Stability analysis of a sliding observer based robust output tracking control design for a nonlinear system

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Abstract

In this paper, an observer based robust output tracking controller design is proposed for a class of nonlinear systems which are input-output linearizable. An ultimate boundedness analysis is presented for an equivalent control based sliding observer where the estimation accuracy is eventually expressed in terms of a single parameter. The observer is incorporated into the closed loop to implement an ideal tracking control law and a complete Lyapunov observer/controller synthesis is performed to prove the ultimate boundedness of the tracking error.

1 Introduction

Feedback linearizable systems constitute a major class of nonlinear systems which are transformable into linear controllable ones via coordinate transformation and state feedback. For output tracking problems, the linearizability of the full input-state response is not necessary, instead; input-output linearization easily suffices to serve as the first step of a tracking controller design to form a linearized input-output map ([1], [4], [9]). The internal stability issues related to the zero dynamics and possible computation errors which might arise during the successive differentiations of the output due to the system being uncertain are the two major problems to be resolved.

In this paper, we study an observer based robust output tracking controller design for an uncertain SISO nonlinear system. The uncertainty in the system is not required to satisfy a certain structural property as long as the relative degree is preserved. For simplicity, the system is further assumed to have full relative degree. The proposed design employs a sliding observer to implement an ideal tracking controller. The observer is based on the *equivalent control methodology* which has been introduced originally as a regularization technique to analyze the sliding motion for systems affine in the control and has also been used for state estimation purposes ([11]). The stability of the resulting closed loop system is proven by utilizing the observer/controller synthesis tools developed in [9].

The organization of the paper is as follows: In Section 2, the problem formulation and the assumptions imposed are given. Using the input-output linearization approach and a continuous min-max design method, an ideal tracking controller which requires the output coordinates for implementation is developed in Section 4. The observer design is presented in Section 4 and the ultimate boundedness of the estimation errors is proven upon the selection of the filter time constants according to a rule. The stability of the composite system is studied in Section 5.

2 Problem Formulation

Consider a SISO nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \quad (1)$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ and $y \in \mathbf{R}$ are the state, input and output variables, respectively and the vector fields $f : D_x \to \mathbf{R}^n$, $g : D_x \to \mathbf{R}^n$ and the output function $h : D_x \to \mathbf{R}$ are smooth on a domain $D_x \subset \mathbf{R}^n$. The system is allowed to be uncertain and its nominal model is defined as follows:

$$\begin{aligned} \dot{x} &= \bar{f}(x) + \bar{g}(x)u \\ y &= h(x) \end{aligned} \quad (2)$$

where $\bar{f} : D_x \to \mathbf{R}^n$, $\bar{g} : D_x \to \mathbf{R}^n$, which represent the known information on f and g, respectively, are also smooth on D_x and the output function h is assumed to be known exactly.

The problem considered is to force the output y to track a given reference signal $y_r(t)$ in the presence of uncertainties where $y_r, \dot{y}_r, \cdots, y_r^{(n)}$ are known and uniformly bounded. We further assume the following:

Assumption 1 *The actual system of Eq. (1) and its nominal model represented in Eq. (2) have both relative* *degree n on D_x, i.e;*

$$L_g L_f^{k-1} h(x) = 0 \quad , \quad L_{\bar{g}} L_{\bar{f}}^{k-1} h(x) = 0 \qquad k = 1, \cdots, n-1$$
$$L_g L_f^{n-1} h(x) \neq 0 \quad , \quad L_{\bar{g}} L_{\bar{f}}^{n-1} h(x) \neq 0 \qquad \text{for} \ \ \forall x \in D_x \quad (3)$$

where $L_f h(x)$ denotes the Lie derivative of h along f ([7]) so that they are both input-state and input-output linearizable, i.e; there exist diffeomorphisms T_y, T_z such that the systems of Eq. (1) and Eq. (2) can be transformed into

$$\begin{aligned} \dot{y}_k &= y_{k+1} \qquad\qquad k = 1, \cdots, n-1 \\ \dot{y}_n &= F(x) + G(x)u, \end{aligned} \qquad (4)$$

$$\begin{aligned} \dot{z}_k &= z_{k+1} \qquad\qquad k = 1, \cdots, n-1 \\ \dot{z}_n &= \bar{F}(x) + \bar{G}(x)u \end{aligned} \qquad (5)$$

respectively, where $y_k = L_f^{k-1} h(x)$, $F(x) = L_f^n h(x)$, $G(x) = L_g L_f^{n-1} h(x)$, $z_k = L_{\bar{f}}^{k-1} h(x)$, $\bar{F}(x) = L_{\bar{f}}^n h(x)$, $\bar{G}(x) = L_{\bar{g}} L_{\bar{f}}^{n-1} h(x)$ and $G(x), \bar{G}(x) \neq 0$ on D_x.

Assumption 2 *The difference between the transformed coordinate variables (y_k's and z_k's) for the systems of Eq. (4)-(5) can be bounded on D_x according to*

$$\begin{aligned} \Psi_k(x) &\triangleq y_{k+1} - z_{k+1} = L_f^k h(x) - L_{\bar{f}}^k h(x) \\ |\Psi_k(x)| &\leq \rho_k(x) \quad \text{for} \ \ k = 1, \cdots, n-1 \end{aligned} \qquad (6)$$

where $\rho_k(x)$'s are known bounding functions.

3 An Ideal Output Tracking Controller

Consider the system of Eq. (4). Since the vector fields f, g are only partially known, $F(x)$ and $G(x)$ cannot be computed exactly. However, they can be approximated by $\bar{F}(x)$ and $\bar{G}(x)$, respectively using the nominal model and the uncertainty in this representation satisfies the standard (generalized) matching condition. The robust stability problem for an uncertain system when the uncertainty satisfies the matching condition has already been studied extensively and several methods are available in the literature. In this paper, a *continuous min-max design* [2] is employed, [7], [9] to develop an ideal output tracking controller assuming that all y_k's are available for feedback. To this end, define

$$\xi_k = y_k - y_r^{(k-1)}(t) \qquad \text{for} \ \ k = 1, \cdots n \qquad (7)$$

and write the system in terms of these tracking error coordinate variables as follows:

$$\dot{\xi} = A\xi + B \left[\ \bar{F}(x) + \Delta F(x) + \bar{G}(x) \ u \ + \Delta G(x) \ u \ - y_r^{(n)} \ \right] \qquad (8)$$

where $\xi = [\xi_1, \cdots, \xi_n]^T$, $\Delta F = F - \bar{F}$, $\Delta G = G - \bar{G}$ and (A, B) is a canonical representation of a chain of n integrators.

Since the pair (A, B) is controllable one can assign the eigenvalues of $(A + BK)$ as desired. Let $P = P^T$ be the unique, p.d. solution to the following Lyapunov matrix equation

$$P(A + BK) + (A + BK)^T P = -Q \qquad (9)$$

where $Q = Q^T$ is a user-selected p.d. matrix and take $V_\xi = \xi^T P \xi$ as the Lyapunov function candidate for the tracking error dynamics. The stability analysis is performed over the set $\mathcal{E}_{b_1} \triangleq \{\xi \in \mathbf{R}^n : V_\xi \leq b_1\}$ for reference signals belonging to $\mathcal{Y}_{r,c} \triangleq \{Y_r \in \mathbf{R}^n : V_{Y_r} \leq c\}$ where $Y_r = [y_r, \dot{y}_r, \cdots, y_r^{(n-1)}]^T$, $V_{Y_r} = Y_r^T P Y_r$ and b_1, c are arbitrary provided that the state vector x is confined to D_x when $\xi \in \mathcal{E}_{b_1}$ and $Y_r \in \mathcal{Y}_{r,c}$. Noting that $Y - Y_r = \xi$ where $Y = [y_1, \cdots, y_n]^T$, the conditions $\xi \in \mathcal{E}_{b_1}$ and $Y_r \in \mathcal{Y}_{r,c}$ imply that $Y \in \mathcal{Y}_d$ for any $d \geq [b_1 + c + 2\sqrt{b_1 c}\lambda_{max}(P)/\lambda_{min}(P)]$ where $\mathcal{Y}_d \triangleq \{Y \in \mathbf{R}^n : V_Y \leq d\}$ and $V_Y = Y^T P Y$.

The control input is selected as follows:

$$u(t, x, \xi) = \bar{G}^{-1}[-\bar{F} + K\xi + y_r^{(n)} + \bar{u}] \qquad (10)$$

It consists of there components: a linearizing term to cancel the nominal nonlinearities, a linear feedback term which would achieve the asymptotic tracking objective in case the available information represents the actual system exactly and an additional robustifying term $\bar{u}$ which is to be determined so as to counteract the effect of the uncertainties. It is also assumed that

$$|\Delta F - \Delta G \bar{G}^{-1} \bar{F} + \Delta G \bar{G}^{-1} K \xi + \Delta G \bar{G}^{-1} y_r^{(n)}| \leq D(t, x, \xi)$$
$$\bar{G}^{-1} G H(x) \geq 1 \qquad (11)$$

over the domain of interest where $D(t, x, \xi)$ and $H(x)$ are known. $D(t, x, \xi)$ is further required to be locally Lipschitz in ξ due to the stability analysis of the composite system as it will be clarified later. Note that, since a regional analysis is employed one could select $D(\bullet)$ and $H(\bullet)$ as sufficiently large constants. However, the conservatism of the design generally decreases if state dependent bounds are allowed. Since both the actual and the nominal systems are feedback linearizable, $G \neq 0$, $\bar{G} \neq 0$ and $\bar{G}G^{-1}$ is sign definite over D_x. Therefore a sign definite function or a constant can easily be found for $H(x)$. Let

$$\bar{u} = -D(t, x, \xi) H(x) \tanh\left(2D(t, x, \xi) B^T P \xi / \mu\right) \qquad (12)$$

where tanh denotes the hyperbolic tangent function which is a smooth approximate for the *sgn* function with the following property reported in [10]:

$$0 \leq |y| - y \tanh(y/\mu) \leq \kappa\mu \qquad (13)$$

for any $\mu > 0$ and any real y where $\kappa = 0.2785$. Using Eq. (12)-(13), one gets $\dot{V}_\xi \leq -\xi^T Q \xi + \kappa\mu$. Selecting $\mu \in (0, \mu^*]$ where $\mu^* = b_1 \lambda_{min}(Q)/\kappa\lambda_{max}(P)$

it can be easily shown that $\mathcal{E}_{b_1}$ is positively invariant and all the trajectories starting inside this set are also ultimately bounded to $\mathcal{E}_{b_2}$ for any b_2 satisfying $\mu\kappa\lambda_{max}(P)/\lambda_{min}(Q) < b_2 \leq b_1$. Furthermore, if we let $\mu = \mu_0 e^{-(t-t_0)/\varrho}$ with $\mu_0 \in (0, \mu^*]$ and $\varrho > 0$, V_ξ converges to zero exponentially satisfying

$$\dot{V}_\xi + \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} V_\xi \leq \kappa\mu_0 \, e^{-(t-t_0)/\varrho} \tag{14}$$

and the tracking objective can indeed be achieved exponentially.

4 Sliding Observer Design

The tracking controller design of Section 3 assumes the availability of the full output vector. In this section, we present an estimator design which provides an estimate for the output vector $Y = [y_1, \cdots, y_n]^T$ with an accuracy controllable by the designer in the closed loop. The estimator is theoretically an equivalent control based sliding observer which recursively uses equivalent control operators to extract additional information from the system. The overall design problem is transformed into independent smaller order stabilization problems which share information with each other through the equivalent control operators [11].

Let

$$\dot{\hat{y}}_1 = z_2 + (\rho_1(x) + \eta_1)\, sgn\,(y_1 - \hat{y}_1) \tag{15}$$

be the observer equation for the variable y_1 which satisfies

$$\dot{y}_1 = z_2 + \Psi_1(x) \tag{16}$$

where z_2 represents the available information on $\dot{y}_1$ which can be computed using the nominal model and $\Psi_1(x)$ is the computation error. Subtracting Eq. (15) from Eq. (16), one gets

$$\dot{\tilde{y}}_1 = \Psi_1(x) - (\rho_1(x) + \eta_1)\, sgn\, \tilde{y}_1 \tag{17}$$

where $\tilde{y}_1 = y_1 - \hat{y}_1$ and it can be easily shown that a sliding motion occurs on $\tilde{y}_1 = 0$ in finite time for any $\eta_1 > 0$. The reaching phase of this motion can further be eliminated with the initial condition selection of $\hat{y}_1(t_0) = y_1(t_0)$ where t_0 denotes the initial time and $\tilde{y}_1$ is kept at zero ideally $\forall t \geq t_0$. According to equivalent control methodology,

$$[\,(\rho_1(x) + \eta_1)\, sgn\, \tilde{y}_1\,]_{eq} = \Psi_1(x) \tag{18}$$

so that an alternative expression for y_2 is obtained as follows:

$$y_2 = z_2 + [\,(\rho_1(x) + \eta_1)\, sgn\, \tilde{y}_1\,]_{eq} \tag{19}$$

where the operator $[\bullet]_{eq}$ outputs the equivalent value of its discontinuous argument which is defined as the continuous injection which would satisfy the invariance conditions of the sliding motion ($\tilde{y}_1 = 0$, $\dot{\tilde{y}}_1 = 0$) that this discontinuous input induces. The equivalent value operator $[\bullet]_{eq}$ can be implemented by an high bandwidth low-pass filter ([11]).

At the next step, we select

$$\dot{\hat{y}}_2 = z_3 + (\rho_2(x) + \eta_2)\, sgn\,(y_2 - \hat{y}_2) \tag{20}$$

as the observer equation for the variable y_2 and rewrite it for implementation as follows:

$$\dot{\hat{y}}_2 = z_3 + (\rho_2(x)+\eta_2)\, sgn\,(z_2 + [\,(\rho_1(x)+\eta_1)\, sgn\, \tilde{y}_1\,]_{eq} - \hat{y}_2) \tag{21}$$

using the equivalent representation of y_2 in Eq. (19) which can be realized by low-pass filtering. For the sake of clarity, the estimator design algorithm is next carried out in terms of the equivalent value operators by postponing the detailed analysis of equivalent value filtering in the realization of sequential equivalent value operators to Theorem 1.

Noting that

$$\dot{y}_2 = z_3 + \Psi_2(x) \tag{22}$$

where z_3 represents the computable ideal value of $\dot{y}_2$ with $\Psi_2(x)$ being the error in this computation and subtracting Eq. (21) from Eq. (22), one gets

$$\dot{\tilde{y}}_2 = \Psi_2(x) - (\rho_2(x) + \eta_2)\, sgn\, \tilde{y}_2 \tag{23}$$

where $\tilde{y}_2 = y_2 - \hat{y}_2$ is steered to zero in finite time and kept at zero ideally for any $\eta_2 > 0$. In sliding mode,

$$[\,(\rho_2(x) + \eta_2)\, sgn\, \tilde{y}_2\,]_{eq} = \Psi_2(x) \tag{24}$$

and y_3 can also be written as follows:

$$y_3 = z_3 + [\,(\rho_2(x) + \eta_2)\, sgn\, \tilde{y}_2\,]_{eq} \tag{25}$$

Continuing in this manner, at the kth step, the observer equation for the variable y_k is chosen as follows:

$$\dot{\hat{y}}_k = z_{k+1} + \Gamma_k \tag{26}$$

$$\Gamma_k = (\rho_k(x) + \eta_k)\, sgn\,(z_k + [\,\Gamma_{k-1}\,]_{eq} - \hat{y}_k) \tag{27}$$

for any $k = 1, \cdots, n-1$ where $[\,\Gamma_0\,]_{eq} = 0$ and $\eta_k > 0$. With this selection, $\tilde{y}_k$ is also steered to zero in finite time, the equivalent value of the discontinuous input of that step gives information required for the next step and so on and so forth. The recursive design is terminated at the $(n-1)$th step and y_n is estimated using the relation

$$y_n = z_n + [\,\Gamma_{n-1}\,]_{eq} \tag{28}$$

while an estimate for each y_k for $k = 1, \cdots, n-1$ can be read through the related observer variable $\hat{y}_k$.

Note that, the previous design assumes that the equivalent control operators can be implemented exactly. Due to the recursive nature of the design, an implementation error of the current step affects not only the accuracy of this step but also that of all the subsequent ones and itself can be an instability source for the closed loop system. Theorem 1 summarizes the result of a complete stability analysis of the observer when the equivalent value operators are implemented by first order high bandwidth low-pass filters by expressing the estimation accuracy in terms of a single design parameter which couples the equivalent value filter time constants.

Theorem 1 *Consider*

$$\dot{\hat{y}}_k = z_{k+1} + \bar{\Gamma}_k \quad \text{for} \quad k = 1, \cdots, n-1 \tag{29}$$

$$y_n = z_n + v_{n-1} \tag{30}$$

where

$$\bar{\Gamma}_k = (\rho_k(x) + \eta_k)\, sgn\,(z_k + v_{k-1} - \hat{y}_k) \tag{31}$$

$$\tau_k \dot{v}_k + v_k = \bar{\Gamma}_k \tag{32}$$

with $v_0 = 0$. Suppose that $Y(t) \in \mathcal{Y}_d$ with an arbitrary d, the control input is uniformly bounded $\forall t \geq t_0$ and let $\tau_{n-k} = \epsilon^{2^{k-1}}$ for $k = 1, \cdots, n-1$. For any $\Delta > 0$ and $T_c > t_0$, there exist an ϵ^*, a sufficiently large switching frequency so as to guarantee $|\tilde{y}_1| \leq \delta_0 < \epsilon^{2^{n-1}}$ and positive η_k's such that $\|Y - \hat{Y}\| \leq \Delta$ $\forall t \geq T_c$ selecting $\epsilon \in (0, \epsilon^*]$ where $\hat{Y} = [\hat{y}_1, \cdots, \hat{y}_n]^T$ and $\| \bullet \|$ denotes any vector norm.

Proof: See Appendix. ■

5 Stability Analysis of the Composite System

The control law of Eq. (10) needs the full output vector for implementation. In this section, we study the stability of the composite system when the controller is implemented using the estimated output vector provided by the observer. The estimation accuracy of the observer has already been proven to be controllable by a free parameter provided that the control input is uniformly bounded and $Y \in \mathcal{Y}_d$.

Assume that the initial conditions are such that $\xi_0 \in \mathcal{E}_{b_0}$ and restrict the stability analysis to the compact set $\xi \in \mathcal{E}_{b_1}$ with a $b_1 > b_0$ as before. For any allowable reference signal, $\xi \in \mathcal{E}_{b_1}$ implies that $Y \in \mathcal{Y}_d$ for some d so that the positive invariance of the set $\mathcal{E}_{b_1}$ actually meets one of the requirements of Theorem 1. The key point of the analysis is to saturate the control by a suitable value using the *globally bounded control idea* [3] so as to guarantee that the system trajectories do not escape $\mathcal{E}_{b_1}$ during the initial convergence phase of the observer and the tracking error can further be confined to an arbitrarily small vicinity of zero after this initial phase by employing an analysis similar to [3], [9]. To this end, the control input is modified as follows:

$$\hat{u} = \begin{cases} \bar{G}^{-1}[-\bar{F} + K\hat{\xi} + y_r^{(n)} + \bar{u}] & \text{if} \quad |\hat{u}| \leq N_u \\ N_u & \text{otherwise} \end{cases} \tag{33}$$

where $N_u = \sup_{\xi \in \mathcal{E}_{b_1}} |u(t, x, \xi)|$,

$$\bar{u} = -D(\bullet) H(x) \tanh(2D(\bullet) B^T P \hat{\xi}/\mu) \tag{34}$$

$\hat{\xi} = \hat{Y} - Y_r$ and $D(\bullet)$, $H(\bullet)$ are as before.

Since the control input of Eq. (33) is uniformly bounded, one can use its maximum value to get a conservative bound for V_ξ at any time instant. Let

$$|F(x) + G(x)\, \hat{u}(t, x, \hat{\xi}) - K\xi - y_r^{(n)}| < \gamma_1 \tag{35}$$

for $\xi \in \mathcal{E}_{b_1}$ and calculate the derivative of $V_\xi = \xi^T P \xi$ along the system trajectories using γ_1 to obtain $\dot{V}_\xi \leq -\gamma_2 V_\xi + \gamma_3 \sqrt{V_\xi}$ where $\gamma_2 = \lambda_{min}(Q)/\lambda_{max}(P)$, $\gamma_3 = 2\gamma_1 \|PB\| / \sqrt{\lambda_{min}(P)}$. If $b_1 \geq \gamma_3^2/\gamma_2^2$, $\xi \in \mathcal{E}_{b_1}$ $\forall t \geq t_0$. Otherwise, there exists a time instant T given by

$$T = \frac{2}{\gamma_2} \ln\left(\frac{\gamma_3/\gamma_2 - \sqrt{V_{\xi_0}}}{\gamma_3/\gamma_2 - \sqrt{b_1}}\right) + t_0 \tag{36}$$

such that $\xi \in \mathcal{E}_{b_1}$ $\forall t \in [t_0, T)$. Since, for any $b_1 > b_0$, the conditions $\xi \in \mathcal{E}_{b_1}$ and $Y \in \mathcal{Y}_d$ are at least satisfied during the time interval $[t_0, T)$, the observer gains (η_k's) need to be selected sufficiently large depending on b_0, b_1 with $T_c < T$ so as to assure that the trajectories do not escape $\xi \in \mathcal{E}_{b_1}$ during the initial phase. The existence of such η_k's has been explored in the proof of Theorem 1 in detail. Specifically, one can even use Eq. (50) to generate a suitable set of η_k's if some information on N_{ρ_k}'s and T are available.

Consider the closed loop system with the modified control of Eq. (33) for $t \geq T$. Noting that

$$\hat{u}(t, x, \xi) = u(t, x, \xi) \tag{37}$$

$$\|BG(x)\ (\ u(t, x, \hat{\xi}) - u(t, x, \xi)\ )\| \leq \alpha_1 \|\xi - \hat{\xi}\| \leq \alpha_1 \alpha_2 \epsilon \tag{38}$$

for $\xi \in \mathcal{E}_{b_1}$ where α_1 is the related Lipschitz constant for the control, α_2 is an additional parameter which quantifies the $\mathcal{O}(\epsilon)$ estimation accuracy of the observer for $t > T$ according to Theorem 1 and using the stability result of Section 3 for the ideal control, one gets $\dot{V}_\xi \leq -\xi^T Q \xi + \kappa\mu + \alpha_3 \epsilon$ for $\xi \in \mathcal{E}_{b_1}$ where $\alpha_3 = 2\|\sqrt{P}\| \sqrt{b_1} \alpha_1 \alpha_2$. Let $\mu \in (0, \mu^*]$, $\epsilon \in (0, \epsilon^*]$ where $\mu^* = \lambda_{min}(Q) b_1 / 2\kappa\lambda_{max}(P)$ and $\epsilon^* = \kappa\mu/\alpha_3$. With these selections, the Lyapunov derivative becomes $\dot{V}_\xi \leq -\xi^T Q \xi + 2\kappa\mu$ and the positive invariance of $\mathcal{E}_{b_1}$ can easily be seen to be guaranteed. Therefore, the tracking error ξ_1 is uniformly bounded and it further converges inside the layer of thickness $\sqrt{b_2/\lambda_{min}(P)}$ in finite time for any b_2 satisfying $2\kappa\mu\lambda_{max}(P)/\lambda_{min}(Q) < b_2 \leq b_1$. Since μ is the ultimate error bound parameter it is desired to be as small as possible. However, this also requires ϵ to be small because α_1 usually increases with a decrease in μ due to its role in the tanh function. Nevertheless, it has just been proven that the tracking error is always bounded and can indeed be made arbitrarily small with suitable design parameters at the expense of possibly increasing the conservatism of the design. The final result is summarized in Theorem 2.

Theorem 2 *Consider the system of Eq. (1) with the control input given by Eq. (33) implemented by the observer of Theorem 4. Let $\xi(t_0) \in \mathcal{E}_{b_0}$, $Y_r \in \mathcal{Y}_{r,c}$ and consider the set $\xi(t) \in \mathcal{E}_{b_1}$ with an arbitrary $b_1 > b_0$ provided that $x \in D_x$ on which Assumption 1-2 hold when $\xi(t) \in \mathcal{E}_{b_1}$ and $Y_r \in \mathcal{Y}_{r,c}$. For any $b_0, c, b_1, P, Q > 0$ there exist suitable η_k's to guarantee $T_c < T$, a μ^* and a resulting ϵ^* such that*
(i) $|y(t) - y_r(t)|$ is uniformly bounded by $\sqrt{b_1/\lambda_{min}(P)}$ for all $t \geq t_0$ selecting $\mu \in (0, \mu^*]$ and $\epsilon \in (0, \epsilon^*]$ where $\mu^* = \lambda_{min}(Q)b_1/2\kappa\lambda_{max}(P)$ and $\epsilon^* = \kappa\mu/\alpha_3$.
(ii) Given a $b_2 \in (0, b_1]$, there exists a time instant T_s given by

$$T_s = \frac{\lambda_{max}(P)}{\lambda_{min}(Q)} \ln\left[\frac{b_1 - b_3}{b_2 - b_3}\right] + T \qquad (39)$$

such that $|y(t) - y_r(t)| \leq \sqrt{b_2/\lambda_{min}(P)}$ for all $t \geq T_s$ selecting $\mu \in (0, \mu^*]$ and $\epsilon \in (0, \epsilon^*]$ where T is as in Eq. (36), $\mu^* = \lambda_{min}(Q)b_3/2\kappa\lambda_{max}(P)$ and $\epsilon^* = \kappa\mu/\alpha_3$ for any $b_3 \in (0, b_2)$.

Remark 1 Since α_1 is related to the smoothness of the control, it is directly affected by μ. Furthermore, although the estimation accuracy has been proven to be $\mathcal{O}(\epsilon)$ the actual value of the estimation error also depends on the control input but not on its derivatives by the nature of the estimator. However, when the linear feedback gain K, the gains $D(\bullet)$, $H(\bullet)$ and μ has been selected, there will be a maximum value for α_1 over the restricted domain considered. Although α_i's are not known, Theorem 2 guarantees the existence of an ϵ^* such that the ultimate boundedness result is valid for all $\epsilon \in (0, \epsilon^*]$.

Remark 2 Although each observer gain has been expressed as the sum of a state dependent bounding function $\Psi_k(x)$ and a constant gain η_k, they could be replaced by a single sufficiently large constant since the whole analysis is employed over a compact set. Furthermore, even if the state vector is not available, the observer can still be used by replacing all z_k's with zero and selecting the observer gains so as to bound y_k's instead of just their differences from the computed ones. The state dependence of $D(\bullet)$ and $H(\bullet)$ can also be overcome due to the regional analysis to achieve the tracking objective using only the output feedback.

Remark 3 Unlike the previous results on an equivalent control based observer design, Theorem 1 proposes a selection rule for the filter time constants and relates the estimation accuracy to them by considering the filtering effects in the analysis at the first place. The resulting observer can also be linked to an high-gain observer considering that a multiple time scale behavior is created for both observers where this is achieved by the proposed filter time constant selection for the former whereas by a suitable observer gain selection for the latter. For an high-gain observer, a direct trade-off exists between the steady state estimation accuracy and the transient performance leading to the well-known *peaking phenomena*. However, for the proposed observer, the estimation errors are always bounded during the convergence transient irrespective of ϵ. Furthermore, the convergence time behavior and the steady state accuracy can indeed be controlled independently by the design parameters.

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6 Appendix

Let $N_{\dot{\Psi}_k} \triangleq \sup_{Y \in \mathcal{Y}_d} |\dot{\Psi}_k(x)|$, $N_{\rho_k} \triangleq \sup_{Y \in \mathcal{Y}_d} \rho_k(x)$ where

$$\dot{\Psi}_k = y_{k+2} - z_{k+2} - w_{k+1}(x,u) \quad (40)$$

$$\dot{\Psi}_{n-1} = \Delta F(x) + \Delta G(x) u - w_n(x,u) \quad (41)$$

for $k = 1, \cdots, n-2$ and

$$w_k = L_{\tilde{f}} L_{\bar{f}}^{k-1} h(x) + [L_{\tilde{g}} L_{\bar{f}}^{k-1} h(x)] u \quad (42)$$

for $k = 2, \cdots, n$ by a direct computation. Note that, y_k's and z_k's are related to the original state vector x through the diffeomorphisms T_y and T_z. Therefore, x and z_k's are also bounded when $Y \in \mathcal{Y}_d$. Since the functions $f, g, \bar{f}, \bar{g}$ are all sufficiently smooth, the control input is assumed to be uniformly bounded and the supremum is taken over a compact set, each $N_{\dot{\Psi}_k}$ and N_{ρ_k} are finite and cannot exceed certain values independent of the design parameters over the region of interest. Consider

$$\dot{v}_1(t) = -(1/\tau_1) v_1(t) + [\Psi_1(x) - \dot{\tilde{y}}_1]/\tau_1 \quad (43)$$

where $|\tilde{y}_1| \leq \delta_0 \ \forall t \geq t_0$ with δ_0 being an arbitrarily small constant controllable by the switching frequency which quantifies the accuracy of the sliding motion on $\tilde{y}_1 = 0$ as before. As in [11] integration by part yields an ultimate bound for the first equivalent value filtering error as follows:

$$|v_1(t) - \Psi_1(x)| \leq |v_1(t_0) - \Psi_1(x_0)|\, e^{-(t-t_0)/\tau_1} + \tau_1 N_{\dot{\Psi}_1} + 3\delta_0/\tau_1 \quad \forall t \geq t_0 \quad (44)$$

where $x_0 = x(t_0)$. Note that the ultimate error bound converges to a constant value where both this constant and the convergence time can be made arbitrarily small by reducing ϵ. The actual error equation for the variable y_2 satisfies

$$\dot{\tilde{y}}_2 = \Psi_2(x) - (\rho_2(x) + \eta_2)\, sgn\,(\tilde{y}_2 + v_1 - \Psi_1(x)) \quad (45)$$

Pick any $t_1' > t_0$. Since $\tilde{y}_2 \dot{\tilde{y}}_2 \leq -\eta_2 |\tilde{y}_2|$ for $|\tilde{y}_2| > |v_1(t) - \Psi_1(x))|$ and $|v_1(t) - \Psi_1(x))| \leq \delta_1(t_1') \ \forall t \geq t_1'$ where $\delta_1(t)$ denotes the ultimate bound for $|v_1(t) - \Psi_1(x)|$ given by Eq. (44), there exists a t_1 satisfying

$$0 \leq t_1 - t_1' \leq \frac{|\tilde{y}_2(t_0)| + 2N_{\rho_2}(t_1' - t_0)}{\eta_2} + (t_1' - t_0) \quad (46)$$

such that $|\tilde{y}_2| \leq \delta_1(t_1') \ \forall t \geq t_1$ where the maximum possible bound for $\tilde{y}_2$ at the end of the interval $[t_0\,,\, t_1']$ has been used in Eq. (46) considering the worst-case scenario. Using the previous generic solution of Eq. (44) with $|\tilde{y}_2| \leq \delta_1(t_1')$ to bound the second equivalent value filtering error, one gets

$$|v_2(t) - \Psi_2(t)| \leq |v_2(t_1) - \Psi_2(t_1)|\, e^{-(t-t_1)/\tau_2} + \frac{3}{\tau_2} |v_1(t_0) - \Psi_1(t_0)|\, e^{-(t_1'-t_0)/\tau_1} + \tau_2 N_{\dot{\Psi}_2} + 3N_{\dot{\Psi}_1}\frac{\tau_1}{\tau_2} + \frac{9}{\tau_1\tau_2}\delta_0 \quad (47)$$

$\forall t \geq t_1$. Note that the second term does not cause any singularity problems as ϵ goes to zero for any $t_1' > t_0$ due to the proposed filter time selection and all terms are indeed controllable by ϵ.

At the second step, we first froze the ultimate bound of $|v_2(t) - \Psi_2(t)|$, denoted by $\delta_2(t)$, at a given $t_2' > t_1$ and consider the actual error dynamic of $\tilde{y}_3$ given by

$$\dot{\tilde{y}}_3 = \Psi_3(x) - (\rho_3(x) + \eta_3)\, sgn\,(\tilde{y}_3 + v_2 - \Psi_2(x)) \quad (48)$$

to conclude that there also exists a t_2 satisfying

$$0 \leq t_2 - t_2' \leq \frac{|\tilde{y}_3(t_0)| + 2N_{\rho_3}(t_2' - t_0)}{\eta_3} + (t_2' - t_0) \quad (49)$$

such that $|\tilde{y}_3| \leq \delta_2(t_2') \ \forall t \geq t_2$. Proceeding in this manner, one can easily find an ultimate bound for each $\tilde{y}_k$ according to $|\tilde{y}_k| \leq \delta_{k-1}(t_{k-1}') \ \forall t \geq t_{k-1}$ where $\delta_k(t)$ denotes the ultimate bound for $|v_k(t) - \Psi_k(x)|$.

The resulting ultimate bound expressions for the estimation errors have indeed a certain pattern with the proposed filter time constant selection which can be summarized as follows:

$$|\tilde{y}_k| \leq \begin{cases} |\tilde{y}_k(t_0)| + (t_{k-1} - t_0)(2N_{\rho_k} + \eta_k) & t < t_{k-1} \\ \delta_{k-1}(t_{k-1}') & t \geq t_{k-1} \end{cases}$$

for $k = 2, \cdots, n-1$ and

$$|\tilde{y}_n| \leq \begin{cases} |\tilde{y}_n(t_0)| + (t_{n-1} - t_0)(2N_{\rho_n} + \eta_n) & t < t_{n-2} \\ \delta_{n-1}(t_{n-2}) & t \geq t_{n-2} \end{cases}$$

where

$$\delta_k(t) = |v_k(t_{k-1}) - \Psi_k(x_{k-1})| e^{-(t-t_{k-1})/\tau_k} + \mathcal{O}(\epsilon^{2^{n-k-1}})$$

$\forall t \geq t_{k-1}$,

$$0 \leq t_k - t_k' \leq \frac{|\tilde{y}_{k+1}(t_0)| + 2N_{\rho_{k+1}}(t_k' - t_0)}{\eta_{k+1}} + (t_k' - t_0) \quad (50)$$

for $k = 1, \cdots, n-2$ and $x_k = x(t_k)$. Note that each δ_k can be made arbitrarily small within an arbitrarily short time interval by decreasing ϵ. Therefore, each $\tilde{y}_k$ is ultimately bounded to an arbitrarily small vicinity of zero after a finite time with ϵ being the ultimate bound parameter. Furthermore, for any given $T_c > t_0$ and $\Delta > 0$ there always exist sufficiently large η_k's so as to guarantee a time sequence $t_0 < t_1' \leq t_1 < t_2' \leq \cdots < t_{n-2}' \leq t_{n-2} < T_c$ satisfying Eq. (50). With this sequence an ultimate bound is obtained for each $|\tilde{y}_k|$ in the form of Eq. (6)-(6) so that there exists an ϵ^* which satisfies $\|Y - \hat{Y}\| \leq \Delta \ \forall t \geq T_c$ with any $\epsilon \in (0, \epsilon^*]$. This completes the proof.