

# Identification of Relative Position and Orientation of Two Cameras from Motion and Shape Parameters of Moving Rigid Body

Hiroyuki Kano<sup>†</sup> and Bijoy K. Ghosh<sup>††</sup>

<sup>†</sup> Department of Information Sciences, Tokyo Denki University, Hatoyama, Hiki-Gun, Saitama 350-0394, Japan. E-mail: kano@j.dendai.ac.jp

<sup>††</sup> Department of Systems Science and Mathematics, Washington University, St. Louis, Missouri 63130, USA. E-mail: ghosh@zach.wustl.edu

## Abstract

In this paper we obtain the *calibration parameters* between a pair of cameras from motion and shape cues obtained from a moving rigid body in space. We consider a body moving rigidly in space and assume that they are observed using a pair of cameras. We assume furthermore that the relative positions of the two cameras are unknown i.e. we are in an *uncalibrated stereo* situation. It is then proved that, under certain generic conditions, relative orientation can be determined uniquely, whereas the relative position can be recovered only upto a one parameter ambiguity. The general problem is to be able to estimate the motion and shape parameters from the environment inspite of the lack of calibration between the two cameras.

## 1 Introduction

In this paper we consider the so called *uncalibrated stereo problem* wherein the problem is to ascertain the shape and motion parameters of rigid objects moving in space using a pair of cameras. We assume that the relative position of the two cameras are unknown, hence any technique that utilizes the standard stereo correspondence between features in each of the two cameras, would fail.

The problem of estimating positions and velocities of moving features in space, leading to the estimation of motion and shape parameters (e.g., [1], [4], [5]) is an important problem in *robotics* and *machine vision* [6], [7], [8]. A typical approach is to use a pair of cameras to locate the precise position of features in space, a method that assumes that the cameras are calibrated. If the cameras are not calibrated, one would hope to calibrate the cameras from additional features in the environment. Camera calibration is therefore an intrinsic step in the standard stereo approach to many problems in machine vision [3].

The *uncalibrated stereo problem* has been introduced in [2] wherein it is assumed that each of the two cameras gather and process information independant of each other in its own coordinate frames. The relative transformation between the two camera frames is assumed to be known and is used at the final step. It is shown in [2] that in principle, the motion and shape parameters are uniquely identifiable. The problem wherein the calibration parameters are unknown is the subject matter of this paper.

## 2 Rigid Body Motion and Calibration Parameters

Let  $(X_1 \ Y_1 \ Z_1)$  be a coordinate frame fixed with a camera, and consider a rigid motion dynamics described as follows,

$$\frac{d}{dt} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = \Omega \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} + b, \quad (1)$$

where

$$\Omega = \begin{pmatrix} 0 & \omega_1 & \omega_2 \\ -\omega_1 & 0 & \omega_3 \\ -\omega_2 & -\omega_3 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad (2)$$

and we assume that  $(\omega_1 \ \omega_2 \ \omega_3) \neq 0$ . The parameters  $(\omega_1, \omega_2, \omega_3, b_1, b_2, b_3)$  are called motion parameters. Writing the vector  $(X_1 \ Y_1 \ Z_1)$  in terms of homogeneous coordinates  $(\bar{X}_1 \ \bar{Y}_1 \ \bar{Z}_1 \ \bar{W}_1)$  as

$$X_1 = \frac{\bar{X}_1}{\bar{W}_1}, \quad Y_1 = \frac{\bar{Y}_1}{\bar{W}_1}, \quad Z_1 = \frac{\bar{Z}_1}{\bar{W}_1} \quad (3)$$

we obtain

$$\frac{d}{dt} \begin{pmatrix} \bar{X}_1 \\ \bar{Y}_1 \\ \bar{Z}_1 \\ \bar{W}_1 \end{pmatrix} = \begin{pmatrix} \Omega & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{X}_1 \\ \bar{Y}_1 \\ \bar{Z}_1 \\ \bar{W}_1 \end{pmatrix} \quad (4)$$

Moreover, we consider a plane in  $\mathbf{R}^3$  characterized by the so-called shape parameters  $(p_1, q_1, r_1)$  as

$$p_1 X_1 + q_1 Y_1 + r_1 = Z_1 \quad (5)$$

Introducing the following homogeneous coordinates  $(\bar{p}_1 \bar{q}_1 \bar{r}_1 \bar{s}_1)$  for  $(p_1 q_1 r_1)$ ,

$$p_1 = \frac{\bar{p}_1}{-\bar{s}_1}, q_1 = \frac{\bar{q}_1}{-\bar{s}_1}, r_1 = \frac{\bar{r}_1}{-\bar{s}_1}, \quad (6)$$

the plane (5) is written as

$$\bar{p}_1 \bar{X}_1 + \bar{q}_1 \bar{Y}_1 + \bar{s}_1 \bar{Z}_1 + \bar{r}_1 \bar{W}_1 = 0 \quad (7)$$

Differentiating (7) with respect to time  $t$  and using (4), we obtain the following shape dynamics,

$$\frac{d}{dt} \begin{pmatrix} \bar{p}_1 \\ \bar{q}_1 \\ \bar{s}_1 \\ \bar{r}_1 \end{pmatrix} = \begin{pmatrix} \Omega & 0 \\ -b^T & 0 \end{pmatrix} \begin{pmatrix} \bar{p}_1 \\ \bar{q}_1 \\ \bar{s}_1 \\ \bar{r}_1 \end{pmatrix} \quad (8)$$

Next we consider a second camera with a coordinate frame  $(X_2 Y_2 Z_2)$ , where we assume that

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = R \begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} + \delta \quad (9)$$

Here, a rotation matrix  $R$  and a translation vector  $\delta \in \mathbf{R}^3$  are calibration parameters that are assumed to be unknown.

Let the motion in (1) be written in the coordinate  $(X_2 Y_2 Z_2)$  as

$$\frac{d}{dt} \begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} = \Omega' \begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} + b' \quad (10)$$

where

$$\Omega' = \begin{pmatrix} 0 & \omega'_1 & \omega'_2 \\ -\omega'_1 & 0 & \omega'_3 \\ -\omega'_2 & -\omega'_3 & 0 \end{pmatrix}, \quad b' = \begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix}, \quad (11)$$

and the plane in (5) be

$$p_2 X_2 + q_2 Y_2 + r_2 Z_2 = 0 \quad (12)$$

Now, in terms of homogeneous coordinates  $(\bar{X}_1 \bar{Y}_1 \bar{Z}_1 \bar{W}_1)$  for  $(X_1 Y_1 Z_1)$ , and  $(\bar{X}_2 \bar{Y}_2 \bar{Z}_2 \bar{W}_2)$  for  $(X_2 Y_2 Z_2)$ , rewrite (9) as

$$\begin{pmatrix} \bar{X}_1 \\ \bar{Y}_1 \\ \bar{Z}_1 \\ \bar{W}_1 \end{pmatrix} = \begin{pmatrix} R & \delta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{X}_2 \\ \bar{Y}_2 \\ \bar{Z}_2 \\ \bar{W}_2 \end{pmatrix} \quad (13)$$

and, accordingly, (10) as

$$\frac{d}{dt} \begin{pmatrix} \bar{X}_2 \\ \bar{Y}_2 \\ \bar{Z}_2 \\ \bar{W}_2 \end{pmatrix} = \begin{pmatrix} \Omega' & b' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{X}_2 \\ \bar{Y}_2 \\ \bar{Z}_2 \\ \bar{W}_2 \end{pmatrix} \quad (14)$$

Then, by (13) and (14), we have

$$\frac{d}{dt} \begin{pmatrix} \bar{X}_1 \\ \bar{Y}_1 \\ \bar{Z}_1 \\ \bar{W}_1 \end{pmatrix} = \begin{pmatrix} R\Omega'R^T & Rb' - R\Omega'R^T\delta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{X}_1 \\ \bar{Y}_1 \\ \bar{Z}_1 \\ \bar{W}_1 \end{pmatrix} \quad (15)$$

Comparing (15) with (4) yields

$$\Omega = R\Omega'R^T \quad (16)$$

$$b = Rb' - R\Omega'R^T\delta \quad (17)$$

Moreover, with homogeneous coordinates  $(\bar{p}_2 \bar{q}_2 \bar{r}_2 \bar{s}_2)$  for  $(p_2 q_2 r_2)$ ,

$$p_2 = \frac{\bar{p}_2}{-\bar{s}_2}, q_2 = \frac{\bar{q}_2}{-\bar{s}_2}, r_2 = \frac{\bar{r}_2}{-\bar{s}_2}, \quad (18)$$

the plane (12) is written as

$$\bar{p}_2 \bar{X}_2 + \bar{q}_2 \bar{Y}_2 + \bar{s}_2 \bar{Z}_2 + \bar{r}_2 \bar{W}_2 = 0. \quad (19)$$

Then by (7), (13) and (19), we have

$$\begin{pmatrix} \bar{p}_2 \\ \bar{q}_2 \\ \bar{s}_2 \\ \bar{r}_2 \end{pmatrix} = \begin{pmatrix} R^T & 0 \\ \delta^T & 1 \end{pmatrix} \begin{pmatrix} \bar{p}_1 \\ \bar{q}_1 \\ \bar{s}_1 \\ \bar{r}_1 \end{pmatrix}, \quad (20)$$

or in terms of the original coordinates  $(p_1 q_1 r_1)$  and  $(p_2 q_2 r_2)$ ,

$$-\bar{s}_2 \begin{pmatrix} p_2 \\ q_2 \\ -1 \\ r_2 \end{pmatrix} = -\bar{s}_1 \begin{pmatrix} R^T & 0 \\ \delta^T & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \\ -1 \\ r_1 \end{pmatrix} \quad (21)$$

Thus with  $\gamma = \bar{s}_2/\bar{s}_1$ , we get

$$R^T \begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix} = \gamma \begin{pmatrix} p_2 \\ q_2 \\ -1 \end{pmatrix} \quad (22)$$

$$\delta^T \begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix} + r_1 = \gamma r_2 \quad (23)$$

Here, since  $R$  is an orthogonal matrix, it must hold in (22) that

$$\left\| \begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix} \right\| = |\gamma| \cdot \left\| \begin{pmatrix} p_2 \\ q_2 \\ -1 \end{pmatrix} \right\| \quad (24)$$

with  $\|\cdot\|$  denoting vector Euclidean norm. Hence we get

$$\gamma = \pm\gamma_0 \quad (25)$$

where  $\gamma_0$  is defined by

$$\gamma_0 = \sqrt{\frac{p_1^2 + q_1^2 + 1}{p_2^2 + q_2^2 + 1}} \quad (26)$$

In conclusion, the rotation matrix  $R$  and the translation vector  $\delta$  in (9) satisfy (16), (17), (22) and (23).

### 3 Identification of Calibration Parameters

Let us assume that a set of feature points on the rigid body are observed by each camera via perspective projections. It is then known [1] that, among the motion and shape parameters  $(\Omega, p_1, q_1, r_1, b)$  and  $(\Omega', p_2, q_2, r_2, b')$ , the parameters or functions of parameters that are identifiable from the image data are

$$(\Omega, p_1, q_1, c) \text{ and } (\Omega', p_2, q_2, c') \quad (27)$$

where  $c$  and  $c'$  are 3-dimensional vectors defined by

$$c = \frac{1}{r_1}b, \quad c' = \frac{1}{r_2}b' \quad (28)$$

In the sequel, we consider to identify the rotation matrix  $R$  and the translation vector  $\delta$  in (9) assuming that the parameters in (27) are known.

#### 3.1 Rotation Matrix $R$

The matrix  $R$  is obtained as a rotation matrix satisfying (16) and (22). First we consider (16), i.e.

$$\Omega R - R\Omega' = 0 \quad (29)$$

Here it is noted that  $\Omega$  and  $\Omega'$  are similar and that the eigenvalues are  $\{\pm\omega, 0\}$  with

$$\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{\omega'^2_1 + \omega'^2_2 + \omega'^2_3} (\neq 0) \quad (30)$$

Now let  $P$  and  $P'$  be orthogonal matrices such that

$$\Omega = P\Lambda P^T, \quad |P| = 1 \quad (31)$$

$$\Omega' = P'\Lambda P'^T, \quad |P'| = 1 \quad (32)$$

where

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \quad (33)$$

Also let the columns of  $P$  and  $P'$  be denoted as  $u_i$  and  $u'_i$ ,  $i = 1, 2, 3$ , i.e.

$$P = (u_1 \ u_2 \ u_3), \quad P' = (u'_1 \ u'_2 \ u'_3) \quad (34)$$

with, in particular,

$$u_3 = \frac{1}{\omega} \begin{pmatrix} \omega_3 \\ -\omega_2 \\ \omega_1 \end{pmatrix}, \quad u'_3 = \frac{1}{\omega} \begin{pmatrix} \omega'_3 \\ -\omega'_2 \\ \omega'_1 \end{pmatrix} \quad (35)$$

Substituting (31) and (32) into (29) yields

$$\Lambda Q - Q\Lambda = 0 \quad (36)$$

where

$$Q = P^T R P' \quad (37)$$

is also a rotation matrix, i.e.  $QQ^T = I$  and  $|Q| = 1$ . Using (33) in (36), it is easily seen that  $Q$  satisfying (36) is given in the form of

$$Q = \begin{pmatrix} R_2(\theta) & 0 \\ 0 & 1 \end{pmatrix} \quad (38)$$

where  $R_2(\theta)$  is any 2-dimensional rotation matrix parametrized by  $\theta$ , i.e.

$$R_2(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (39)$$

Thus, the matrix  $R$  satisfying (29) is given by

$$R = P Q P'^T \quad (40)$$

with  $Q$  in (38).

In order to determine  $R_2(\theta)$  in (38), we now consider (22). Substituting (40) into (22) yields

$$Q^T \begin{pmatrix} \hat{p}_1 \\ \hat{q}_1 \\ \hat{s}_1 \end{pmatrix} = \gamma \begin{pmatrix} \hat{p}_2 \\ \hat{q}_2 \\ \hat{s}_2 \end{pmatrix} \quad (41)$$

where

$$\begin{pmatrix} \hat{p}_1 \\ \hat{q}_1 \\ \hat{s}_1 \end{pmatrix} = P^T \begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} \hat{p}_2 \\ \hat{q}_2 \\ \hat{s}_2 \end{pmatrix} = P'^T \begin{pmatrix} p_2 \\ q_2 \\ -1 \end{pmatrix} \quad (42)$$

Then (41) is written as

$$R_2^T(\theta) \begin{pmatrix} \hat{p}_1 \\ \hat{q}_1 \end{pmatrix} = \gamma \begin{pmatrix} \hat{p}_2 \\ \hat{q}_2 \end{pmatrix} \quad (43)$$

$$\hat{s}_1 = \gamma \hat{s}_2 \quad (44)$$

Here, by (42), (34) and (35),  $\hat{s}_1$  and  $\hat{s}_2$  are given by

$$\hat{s}_1 = u_3^T \begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix} = \frac{1}{\omega} (\omega_3 p_1 - \omega_2 q_1 - \omega_1) \quad (45)$$

$$\hat{s}_2 = u_3'^T \begin{pmatrix} p_2 \\ q_2 \\ -1 \end{pmatrix} = \frac{1}{\omega} (\omega'_3 p_2 - \omega'_2 q_2 - \omega'_1) \quad (46)$$

Thus, the matrix  $R_2(\theta)$  is obtained by solving (43), and we get the rotation matrix  $R$  using (40) and (38).

In the above, the choices of orthogonal matrices  $P$  and  $P'$  in (31) and (32) are immaterial as shown in the following lemma.

**Lemma 1** *The rotation matrix  $R$  obtained by (40) is independent of the choices of orthogonal matrices  $P$  in (31) and  $P'$  in (32).*

(Proof) First notice that  $\bar{P}$  and  $\bar{P}'$ , an another set of orthogonal matrices, are related to  $P$  and  $P'$  as

$$\bar{P} = P S, \quad \bar{P}' = P' S' \quad (47)$$

respectively, where

$$S = \begin{pmatrix} R_2(\phi) & 0 \\ 0 & 1 \end{pmatrix}, \quad S' = \begin{pmatrix} R_2(\psi) & 0 \\ 0 & 1 \end{pmatrix} \quad (48)$$

Here  $R_2(\phi)$  and  $R_2(\psi)$  are defined as in (39).

Now let  $\bar{R}$  be a rotation matrix computed similarly as (40), i.e.,

$$\bar{R} = \bar{P}\bar{Q}\bar{P}'^T = PS\bar{Q}S'^T P'^T \quad (49)$$

where  $\bar{Q}$  is defined by

$$\bar{Q} = \begin{pmatrix} R_2(\bar{\theta}) & 0 \\ 0 & 1 \end{pmatrix} \quad (50)$$

In this case, the counterparts of (43) and (44) are written as

$$R_2^T(\bar{\theta}) \begin{pmatrix} \bar{p}_1 \\ \bar{q}_1 \end{pmatrix} = \gamma \begin{pmatrix} \bar{p}_2 \\ \bar{q}_2 \end{pmatrix} \quad (51)$$

$$\bar{s}_1 = \gamma \bar{s}_2 \quad (52)$$

where

$$\begin{pmatrix} \bar{p}_1 \\ \bar{q}_1 \\ \bar{s}_1 \end{pmatrix} = \bar{P}^T \begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} \bar{p}_2 \\ \bar{q}_2 \\ \bar{s}_2 \end{pmatrix} = \bar{P}'^T \begin{pmatrix} p_2 \\ q_2 \\ -1 \end{pmatrix} \quad (53)$$

Substituting (47) and (48) into (53), and using (42), we see that  $\bar{s}_1 = \hat{s}_1$ ,  $\bar{s}_2 = \hat{s}_2$ , and

$$\begin{pmatrix} \bar{p}_1 \\ \bar{q}_1 \end{pmatrix} = R_2^T(\phi) \begin{pmatrix} \hat{p}_1 \\ \hat{q}_1 \end{pmatrix}, \quad \begin{pmatrix} \bar{p}_2 \\ \bar{q}_2 \end{pmatrix} = R_2^T(\psi) \begin{pmatrix} \hat{p}_2 \\ \hat{q}_2 \end{pmatrix} \quad (54)$$

hold. Thus, (52) coincides with (44), and by (51), (54) and (43), we obtain

$$R_2(\theta) = R_2(\phi)R_2(\bar{\theta})R_2^T(\psi) \quad (55)$$

In (49), it then holds that

$$S\bar{Q}S'^T = \begin{pmatrix} R_2(\theta) & 0 \\ 0 & 1 \end{pmatrix} = Q \quad (56)$$

Thus  $\bar{R} = PQP'^T = R$ . (Q.E.D.)

Regarding the parameter  $\gamma$  in (43) and (44), we prepare the following lemma.

**Lemma 2** *The following conditions (i)–(iv) are all equivalent: (i)  $(p_1 \ q_1 \ -1)^T \in \text{Im } \Omega$ , (ii)  $(p_2 \ q_2 \ -1)^T \in \text{Im } \Omega'$ , (iii)  $\omega_3 p_1 - \omega_2 q_1 - \omega_1 = 0$ , and (iv)  $\omega'_3 p_2 - \omega'_2 q_2 - \omega'_1 = 0$ .*

(Proof) It is easily seen that  $\text{Im } \Omega$  is a plane in  $\mathbb{R}^3$  given as

$$\text{Im } \Omega = \{x \in \mathbb{R}^3 \mid (\omega_3 \ -\omega_2 \ \omega_1)x = 0\} \quad (57)$$

which readily yields (i)  $\Leftrightarrow$  (iii), and similarly (ii)  $\Leftrightarrow$  (iv). Thus it suffices to show that (i)  $\Leftrightarrow$  (ii). For simplicity, let  $x_1 = (p_1 \ q_1 \ -1)^T$  and  $x_2 = (p_2 \ q_2 \ -1)^T$ , and we show that (i)  $\Rightarrow$  (ii). Let  $x_1 \in \text{Im } \Omega$  and  $x_1 = \Omega y_1$  for some  $y_1 \in \mathbb{R}^3$ . Then, using (22) and (29), we obtain

$$\gamma x_2 = R^T \Omega y_1 = \Omega' R^T y_1$$

implying  $x_2 \in \text{Im } \Omega'$  since  $\gamma \neq 0$ . The converse follows similarly. (Q.E.D.)

Then, if the condition  $(p_2 \ q_2 \ -1)^T \notin \text{Im } \Omega'$ , or equivalently  $(p_1 \ q_1 \ -1)^T \notin \text{Im } \Omega$  holds, Lemma 2 implies  $\hat{s}_2 \neq 0$  (and  $\hat{s}_1 \neq 0$ ) in (44), and  $\gamma$  is determined uniquely as

$$\gamma = \frac{\hat{s}_1}{\hat{s}_2} = \frac{\omega_3 p_1 - \omega_2 q_1 - \omega_1}{\omega'_3 p_2 - \omega'_2 q_2 - \omega'_1} \quad (58)$$

It is noted that the expression (58) for  $\gamma$  is then consistent with (25) in the sense that

$$\left| \frac{\omega_3 p_1 - \omega_2 q_1 - \omega_1}{\omega'_3 p_2 - \omega'_2 q_2 - \omega'_1} \right| = \sqrt{\frac{p_1^2 + q_1^2 + 1}{p_2^2 + q_2^2 + 1}} \quad (59)$$

holds, which can be shown as follows. Using (40), (35) and (38), we get

$$R^T \begin{pmatrix} \omega_3 \\ -\omega_2 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} \omega'_3 \\ -\omega'_2 \\ \omega'_1 \end{pmatrix} \quad (60)$$

Now, for simplicity, let  $x_1 = (p_1 \ q_1 \ -1)^T$ ,  $x_2 = (p_2 \ q_2 \ -1)^T$ ,  $y_1 = (\omega_3 \ -\omega_2 \ \omega_1)^T$  and  $y_2 = (\omega'_3 \ -\omega'_2 \ \omega'_1)^T$ . Then (22) and (60) are written as

$$R^T x_1 = \gamma x_2, \quad R^T y_1 = y_2 \quad (61)$$

Letting  $\beta_1$  be the angle between the vectors  $x_1$  and  $y_1$ , it holds that

$$x_1^T y_1 = \|x_1\| \cdot \|y_1\| \cos \beta_1 \quad (62)$$

and similarly,  $(\gamma x_2)^T y_2 = \|\gamma x_2\| \cdot \|y_2\| \cos \beta_2$ , or

$$x_2^T y_2 = \frac{1}{\gamma} |\gamma| \cdot \|x_2\| \cdot \|y_2\| \cos \beta_2 \quad (63)$$

where  $\beta_2$  is the angle between  $\gamma x_2$  and  $y_2$ . Here, (61) shows that rotating the vectors  $x_1$  and  $y_1$  by the rotation matrix  $R$  results in the vectors  $\gamma x_2$  and  $y_2$ , respectively, implying  $\beta_1 = \beta_2$ . Noting that  $\|y_1\| = \|y_2\| (= \omega)$ , we then have

$$\left| \frac{\hat{s}_1}{\hat{s}_2} \right| = \left| \frac{\omega x_1^T y_1}{\omega x_2^T y_2} \right| = \frac{\|x_1\|}{\|x_2\|} = \gamma_0 \quad (64)$$

yielding (59).

We then have the following theorem, where  $(\text{Im } \Omega)^\perp$ , the complementary orthogonal subspace of  $\text{Im } \Omega$ , is given by

$$(\text{Im } \Omega)^\perp = \{x \in \mathbb{R}^3 \mid x = \alpha(\omega_3 \ -\omega_2 \ \omega_1)^T, \alpha \in \mathbb{R}\} \quad (65)$$

**Theorem 1** Assume that  $(\Omega, p_1, q_1, c)$  and  $(\Omega', p_2, q_2, c')$  are given. Then, generically, i.e. under the conditions

$$\begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix} \notin \text{Im } \Omega \quad \text{and} \quad \begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix} \notin (\text{Im } \Omega)^\perp, \quad (66)$$

the rotation matrix  $R$  is determined uniquely.

(Proof) Notice that, under the assumption of  $(p_1 \ q_1 \ -1)^T \notin \text{Im } \Omega$ , the parameter  $\gamma$  is obtained uniquely by (58). Moreover, in (43), the matrix  $R_2(\theta)$  or the parameter  $\theta$  is determined uniquely if it holds that

$$\left\| \begin{pmatrix} \hat{p}_1 \\ \hat{q}_1 \end{pmatrix} \right\| = \left\| \gamma \begin{pmatrix} \hat{p}_2 \\ \hat{q}_2 \end{pmatrix} \right\| \quad (67)$$

and  $(\hat{p}_1 \ \hat{q}_1) \neq (0 \ 0)$ . Then by Lemma 1, we obtain a unique  $R$ .

As for (67), using (24) and (44), it holds that

$$\left\| \begin{pmatrix} \hat{p}_1 \\ \hat{q}_1 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} \hat{p}_1 \\ \hat{q}_1 \\ \hat{s}_1 \end{pmatrix} \right\|^2 - \hat{s}_1^2 = \gamma^2 \left\| \begin{pmatrix} \hat{p}_2 \\ \hat{q}_2 \\ \hat{s}_2 \end{pmatrix} \right\|^2 - \gamma^2 \hat{s}_2^2 \quad (68)$$

from which (67) results.

Next we consider the condition  $(\hat{p}_1 \ \hat{q}_1) \neq (0 \ 0)$ . By definition of  $\hat{p}_1, \hat{q}_1$  in (42),  $(\hat{p}_1 \ \hat{q}_1) = (0 \ 0)$  is written as

$$u_1^T \begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix} = 0, \quad u_2^T \begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix} = 0 \quad (69)$$

where  $u_1$  and  $u_2$  are defined in (34). Thus

$$\begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix} = \alpha u_3 = \frac{\alpha}{\omega} \begin{pmatrix} \omega_3 \\ -\omega_2 \\ \omega_1 \end{pmatrix}$$

for nonzero scalar  $\alpha$ , and  $(p_1 \ q_1 \ -1)^T \in (\text{Im } \Omega)^\perp$  by (65), contradicting the assumption in the theorem. Thus  $(\hat{p}_1 \ \hat{q}_1) \neq (0 \ 0)$ . (Q.E.D.)

### 3.2 Translation Vector $\delta$

The translation vector  $\delta$  in (9) is determined so that (17) and (23), or

$$\Omega \delta = Rb' - b \quad (70)$$

$$(p_1 \ q_1 \ -1) \delta = \gamma r_2 - r_1 \quad (71)$$

are satisfied. Here note that, by each camera, we can identify  $b, r_1$  and  $b', r_2$  only in terms of  $c$  and  $c'$  in (28), and the following result holds.

**Lemma 3** Given  $\Omega, R, p_1, q_1, c$  and  $c'$ , the possible extent to which the vector  $\delta$  is identifiable by (70) and (71) is up to its scalar multiple.

(Proof) For nonzero scalar  $\alpha$ , (70) and (71) are written as

$$\Omega(\alpha\delta) = R\hat{b}' - \hat{b}, \quad (p_1 \ q_1 \ -1)(\alpha\delta) = \gamma\hat{r}_2 - \hat{r}_1$$

with  $\hat{b}' = \alpha b', \hat{b} = \alpha b, \hat{r}_2 = \alpha r_2$  and  $\hat{r}_1 = \alpha r_1$ . But, in this case, we have

$$\hat{c}' \triangleq \frac{1}{\hat{r}_2} \hat{b}' = \frac{1}{r_2} b' = c', \quad \hat{c} \triangleq \frac{1}{\hat{r}_1} \hat{b} = \frac{1}{r_1} b = c$$

implying that if  $\delta$  is a solution of (70) and (71), then so is the vector  $\alpha\delta$  for any nonzero value of  $\alpha$ . (Q.E.D.)

Now premultiplying  $c$  to (71) and using (28) yield

$$c(p_1 \ q_1 \ -1)\delta = \gamma r_2 c - b$$

which can be manipulated as

$$[\Omega - c(p_1 \ q_1 \ -1)]\delta_0 = Rc' - \gamma c \quad (72)$$

where

$$\delta_0 \triangleq \frac{1}{r_2} \delta \quad (73)$$

Thus, if

$$|\Omega - c(p_1 \ q_1 \ -1)| \neq 0 \quad (74)$$

then  $\delta_0$  is determined uniquely as

$$\delta_0 = [\Omega - c(p_1 \ q_1 \ -1)]^{-1}(Rc' - \gamma c) \quad (75)$$

Since  $\delta$  can be determined up to a scalar multiple, let us assume that  $\|\delta\|$ , the distance of two cameras, or equivalently

$$r_2 = \frac{\|\delta\|}{\|\delta_0\|} \quad (76)$$

is known. Then  $\delta$  is obtained as

$$\delta = r_2 \delta_0 = \frac{\|\delta\|}{\|\delta_0\|} \delta_0 \quad (77)$$

Regarding (74), the following lemma holds.

**Lemma 4** The matrix  $\Omega - c(p_1 \ q_1 \ -1)$  in (72) is nonsingular if and only if  $b \notin \text{Im } \Omega$  and  $\begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix} \notin \text{Im } \Omega$  hold.

(Proof) Let  $d \triangleq (p_1 \ q_1 \ -1)^T$ , and let

$$S \triangleq \Omega - c(p_1 \ q_1 \ -1) = \Omega - cd^T$$

(‘only if’ part) If  $b \in \text{Im } \Omega$ , or  $c = \frac{1}{r_1} b \in \text{Im } \Omega$ , then  $c = \Omega x$  for some  $x$ , and we get  $S = \Omega - \Omega x d^T = \Omega(I - x d^T)$

implying  $|S| = 0$ . Similarly, if  $d \in \text{Im } \Omega$ , then with  $d = \Omega x$ , we have

$$S = \Omega - c(\Omega x)^T = \Omega + cx^T \Omega = (I + cx^T)\Omega$$

and  $|S| = 0$ . Thus  $|S| \neq 0$  implies  $b \notin \text{Im } \Omega$  and  $d = \begin{pmatrix} p_1 & q_1 & -1 \end{pmatrix}^T \notin \text{Im } \Omega$ .

(‘if’ part) Assume on the contrary that  $|S| = 0$  and let  $Sx = (\Omega - cd^T)x = 0$ ,  $x \neq 0$ , or

$$\Omega x = cd^T x \quad (78)$$

Then, noting that  $\Omega$  is skew-symmetric, we have  $0 = x^T \Omega x = (x^T c)(d^T x)$  resulting in  $x^T c = 0$  or  $d^T x = 0$ . Here if  $d^T x = 0$ , then by (78), we get  $\Omega x = 0$  and

$$\begin{pmatrix} \Omega \\ d^T \end{pmatrix} x = 0$$

Noting  $\text{rank } \Omega = 2$ , we have  $d^T = y^T \Omega$  or  $d = \Omega(-y)$  for some  $y$ , which contradicts  $d \notin \text{Im } \Omega$ . Thus  $d^T x \neq 0$ . In this case, (78) yields

$$c = \Omega x' \text{ with } x' = \frac{1}{d^T x} x (\neq 0)$$

contradicting  $c \notin \text{Im } \Omega$  or  $b \notin \text{Im } \Omega$ . Consequently,  $b \notin \text{Im } \Omega$  and  $d \notin \text{Im } \Omega$  imply  $|S| \neq 0$ . (Q.E.D.)

Thus, regarding the identifiability of  $\delta$ , we have the following result.

**Theorem 2** *Assume that  $(\Omega, p_1, q_1, c)$ ,  $(\Omega', p_2, q_2, c')$  and  $R$  are given. Then, the translation vector  $\delta$  is determined uniquely up to the scalar multiple if and only if the following conditions hold:*

$$b \notin \text{Im } \Omega \text{ and } \begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix} \notin \text{Im } \Omega \quad (79)$$

**Lemma 5**  *$b' \in \text{Im } \Omega'$  if and only if  $b \in \text{Im } \Omega$ .*

(Proof) We show that  $b' \in \text{Im } \Omega'$  implies  $b \in \text{Im } \Omega$ . Let  $b' \in \text{Im } \Omega'$  and let  $b' = \Omega' x$  for some  $x$ . Then (17) is written as

$$b = R\Omega' x - R\Omega' R^T \delta$$

Since  $R\Omega' = \Omega R$  by (16), we have  $b = \Omega(Rx - \delta)$ , and  $b \in \text{Im } \Omega$ . The converse follows similarly. (Q.E.D.)

We now give the following remark on Theorem 2.

**Remark 1** *As we see in Theorem 2, one parameter ambiguity remains for determining the vector  $\delta$ . The parameter could be any one of  $\|\delta\|$ ,  $r_1$ , and  $r_2$ .*

From Theorems 1 and 2, we obtain the following result.

**Theorem 3** *Assume that a planar surface undergoing the rigid body motion is observed by perspective projections by two cameras, and that the parameters  $(\Omega, p_1, q_1, c)$  and  $(\Omega', p_2, q_2, c')$  are identified. Then, generically, i.e. under the conditions*

$$b \notin \text{Im } \Omega, \begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix} \notin \text{Im } \Omega \text{ and } \begin{pmatrix} p_1 \\ q_1 \\ -1 \end{pmatrix} \notin (\text{Im } \Omega)^\perp,$$

*the calibration parameters  $R$  and  $\delta$  (within the scalar multiple) are determined uniquely.*

## 4 Concluding Remarks

The main result we show in this paper is that when the calibration parameters between a pair of cameras is unknown, it is possible to recover these parameters only upto a one parameter ambiguity. Thus the depth ambiguity cannot be resolved even with a pair of cameras.

## References

- [1] Ghosh, B. K. and E. P. Loucks, A Perspective Theory for Motion and Shape Estimation in Machine Vision, SIAM J. Control and Optimization, 1995.
- [2] Ghosh, B. K., H. Inaba and S. Takahashi, Identification of Riccati Dynamics under Perspective and Orthographic Observations, IEEE Transactions on Automatic Control, to appear in 2000.
- [3] Kanatani, K., Group-Theoretical Methods in Image Understanding, Springer Verlag, 1990.
- [4] Kano, H., B. K. Ghosh and H. Kanai, Recursive Estimation of Motion and Shape Parameters of A Moving Rigid Body from Noisy Image Data under Perspective Projection, Proc. of the 1998 International Symposium on Mathematical Theory of Networks and Systems, pp.1039-1042, Padova, Italy, July 6-10, 1998.
- [5] Loucks, E.P., A Perspective Systems Approach to Motion and Shape Estimation in Machine Vision, Ph.D. Thesis, Washington University, 1994.
- [6] Soatto, S., A Geometric Framework for Dynamic Vision, Ph.D. Thesis, California Institute of Technology, 1996.
- [7] Subbarao, M. and A. M. Waxman, On the Uniqueness of Image Flow Solutions for Planar Surfaces in Motion, Technical Report 114, College Park, University of Maryland, Center for Automation Research, 1985.
- [8] Waxman, A. M. and S. Ullman, Surface Structure and 3-D Motion from Image Flow : Kinematic Analysis, International J. Robotics Research, vol.4, pp.72-94, 1985.