Topological Structure of the Set of Non-strongly-stabilizable SISO Systems

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Abstract

We study on the robustness of non-strongly stabilizable SISO plants under small perturbations in graph topology. From a new notion of pole-zero shifting, we show that there are two types of non-strongly stabilizable plants : one that can be made strongly stabilizable by arbitrary small perturbations and the other that is essentially non-strongly stabilizable. We provide a simple criterion for this classification and examine its relevance to the simultaneous stabilization problem.

1 Introduction

The strong stabilization problem concerns with the existence of a *stable* controller that stabilizes a given plant. It is now a classical result that every strongly stabilizable plant has a special pattern of nonnegative real pole-zeros, which is known as the parity interlacing property (p.i.p.) [7, 9, 10]. It is remarkable that the strong stabilizability does not depend on any non-real or open LHP (left half plane) pole-zeros. With this observation, at a glance, we might guess that the strong stabilization is a purely algebraic property.

In fact, there are several topological aspects of this problem that have obvious engineering significance. For instance, [9] proved that the set of strongly stabilizable plants $P \in R^{m\times n}(s)$ ($:= (m\times n)$ matrix whose elements are rational functions) is *dense* in graph topology, provided that either $n \geq 2$ or $m \geq 2$ holds.

As another example, suppose P is strongly stabilizable and let $\tilde{P}$ be a slight perturbation of P. Then it can be easily shown that $\tilde{P}$ is also strongly stabilizable provided that the perturbation is sufficiently small.

We believe, however, our understanding on the topological aspects of the strong stabilization problem, even in SISO case, is far from its full depth. A simple modification of the previous perturbation problem is enough to show this. That is, *suppose P is not strongly stabilizable. Can we slightly perturb P to a strongly stabilizable $\tilde{P}$?* This problem will be studied in our paper.

Many useful tools for robustness analysis come from the fundamental fact that the set of invertible elements in RH_∞ is open set [5, 6, 8]. However, since we have to consider non-invertible elements in our study, existing tools are not sufficient. This motives a new approach where we investigate all possible *shifting* of nonnegative real pole-zeros, caused by perturbations.

2 Mathematical Backgrounds

$\mathbb{C}$ denotes the complex plane and $\mathbb{C}_+$ denotes the *open* right half plan (RHP) of $\mathbb{C}$. Also $\mathbb{R}_+^e$ denotes the extended nonnegative real axis, i.e., $\mathbb{R}_+^e := [0, \infty]$.

H_∞ denotes the usual Hardy space equipped with the norm $\|\cdot\|_\infty$ and RH_∞ denotes the *real rational* subset of H_∞. $R(s)$ denotes the space of proper rational transfer function. For an open set $\Omega \subset \mathbb{C}$, let $H(\Omega)$ denote the set of analytic functions defined in Ω.

Let γ be a closed path (:= piecewise continuously differentiable closed curve from a compact real interval $\xi \in [\alpha, \beta]$ to $\mathbb{C}$ satisfying $\gamma(\alpha) = \gamma(\beta)$). The range of γ is denoted by $\gamma^* \subset \mathbb{C}$.

Given a closed path γ, define Δ_γ as the complement of γ^* relative to $\mathbb{C}$ and let

$$\mathrm{Ind}_\gamma(z) := \frac{1}{2\pi i}\int_\gamma \frac{d\xi}{\xi - z} \quad , \ z \in \Delta_\gamma \qquad (1)$$

Then $\mathrm{Ind}_\gamma(\cdot)$ is a constant integer on each component of Δ_γ and it is zero in the unbounded component of Δ_γ. In fact $\mathrm{Ind}_\gamma(z)$ denotes the number of times that γ winds around z, the *winding number*. For details see [3] (pp.203).

Theorem 1 (Rouché) *Suppose γ is a closed path in the nonempty open subset Ω of $\mathbb{C}$, such that $\mathrm{Ind}_\gamma(\alpha) = 0$ for every α not in Ω. Suppose also that $\mathrm{Ind}_\gamma(\alpha) = 0$ or 1 for every $\alpha \in \Omega - \gamma^*$, and let Ω_1 be the set of all*

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α *with* $\mathrm{Ind}_\gamma(\alpha)=1$*. For any* $f\in H(\Omega)$*, let* N_f *be the number of zeros of* f *in* Ω_1*, counted according to their multiplicity. If* $g\in H(\Omega)$ *satisfies*

$$|f(s)-g(s)|<|f(s)|\ ,\quad \forall\, s\in\gamma^* \tag{2}$$

then $N_f=N_g$ *holds.*

Proof: See [3] (pp.225) ∎

Theorem 1 says that if $|f-g|$ is strictly smaller than $|f|$ along γ^* pointwisely, then every $g\in H(\Omega)$ has the same number of zeros inside γ^* as f. The next simple corollary of Theorem 1 is adequate for our developments.

Corrollary 1 *With* $f,g\in RH_\infty$*, if a closed path* γ *and an open set* $\Omega\subset\mathbb{C}_+$ *satisfy all hypothesis of Theorem 1 and the next inequality*

$$\|f-g\|_\infty<\inf_{\gamma^*}|f| \tag{3}$$

holds, then we have $N_f=N_g$.

Proof: Since $f-g$ is analytic on $\mathbb{C}_+$, it holds that $\sup_{\gamma^*}|f-g|\le\|f-g\|_\infty$ from the maximum modulus theorem, whatever γ^* is. Thus obviously (3) is a sufficient condition for (2). ∎

3 Problem Definition

Recall that every $P\in R(s)$ has a coprime factor representation $P=ND^{-1}$ with $N,D\in RH_\infty$.

The neighborhood of $X\in RH_\infty$, in H_∞ topology, is defined by

$$B_h(X,\varepsilon):=\{\tilde X\,;\,\|\tilde X-X\|_\infty<\varepsilon\,,\,\varepsilon>0\} \tag{4}$$

and that of $P=ND^{-1}$, in graph topology, is given by

$$B_g(P,\varepsilon):=\left\{\tilde N\tilde D^{-1}\,;\,\left\|\begin{matrix}\tilde N-N\\ \tilde D-D\end{matrix}\right\|_\infty<\varepsilon\,,\,\varepsilon>0\right\} \tag{5}$$

From the next inequalities :

$$\begin{matrix}\|\tilde N-N\|_\infty\le\varepsilon_1\\ \|\tilde D-D\|_\infty\le\varepsilon_2\end{matrix}\ \Rightarrow\ \left\|\begin{matrix}\tilde N-N\\ \tilde D-D\end{matrix}\right\|_\infty\le\sqrt{\varepsilon_1^2+\varepsilon_2^2} \tag{6}$$

$$\left\|\begin{matrix}\tilde N-N\\ \tilde D-D\end{matrix}\right\|_\infty\le\varepsilon_3\ \Rightarrow\ \begin{matrix}\|\tilde N-N\|_\infty\le\varepsilon_3\\ \|\tilde D-D\|_\infty\le\varepsilon_3\end{matrix} \tag{7}$$

with $\varepsilon_1>0,\varepsilon_2>0,\varepsilon_3>0$, it follows that a small perturbation of $P=ND^{-1}$ in graph topology is equivalent to two independent and simultaneous perturbations of N,D in RH_∞ topology.

Now we are ready to explicitly state our problem.

Problem 1 *Suppose* $P\in R(s)$ *is a non-strongly stabilizable SISO plant. Does there exist* $\varepsilon>0$ *such that every element of* $B_g(P,\varepsilon)$ *is not strongly stabilizable ?*

4 Shifting of Nonnegative Real Zeros

Recall that the strong stabilizability of $P=ND^{-1}$ is determined by the $\mathbb{R}^e_+$ zeros of both N and D. Therefore, in order to check the same property for a perturbed plant $\tilde P=\tilde N\tilde D^{-1}\in B_g(P,\varepsilon)$, it is quite natural to investigate all possible $\mathbb{R}^e_+$ zeros of $\tilde N$ and $\tilde D$. Firstly in RH_∞ topology we consider the $\mathbb{R}^e_+$ zeros of $\tilde N$ and $\tilde D$ separately. The easiest way to *grasp* our approach is, we believe, to consider a numerical example below.

In the rest of paper, we identify a closed path γ with its range γ^* without explicitly specifying the mapping $\xi\in[\alpha,\beta]\to\gamma(\xi)$ since it is irrelevant to our developments.

Example 1 *Let*

$$N_1(s)=\frac{s-2}{s+2}\ ,\ \ \gamma_1^*:=\{s\in\mathbb{C};\ |s-2|=1\}. \tag{8}$$

The path γ_1^* *is shown in Fig. 1. Choosing* $\Omega=\mathbb{C}_+$ *and* $\gamma^*=\gamma_1^*$*, we can easily check that all hypotheses of Theorem 1 hold. From the fact*

$$\inf\{|N_1(s)|\,;\,s\in\gamma_1^*\}=0.2, \tag{9}$$

Corollary 1 proves that every plant $\tilde N_1\in B_h(N_1,0.2)$ *has only one zero inside* γ_1^**. Moreover, this zero should stay on the real interval* $[1,3]$ *in order to guarantee* $\tilde N_1$ *to be real rational.*

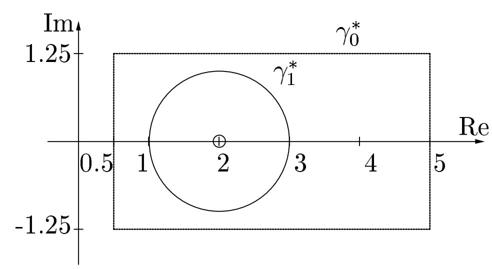

Figure 1: Path γ_0 and γ_1

In fact, we can prove a much stronger result that every $\tilde N_1\in B_h(N_1,0.2)$ *has only one zero (no additional zeros) on* $[0,\infty]$*. Firstly numerical computations give*

$$\inf\{|N_1(s)|\,;\,s\in[0,0.5]\}=0.6 \tag{10}$$

$$\inf\{|N_1(s)|\,;\,s\in[5,\infty]\}=0.4286 \tag{11}$$

Thus it follows that every $\tilde N_1\in B_h(N_1,0.2)$ *has no zeros on these two intervals from the maximum modulus theorem, since* $\tilde N_1-N_1\in H(\mathbb{C}_+)$ *and* $\|\tilde N_1-N_1\|_\infty\le 0.2$*. Finally let's check the possibility of new additional zeros inside* γ_0^* *and outside* γ_1^* *at the same time. Since it holds that*

$$\inf\{|N_1|;s\in\gamma_0^*\}=0.2826, \tag{12}$$

applying Corollary 1 with γ_0^* again, we conclude that $\tilde{N}_1 \in B_h(N_1, 0.2)$ has only one zero inside γ_0^*, which is actually inside γ_1^*.

We can generalize Example 1 in an obvious way for a plant $N \in RH_\infty$ having finite positive real zeros $0 < s_1 < \cdots < s_n < \infty$. It should be noted here that each s_k may be a multiple zero and that N may have zeros at $s = 0$ or $s = \infty$.

We choose paths $\{\gamma_k^*\,;\, k = 0, 1, \cdots, n\}$ as shown in Fig. 2. Here γ_0^* is chosen not to include any complex zeros of N with a sufficiently small $y_m > 0$.

Lemma 1 *Suppose $N \in RH_\infty$ has finite positive real zeros $0 < s_1 < \cdots < s_n < \infty$. Let's choose paths $\{\gamma_k\,;\, k = 0, 1, \cdots, n\}$ as shown in Fig. 2. Then there exists $\varepsilon > 0$ such that every $\tilde{N} \in B_h(N, \varepsilon)$ has the same number of zeros as N inside γ_k^* for all $k = 0, 1, \cdots, n$, counting multiplicity. Moreover, if $N(0) \neq 0\,(N(\infty) \neq 0)$ then we can choose $\varepsilon > 0$ such that every $\tilde{N} \in B_h(N, \varepsilon)$ has no zero in $[0, x_a]$ ($[x_b, \infty]$, respectively).*

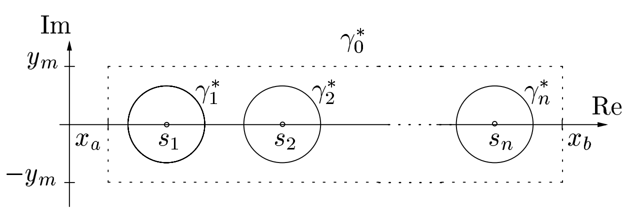

Figure 2: Paths $\{\gamma_k^*\,;\, k = 0, \cdots, n\}$

The core of this lemma is that, if the magnitude of perturbation is sufficiently small, we can group all zeros inside γ_0^* according to each path γ_k^* or, equivalently, to their original location s_k. With this grouping in mind, we can say that all zeros of $\tilde{N}$ inside γ_k^* are *shifted* from $s_k \in \mathbb{R}_+$ without ambiguity.

Since the strong stabilizability depends on the set of nonnegative *real* pole-zeros (see Theorem 2), we are now interested in characterizing the path γ_k^* that includes at least one real zero of $\tilde{N}$ for every $\tilde{N} \in B_h(N, \varepsilon)$ where $\varepsilon > 0$ is determined from Lemma 1. In fact, this task turns out to be fairly easy since we are working in RH_∞, the *real* rational function space. Let's consider a numerical example.

Example 2 *Let*

$$N_2(s) = \frac{(s-2)^2}{(s+2)^2} \quad , \quad \gamma_1^* := \{ s \in \mathbb{C};\ |s-2| = 1 \} \qquad (13)$$

Following the reasoning of Example 1 with Fig. 1, we can prove that every $\tilde{N}_2 \in B_h(N_2, 0.04)$ has two zeros inside γ_0^* and none on $[0,1] \cup [3, \infty]$. However we should note a key difference that these two zeros need not to be real in this case. For instance, let

$$\tilde{N}_{2,\epsilon}(s) := \frac{(s-2)^2 + \varepsilon^2}{(s+2)^2} \quad , \quad \varepsilon > 0 \qquad (14)$$

Then it follows that $\|\tilde{N}_{2,\epsilon} - N_2\|_\infty \to 0$ as $\varepsilon \to 0$ and that $\tilde{N}_{2,\varepsilon}$ has two non-real zeros $s = 2 \pm j\varepsilon$ inside γ_1^*.

A direct generalization can be done with ease. Suppose $N(s_k) = 0$, $s_k \in (0, \infty)$, whose multiplicity is $2p > 0$. Then we can write

$$N = \frac{(s - s_k)^{2p}}{(s+1)^{2p}} M \quad \text{for some } M \in RH_\infty \qquad (15)$$

and choose

$$\tilde{N}_\varepsilon = \frac{(s - s_k)^{2p} + \varepsilon^{2p}}{(s+1)^{2p}} M \ \ (\varepsilon > 0). \qquad (16)$$

Then it holds that $\|\tilde{N}_\varepsilon - N\|_\infty \to 0$ as $\varepsilon \to 0$. Note that each shifted zero of $\tilde{N}_\varepsilon$,

$$\tilde{s}_{\varepsilon,\ell} := s_k \pm \varepsilon \exp\left(\frac{(1+2\ell)\pi}{2p} j \right), \quad \ell = 0, \cdots, p-1 \ (17)$$

satisfies $|\tilde{s}_{\varepsilon,\ell} - s_k| \to 0$ as $\varepsilon \to 0$ for all ℓ but none of them is real unless $\varepsilon = 0$.

Suppose the multiplicity of s_k is an odd number. In this case, the shifted zeros of $\tilde{N}$ inside γ_k^* should be symmetric with respect to the real axis since $\tilde{N}$ is *real* rational. This implies that at least one real zero exists inside γ_k^*.

Lemma 2 *Suppose $N \in RH_\infty$ has positive finite real zeros $0 < s_1 < \cdots < s_n < \infty$. Let's fix paths $\{\gamma_k^*\,;\, k = 0, \cdots, n\}$ as shown in Fig. 2 and choose $\varepsilon > 0$ from Lemma 1. Then, for all $k = 1, \cdots, n$, every $\tilde{N} \in B_h(N, \varepsilon)$ has at least one real zero inside γ_k^* if and only if the multiplicity of s_k is an odd number.*

Now consider the cases where $N(0) = 0$ or $N(\infty) = 0$. Firstly suppose $N(0) = 0$ with multiplicity q. Then we can write $N = s^q/(s+1)^q \cdot M$ for some $M \in RH^\infty$. Let's define $\tilde{N}_\varepsilon = (s+\varepsilon)^q/(s+1)^q \cdot M$ with $\varepsilon > 0$. Then it follows that

$$\left\| \frac{(s+\varepsilon)^q - s^q}{(s+1)^q} \right\|_\infty \le \varepsilon \sum_{k=0}^{q-1} \left\| \frac{s^{q-1-k}(s+\varepsilon)^k}{(s+1)^q} \right\|_\infty \le \varepsilon\, n \left\| \frac{s+\varepsilon}{s+1} \right\|_\infty^q \le \varepsilon\, q. \quad (18)$$

Thus $\|\tilde{N}_\varepsilon - N\|_\infty \to 0$ as $\varepsilon \to 0$ and $\tilde{N}(0) \neq 0$. This shows that the zero $s = 0$ can be shifted to the *negative* real axis with an arbitrary small perturbation regardless of its multiplicity. From the usual mapping

$z = (s-1)/(s+1)$, we can similarly prove that the zero $z = 1$ can be shifted to $z = 1 + \varepsilon$, whose inverse image is outside $\mathbb{C}_+$.

Lemma 3 *Suppose $N \in RH_\infty$, given any $\varepsilon > 0$, we can construct $\tilde{N} \in B_h(N, \varepsilon)$ having the same positive real zeros as N but $\tilde{N}(0) \neq 0$ and $\tilde{N}(\infty) \neq 0$.*

Proof: By a successive application of two zero shiftings: $s = 0$ and $s = \infty$. ■

5 Robustness Problem

5.1 Strong Stabilizable Case

Let $P = ND^{-1}$ be a coprime representation of $P \in R(s)$. The following theorem provides a standard criterion of the strong stabilizability [7, 10].

Theorem 2 *Let $\sigma_1 < \sigma_2 < \cdots < \sigma_l$ be nonnegative possibly infinite ($\sigma_l = \infty$) zeros of N and let ν_i denote the numbers of zeros of D in the interval (σ_i, σ_{i+1}), counting multiplicity. Then P is strongly stabilizable if and only if each ν_i is even for all $i = 1, \cdots, l-1$.*

The described pattern of pole-zeros in Theorem 2 is called as the *parity interlacing property* (p.i.p.).

Fact 1 *Suppose $P \in R(s)$ is strongly stabilizable. Then there exists $\varepsilon > 0$ such that every $\tilde{P} \in B_g(P, \varepsilon)$ is also strongly stabilizable.*

This result can be easily proved from the standard fact that closed loop stability is a robust property [6, 7]. Concretely, we have only to select any *stable* controller for P and then compute the maximum magnitude of perturbation with which the closed stability can be preserved [8].

Our next aim is to develop a new proof of Fact 1 based on our previous developments.

Proof: Let's start with a temporal assumption that $0 < \sigma_1$ and $\sigma_n < \infty$. We choose paths shown in Fig. 3. The path δ_k^* includes all poles in (σ_k, σ_{k+1}) and γ_0^* does not include any complex pole-zeros of P. In addition we assume $D(x_a) \neq 0$ and $D(x_b) \neq 0$. It is obvious that we can choose $\{x_a, x_b, y_m\}$ satisfying these conditions for any $P \in RH_\infty$.

Since the perturbation we are considering is arbitrary small, we may assume that the perturbation of N, say it $\tilde{N}$, satisfies $\tilde{N} \in B_h(N, \varepsilon_1)$ where $\varepsilon_1 > 0$ is decided from Lemma 1 with $\{\gamma_0^*, \cdots, \gamma_n^*\}$. In the same way, we assume $\tilde{D} \in B_h(D, \varepsilon_2)$ where $\varepsilon_2 > 0$ is decided from the paths $\{\gamma_0^*, \delta_1^*, \cdots, \delta_{n-1}^*\}$.

Suppose that the multiplicity of σ_k is odd for all $k = 1, \cdots, n$. From Lemma 2, for every $\tilde{N} \in B_h(N, \varepsilon_1)$, there exists at least one real zero inside γ_k^* for all k. Since P satisfies the p.i.p., for each k, δ_k^* includes an even numbers ($:= \nu_k$ in Theorem 2) of real zeros (poles of P) of D. Now consider the perturbation of D into $\tilde{D} \in B_h(D, \varepsilon_2)$. A key observation here is that, since the shifted zeros inside δ_k^* should be symmetric with respect to the real axis, still there is an even numbers of real zeros inside δ_k^* (even−even=even). This is because the shifted zeros can escape from the real axis only in pair (even). Hence the p.i.p. is conserved for all $\tilde{P} \in B_g(P, \varepsilon)$ with $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.

Suppose the multiplicity of some $\sigma_{1<k^\circ<n}$ is an even number. Then, from Lemma 2, $\gamma_{k^\circ}^*$ may have no real zero. If this happens, the number of real zeros inside $\delta_{k^\circ-1}^*$ and $\delta_{k^\circ}^*$ are added to be an even number again, since each of them is even number for all $\tilde{D} \in B_h(D, \varepsilon_2)$. Thus the p.i.p. *survives* again. The cases where γ_1^* or γ_n^* has no real zeros do not effect on the p.i.p. Thus it follows that small perturbations of P in graph topology does not break the p.i.p. under the assumptions ; $0 < \sigma_1$ and $\sigma_n < \infty$.

Now suppose $\sigma_n = \infty$, i.e. P is strictly proper. In this case we choose $x_b > 0$ strictly greater than the maximum real zero of D. This is always possible because N, D are coprime. Since $D(\infty) \neq 0$, from Lemma 1, we can choose $\varepsilon_3 > 0$ such that every $\tilde{D} \in B_h(D, \varepsilon_3)$ has no zeros in $[x_b, \infty]$. Thus the number of poles (= the zeros of D) in the interval (σ_{n-1}, ∞) is the same with that of (σ_{n-1}, x_b) for all $\tilde{D} \in B_h(D, \varepsilon_3)$. Equipped with this observation, we can resort to the same reasoning as before. Similarly the case $\sigma_1 = 0$ can be dealt. ■

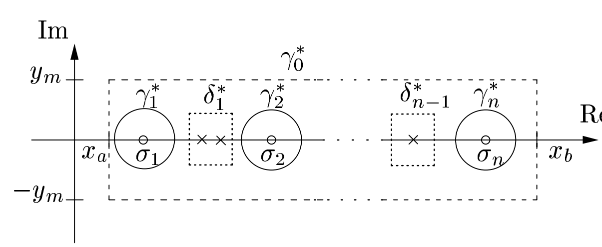

Figure 3: Paths $\{\gamma_{k=0,\cdots,n}^*\}$ and $\{\delta_{k=1,\cdots,n-1}^*\}$

5.2 Non-strongly Stabilizable Case

An essential advantage of our approach is that, with the reasoning developed in the proof of Fact 1, we can tackle the Problem 1. That is, we are considering the possibility that a perturbed system becomes strongly stabilizable by a *clever* pole-zero shifting.

Theorem 3 *Suppose $P = ND^{-1}$ is not strongly stabilizable. Let $0 < \tau_1 < \tau_2 < \cdots < \tau_n < \infty$ be strictly positive finite zeros of N whose multiplicity is odd. We define κ_i as the numbers of poles of P in the interval (τ_i, τ_{i+1}), counting multiplicity. Then there exists $\varepsilon > 0$ such that every $\tilde{P} \in B_g(P, \varepsilon)$ is not strongly stabilizable if and only if κ_i is odd at least one $i \in \{1, \cdots, n-1\}$.*

Proof: For consistence, we use the notations of Fig. 3. With the paths $\{\gamma_k^*, \delta_k^*\}$, we choose a sufficiently small $\varepsilon > 0$ such that both the perturbations $\tilde{N} \in B_h(N, \varepsilon)$ and $\tilde{D} \in B_h(D, \varepsilon)$ satisfy the condition of Lemma 1.

Firstly suppose that the multiplicity of σ_k is odd for all $k = 1, \cdots, n$. Then each $\tilde{N} \in B_h(N, \varepsilon)$ has at least one real zero inside γ_k^* for all k. Fix any $k^\circ \in \{1, \cdots, n-1\}$ such that $\delta_{k^\circ}^*$ includes odd numbers of real zeros of D. Note that this number can be decreased by only an even number because $\tilde{D}$ is real rational. Thus there should be odd number of real zeros inside δ_{k° for all $\tilde{D} \in B_h(D, \varepsilon)$ (odd−even=odd). This implies that the p.i.p. does not hold for all $\tilde{N} \in B_h(N, \varepsilon)$ and $\tilde{D} \in B_h(D, \varepsilon)$.

Let E be the set of all $k \in \{1, \cdots, n\}$ such that the multiplicity of σ_k is even. Choose $\tilde{N}_E \in B_h(N, \varepsilon)$ having no real zeros inside γ_k^* for all $k \in E$ and check the p.i.p. of $\tilde{N}_E D^{-1}$. If the p.i.p. holds, since ε can be arbitrary small, it follows that P can be slightly perturbed to satisfy the p.i.p. in this case.

Now assume $\tilde{N}_E D^{-1}$ does not satisfy the p.i.p. That is, $\tilde{N}_E D^{-1}$ has odd numbers of real poles between γ_{k1}^* and γ_{k2}^* where the multiplicity of both σ_{k1} and σ_{k2} are odd for some $1 \le k_1 < k_2 \le n$. Since this number can change by even numbers for all $\tilde{D} \in B_h(D, \varepsilon)$ from the same reasoning as before, it follows that $\tilde{N}_E \tilde{D}^{-1}$ does not satisfy the p.i.p. either (odd-even=odd).

Finally consider the possibility that a partial restoration of *removable* real zeros of N inside γ_k^* ($k \in E$, $k_1 < k < k_2$) can be helpful for the p.i.p. Suppose that it is, then, in the opposite direction, we come to conclude that a finite addition of even numbers (κ_i of Theorem 3) becomes odd, which is impossible. ∎

At this stage, Theorem 3 and Theorem 2 should be carefully compared. Especially note that we have only to consider the zeros in the *open* interval $(0, \infty)$ whose multiplicity is *odd* number.

Example 3 *Consider two cases shown in Fig. 4 and Fig. 5. Here N_m denotes the multiplicity of order N. Note that both plants do not satisfy the p.i.p. The lower part of each figure shows the pole-zeros of $\tilde{N}_E D^{-1}$ defined in the above proof (unconcerned poles for the p.i.p. are not shown). In the case of Fig. 4, we have $\{\kappa_i\} = \{1\}$ (defined in Theorem 3) and thus we can find some neighborhood whose elements are not strongly stabilizable. However, since $\{\kappa_i\} = \{4\}$ in the second case, this plant can be strongly stabilizable with arbitrary small perturbation.*

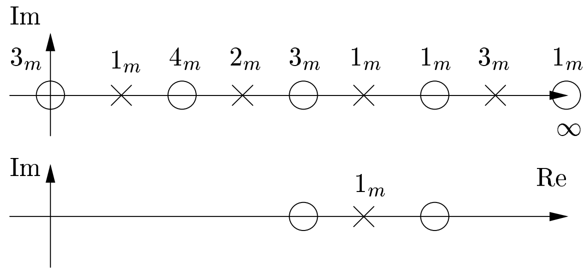

Figure 4: Case 1

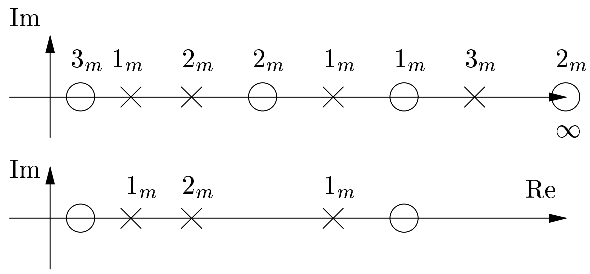

Figure 5: Case 2

Remark 1 *The existence of a non-strongly stabilizable SISO rational plant, which is robust to small perturbations in graph topology, implies that the strong stabilizability is not a* generic *property [9].*

6 Simultaneous Stabilization Problem

Suppose we are given two plants $P_0, P_1 \in R(s)$. The *simultaneous stabilizing* problem concerns with the existence of a common controller $K \in R(s)$ stabilizing both P_0 and P_1 [1, 2, 4, 7].

Let

$$P_0 = N_0 D_0^{-1} \quad \text{and} \quad P_1 = N_1 D_1^{-1} \tag{19}$$

be coprime (within RH_∞) representations and suppose $X_0, Y_0 \in RH_\infty$ satisfy the Bezout identity

$$X_0 N_0 + Y_0 D_0 = 1 \tag{20}$$

It is a standard fact that the pair $\{P_0, P_1\}$ is *simultaneous stabilizable* if and only if the following system;

$$F := BA^{-1}, \quad \begin{bmatrix} B \\ A \end{bmatrix} := \begin{bmatrix} D_0 & -N_0 \\ X_0 & Y_0 \end{bmatrix} \begin{bmatrix} N_1 \\ D_1 \end{bmatrix} \tag{21}$$

is strongly stabilizable.

In this section, we are interested in the next problem;

Problem 2 *Suppose that a plant pair $\{P_0, P_1\} \subset R(s)$ is not simultaneously stabilizable. Is there a slightly perturbed system of P_1 which is now simultaneously stabilizable with P_0 ?*

We believe Problem 2 has importance interpretation not only in theory but in practice. For instance, suppose we are given two plants $\{P_0, P_1\}$ which are not simultaneously stabilizable. Since our system modeling is not perfect in practice, we should ask if this unhappy situation is just a numerical coincidence or a fundamental limitation of our design.

From (21) it holds that

$$\left\| \begin{pmatrix} N_1 \\ D_1 \end{pmatrix} \right\|_\infty \le \left\| \begin{pmatrix} Y_0 & N_0 \\ -X_0 & D_0 \end{pmatrix} \right\|_\infty \left\| \begin{pmatrix} B \\ A \end{pmatrix} \right\|_\infty \tag{22}$$

$$\left\| \begin{pmatrix} B \\ A \end{pmatrix} \right\|_\infty \le \left\| \begin{pmatrix} D_0 & -N_0 \\ X_0 & Y_0 \end{pmatrix} \right\|_\infty \left\| \begin{pmatrix} N_1 \\ D_1 \end{pmatrix} \right\|_\infty \tag{23}$$

From these inequalities, the robustness results of the non-strongly stabilizable plant BA^{-1} can be straightforwardly interpreted for the non-simultaneously stabilizable pair $\{P_0, P_1\}$. Explicitly, as an answer to Problem 2, we have the next results.

Corrollary 2 *The following conditions are equivalent;*

1. *There exists $\varepsilon > 0$ such that every $\tilde{F} \in B_g(F, \varepsilon)$ is not strongly stabilizable.*
2. *There exists $\varepsilon > 0$ such that every $\tilde{P}_1 \in B_g(P_1, \varepsilon)$ is not simultaneously stabilizable with P_0.*

7 Conclusion

We studied on the robustness of non-strongly stabilizable SISO plants in graph topology setting. Developing a new notion of pole-zero shifting, we proved that there are two types of non-strongly stabilizable plants; one is marginal type in the sense that it can be slightly perturbed into a strongly stabilizable plant and the other is essential type since there exists a neighborhood where every plant is not strongly stabilizable. The criterion for this grouping turned out to depend on the strictly positive, finite real pole-zeros of plant. Finally corresponding implications in terms of simultaneous stabilization problem were also presented.

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