control problem is defined by the diffusion process $\bar{X}(t)$ which the stochastic differential equation is

$$d\bar{X}(t) = b(\bar{X}(t), u(\bar{X}(t)))dt + \sqrt{2}\sigma(\bar{X}, u(\bar{X}(t)))dV(t)$$

with the drift and diffusion coefficients

$$b(X,u) = \begin{pmatrix} y \\ -(uy + \beta xy + \gamma \mathrm{signe}(y)) \end{pmatrix}$$

and,

$$\sigma(X,u) = \begin{pmatrix} 0 & 0 \\ 0 & \left(\int_0^\infty E(B(Z_s)B(Z_0))ds\right)^{1/2} \end{pmatrix}$$

We note $\bar{\mathcal{L}}_u$ the infinitesimal generator of $\bar{X}(t)$. So, we can write the Hamilton–Bellman–Jacobi equation as

$$\begin{aligned} -\frac{d\hat{w}}{dt} + \inf_{u \in \mathcal{U}} \{\bar{\mathcal{L}}_u \hat{w} + f(u)\} &= 0 \\ \hat{w}(0) &= 0 \end{aligned}$$

where the cost function is defined as previously. This problem is an 2-dimensional problem.

Now, by substraction, we get

$$\begin{aligned} \frac{\partial \hat{w}_2}{\partial \tau} + [\int_\tau^\infty E(B(Z_s)B(Z_0))ds]\hat{w}_0(t) &= 0 \\ \hat{w}_2(t,0) &= 0 \end{aligned}$$

and, we obtain

$$\hat{w}_2(t,\tau) = -[\int_0^\tau \int_r^\infty E(B(Z_s)B(Z_0))dsdr]\hat{w}_0(t)$$

or,

$$\begin{aligned} \frac{\partial w_2}{\partial \tau} - Lw_2 + B\hat{w}_1 + Bw_1 - \int_{\mathbb{R}} Bw_1 mdz &= 0 \\ w_2(t,0,z) &= 0 \end{aligned}$$

So, we get

$$\begin{aligned} w_2(t,\tau,z) = -\int_0^\tau E_z(B(Z_s))ds\hat{w}_1(t) & \\ +[\int_0^\tau \int_0^r (E_z - E)(B(Z_s)B(Z_r))dsdr]\hat{w}_0(t) & \end{aligned}$$

By similar arguments, we get

$$\begin{aligned} -\frac{\partial \hat{w}_3}{\partial \tau} + \int_{\mathbb{R}} L\phi_2 mdz - \frac{d\hat{w}_1}{dt} + \int_{\mathbb{R}} A\phi_1 mdz & \\ -\int_{\mathbb{R}} B\phi_2 mdz - \int_{\mathbb{R}} D(u_0)\phi_1 mdz &= 0 \end{aligned}$$

We put

$$\begin{aligned} -\frac{d\hat{w}_1}{dt} + [A - \int_0^\infty E(B(Z_s)B(Z_0))ds - D(u_0)]\hat{w}_1(t) & \\ -[\int_0^\infty \int_0^r E(B(Z_r)B(Z_s)B(Z_0))dsdr]\hat{w}_0(t) &= 0 \\ \hat{w}_1(0) &= 0 \end{aligned}$$

We finally get for the quantities $\hat{w}_3$ and w_3

$$\begin{aligned} \hat{w}_3(t,\tau) = -[\int_0^\tau \int_r^\infty E(B(Z_s)B(Z_0))dsdr]\hat{w}_1(t) & \\ +[\int_0^\tau \int_l^\infty \int_0^r E(B(Z_r)B(Z_s)B(Z_0))dsdrdl]\hat{w}_0(t) & \end{aligned}$$

and, to complete the expansion, we have the following equation

$$\begin{aligned} \frac{\partial w_3}{\partial \tau} - Lw_3 + B\hat{w}_2 + Bw_2 - \int_{\mathbb{R}} Bw_2 mdz & \\ +\frac{dw_1}{dt} - Aw_1 + D(u_0)w_1 &= 0 \\ w_3(t,0,z) &= 0 \end{aligned}$$

or, using the expressions of the different quantities w_j for $j \leq 2$,

$$\frac{\partial w_3}{\partial \tau} - Lw_3 = B(Z)[\int_0^\tau \int_r^\infty E(B(Z_r)B(Z_0))dsdr]\hat{w}_0(t)$$

$$
\begin{aligned}
&+[\int_0^\tau \int_0^r E(B(Z_s)B(Z_r)B(Z_0))dsdr]\hat{w}_0(t) \\
&-[\int_0^\tau (E-E_z)(B(Z_s)B(Z_0))ds]\hat{w}_1(t) \\
&+[\int_0^\tau \int_0^r (E-E_z)(B(Z_s)B(Z_r)B(Z_0))dsdr]\hat{w}_0(t) \\
&+[\int_0^\tau E_z(B(Z_s))ds]\frac{d\hat{w}_0}{dt}(t) + A[\int_0^\tau E_z(B(Z_s))ds]\hat{w}_0(t) \\
&-D(u_0)[\int_0^\tau E_z(B(Z_s))ds]\hat{w}_0(t) \\
&w_3(t,0,z) = 0
\end{aligned}
$$

We get the general following equations for $k \geq 4$,

$$
\begin{aligned}
-\frac{\partial \hat{w}_k}{\partial \tau} &= f_k \\
\hat{w}_k(0,\tau,z) &= 0
\end{aligned}
$$

and,

$$
\begin{aligned}
\frac{\partial w_k}{\partial \tau} - Lw_k &= g_k \\
w_k(t,0,z) &= 0
\end{aligned}
$$

where f_k and g_k are functions of $\hat{w}_0$ and $\hat{w}_1$.

Remark We can show that the all integrals are well defined ([5]).
Remark All equations are well defined on $\mathcal{O} = \mathbb{R}\backslash\{y=0\}$.

5 Convergence of the Approximation

In this section, we prove some results about the convergence of the approximation of the solution of the finite horizon control problem.
First, if ϕ^ε and u^ε denote the cost and the feedback control, solutions of the Hamilton–Jacobi–Bellman equation, we write

$$
\begin{aligned}
\phi^\varepsilon_{2k+1}(t) &= \sum_{j=0}^{j=2k+1} \varepsilon^j \phi_j(t, \frac{t}{\varepsilon^2}) \\
u^\varepsilon_{2k+1}(t) &= \sum_{j=0}^{j=k+1} \varepsilon^j u_j(t, \frac{t}{\varepsilon^2})
\end{aligned}
$$

where ϕ_j and u_j are solutions of the previous equations and defined in the section 3.

Proposition 5.1 *Under the previous assumptions, the following estimation*

$$
|\phi^\varepsilon_{2k+1}(t) - \phi^\varepsilon(t)| \leq \varepsilon^{2k+2} C(1+|x|^2+|y|^2+|z|^2)
$$

For that, we put for $j \geq 0$,

$$
\psi_j(u) = -\frac{\partial \phi_j}{\partial t} - \frac{\partial \phi_{j+2}}{\partial \tau} + H_j(u) + L\phi_{j+2}
$$

And we show the following propositions

Proposition 5.2 *Under the previous assumptions, the following estimation hold*

$$
\begin{aligned}
|\sum_{j=0}^{j=2k} \varepsilon^j \psi_j(u^\varepsilon_{2k+1})| &\leq \varepsilon^{2k+1} C(1+|x|^2+|y|^2+|z|^2) \\
|\sum_{j=0}^{j=2k} \varepsilon^j \partial H_j(u^\varepsilon_{2k+1})| &\leq \varepsilon^{k+1} C(1+|x|+|y|+|z|)
\end{aligned}
$$

To show that, we use the previous relations of the section 4 (See [9]).
If we note

$$
\bar{\phi}^\varepsilon_{\cdot}(v) = -\frac{\partial \phi^\varepsilon_{\cdot}}{\partial t} + H^\varepsilon(v) + \frac{1}{\varepsilon^2} L\phi^\varepsilon_{\cdot}
$$

then, we have

Proposition 5.3 *Under the same assumptions, the inequality holds*

$$
\bar{\phi}^\varepsilon_{2k+1}(v) \geq -\varepsilon^{2k} C(1+|x|^2+|y|^2+|z|^2)
$$

where v is a control bounded and fixed.

Proof: We choose a class of controls bounded v. Let be v one of these controls. There exists w such that

$$
\begin{aligned}
v(t) &= \sum_{j=0}^{j=k+1} \varepsilon^j u_j(t, \frac{t}{\varepsilon^2}) + w(t) \\
&= u^\varepsilon_{2k+1}(t) + w(t)
\end{aligned}
$$

We are going to give an approximation

$$
\begin{aligned}
\bar{\phi}^\varepsilon_{2k+1}(u^\varepsilon_{2k+1} + w) &= \sum_{j=0}^{j=2k} \varepsilon^j \psi_j(u^\varepsilon_{2k+1} + w) \\
+\varepsilon^{2k+1} &\left(-\frac{\partial \phi_{2k+1}}{\partial t} + A\phi_{2k+1} - D(u^\varepsilon_{2k+1} + w)\phi_{2k+1} \right) \\
&+\varepsilon^{2k} \left(\frac{\partial \phi_{2k+2}}{\partial \tau} - L\phi_{2k+2} \right)
\end{aligned}
$$

and,

$$
\begin{aligned}
H_j(u^\varepsilon_{2k+1} + w) &= H_j(u^\varepsilon_{2k+1}) + \partial H_j(u^\varepsilon_{2k+1})w \\
&+ \int_0^1 \int_0^1 \lambda \partial^2 H_j(u^\varepsilon_{2k+1} + \lambda\mu w) w^2 d\lambda d\mu
\end{aligned}
$$

So, after some calculations, we get the following approximation

$$\bar{\phi}^{\varepsilon}_{2k+1}(v) \geq -\varepsilon^{2k} C(1 + |x|^2 + |y|^2 + |z|^2 + |v(t)|^2)$$

So, we get the desired result.
Using the Ito formula, we get the following result

$$\begin{aligned}\phi^{\varepsilon}_{2k+1}(0) - \phi^{\varepsilon}_{2k+1}(t) \geq -\varepsilon^{2k} C(1 + |x|^2 + |y|^2 + |z|^2) \\ -E \int_0^t f(X(s), v(s)) ds\end{aligned}$$

And, as we have $\phi^{\varepsilon}_{2k+1}(0) = 0$, and we use $v = u^{\varepsilon}$, we get the following result

$$\phi^{\varepsilon}_{2k+1}(t) - \phi^{\varepsilon}(t) \leq \varepsilon^{2k} C(1 + |x|^2 + |y|^2 + |z|^2)$$

Moreover, we know that

$$\bar{\phi}^{\varepsilon}(u^{\varepsilon}_{2k+1}) \geq 0$$

and,

$$\bar{\phi}^{\varepsilon}_{2k+1}(u^{\varepsilon}_{2k+1}) \leq \varepsilon^{2k} C(1 + |x|^2 + |y|^2 + |z|^2)$$

and, by substraction, and using the Ito formula, we find

$$\phi^{\varepsilon}_{2k+1}(t) - \phi^{\varepsilon}(t) \geq -\varepsilon^{2k} C(1 + |x|^2 + |y|^2 + |z|^2)$$

And, we get the desired result.

6 The Extension

In the section 4, we obtain an asymptotic expansion of the solution of the control problem with finite horizon. Using the assumption about Z^{ε}, we shall show the ergodicity of this problem ([8]).
Using results and considerations from [1], we are going to find the asymptotic expansion of the ergodic control problem ϕ^{ε} called *the asymptotic ergodic expansion.*

7 Remark

In [6], we solved numerically the 3-dimensional ergodic problem applied to the car suspension design by introducing an Ornstein-Ulhenbeck process as the random input of the system described in section 2.

References

[1] M. Airisawa *Ergodic Problem for the Hamilton–Jacobi–Bellmann Equation.* Annales de l'Institut Henri Poincaré, 14, 4, 1997

[2] A. Bensoussan *On Some Singular Perturbations Problems Arising in Stochastic Control.* Stochastic Analysis and Applications, 2, 13-53, 1984

[3] A. Bensoussan - J.L. Lions *Applications des Inéquations Variationnelles en Contrôle Stochastique.* Dunod, Studies in Mathematics and Its Applications, 1978

[4] R.N. Bhattacharya *On Functional Central Limit theorem and the Law of the Iterated Logarithm for Markov Processes* Z. Wahrscheinlichkeitstherie, 60, 185-201, 1982

[5] R.Bouc - E. Pardoux *Asymptotic Analysis of P.D.E.s with Wide-band Noise Disturbances, and Expansion of the Moments.* Stochastic Analysis and Applications, 2, 369-422, 1984

[6] F. Campillo - A. Gon - C. Marchetti - E. Pardoux *Application du Contrôle Stochastique au Problème de Suspension de Véhicule.* Rapport de Fin d'Etude, Regienov, Contrat H6.10.601/INRIA/17, 1987

[7] F. Campillo - F. Legland - E. Pardoux *Approximation d'un Problème de Contrôle Ergodique Dégénéré.* Colloque International Automatique Non Linéaire, CNRS, Nantes, 1988

[8] C. Marchetti *Asymptotic Expansion in the 3-Dimensional Ergodic Control Problem* To be submitted in Stochastoc Analysis and Applications, 2000

[9] S. Peng *Etude de Perturbations et d'Homogénéisation des Systèmes Stochastiques et des Systèmes Périodiques.* Thèse de Doctorat , Université de Marseille , 1986

[10] S. Peng *Analyse Asymptotique et Problème Homogénéisé en Contrôle Optimal avec Vibrations Rapides* Siam J. Control and Optimization, 27, 673-696, 1989

[11] A. Touati *Théorèmes de Limite Centrale Fonctionnels pour les Processus de Markov* Annales de l'Institut Henri Poincaré, vol XIX, 1, 43-55, 1983