Fixed-Structure Controller Synthesis for Real and Complex Multiple Block-Structured Uncertainty

Joseph R. Corrado*, **Wassim M. Haddad**†, **VijaySekhar Chellaboina**‡

* Raytheon Systems Company, 1151 E. Hermans Rd., Bldg. 805, M/S M4, Tucson, AZ 85734-1337
† School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150
‡Mechanical and Aerospace Engineering, University of Missouri-Columbia, Columbia, MO 65211

Abstract

This paper uses a unifying absolute stability result for mixed uncertainty in conjunction with a quasi-Newton numerical optimization routine to obtain fixed-structure controllers and fixed-order stability multipliers which provide robust stability and performance. The robust controller synthesis technique proposed here permits the treatment of fully populated real uncertain blocks which may, in addition, possess internal structure.

1. Introduction

The ability of the structured singular value to account for complex, real, and mixed uncertainty provides a powerful framework for robust stability and performance problems in both analysis and synthesis (see [1] and the numerous references therein). Since exact computation of the structured singular value is, in general, an intractable problem, the development of practically implementable bounds remains a high priority in robust control research. Recent work in this area includes upper and lower bounds for mixed uncertainty [2] as well as LMI-based computational techniques [3].

An alternative approach to developing bounds for the structured singular value is to specialize absolute stability criteria for sector-bounded nonlinearities to the case of linear uncertainty [4]. This approach demonstrates the direct applicability of the classical theory of absolute stability to the modern structured singular value framework. In particular, the rich theory of multiplier-based absolute stability criteria can be seen to have a close and fundamental relationship with recently developed structured singular value bounds.

The objective of this paper is to use the absolute stability results of [4], which unify and extend existing structured singular value bounds for mixed uncertainty, in conjunction with a quasi-Newton numerical optimization routine to obtain fixed-structure controllers and fixed-order stability multipliers which provide robust stability and performance. Using the results of [4], the robust controller synthesis technique proposed here permits the treatment of fully populated real uncertain blocks which may, in addition, possess internal structure. Such problems arise in a variety of applications, such as the study of modal dynamics, in which transformation to 'standard' diagonal form may introduce additional conservatism, computational complexity, as well as destroying the parameter space of the original uncertainty characterization [4]. The ability to address real uncertain blocks is based on the use of an appropriate class of multipliers whose structure is compatible with the real block uncertainty. Hence, tailoring the multipliers to the structure of the uncertainty not only leads to the ability to address more general uncertainty characterizations but can also lead to less conservative controllers than obtained from the standard mixed-μ synthesis techniques. This more general class of multipliers has no counterpart in standard mixed-μ theory.

2. Absolute Stability Criterion with Generalized Positive Real Stability Multipliers

In this section we review the absolute stability criterion for multivariable systems with generalized positive real stability multipliers given in [4]. This criterion involves a square nominal (open-loop or feedback) transfer function $G(s)$ in a negative feedback interconnection with a complex, square, uncertain matrix Δ. Here, we consider the set of block-diagonal matrices with possibly repeated blocks defined by

$$\begin{aligned}\boldsymbol{\Delta}_{\rm bs} \triangleq \{\Delta \in \mathbb{C}^{p\times p}\colon \Delta = \text{block-diag}\,[I_{\psi_1}\otimes\Delta_1^r,\ldots,I_{\psi_r}\otimes\Delta_r^r,\\ I_{\psi_{r+1}}\otimes\Delta_{r+1}^c,\ldots,I_{\psi_{r+c}}\otimes\Delta_{r+c}^c]\,; \Delta_i^r\in\mathbb{R}^{p_i\times p_i},\\ i=1,\ldots,r; \Delta_i^c\in\mathbb{C}^{p_i\times p_i}, i=r+1,\ldots,r+c\},\end{aligned}$$

where the dimension p_i of each block and the number of repetitions ψ_i of each block are given such that $\sum_{i=1}^{v}\psi_i p_i = p$, where $v = r+c$ is the number of *distinct* uncertain blocks and where $\otimes$ denotes the Kronecker product. Furthermore, define the subset $\boldsymbol{\Delta}\subseteq\boldsymbol{\Delta}_{\rm bs}$ consisting of sector-bounded matrices

$$\begin{aligned}\boldsymbol{\Delta} \triangleq \{\Delta\in\boldsymbol{\Delta}_{\rm bs} : 2(\Delta-M_1)^*(M_2-M_1)^{-1}(\Delta-M_1)\\ \le (\Delta-M_1)+(\Delta-M_1)^*\},\end{aligned}$$

where $M_1, M_2\in\boldsymbol{\Delta}_{\rm bs}$ are Hermitian matrices such that $M\triangleq M_2-M_1$ is positive definite. Note that M_1 and M_2 are elements of $\boldsymbol{\Delta}$. Alternate characterizations of $\boldsymbol{\Delta}$ are given in [4].

To draw connections with the structured singular value for real and complex block-structured uncertainty, we specialize the set $\boldsymbol{\Delta}$ to the case of norm-bounded, internally block-structured uncertainty. Specifically, by letting $M_1=-\gamma^{-1}I$ and $M_2=\gamma^{-1}I$, where $\gamma>0$, it follows that $M=2\gamma^{-1}I$ so that $M^{-1}=\frac{1}{2}\gamma I$. In this case, $\boldsymbol{\Delta}$ becomes

$$\begin{aligned}\boldsymbol{\Delta}_\gamma = \{\Delta\in\boldsymbol{\Delta}_{\rm bs} : \gamma(\Delta+\gamma^{-1}I)^*(\Delta+\gamma^{-1}I)\\ \le (\Delta+\gamma^{-1}I)+(\Delta+\gamma^{-1}I)^*\}.\end{aligned}$$

Now, $\Delta\in\boldsymbol{\Delta}_\gamma$ if and only if $\sigma_{\max}(\Delta)\le\gamma^{-1}$. Therefore, $\boldsymbol{\Delta}_\gamma$ is given by $\boldsymbol{\Delta}_\gamma=\{\Delta\in\boldsymbol{\Delta}_{\rm bs} : \sigma_{\max}(\Delta)\le\gamma^{-1}\}$.

Next we give the multivariable absolute stability criterion for sector-bounded uncertain matrices. To state

---

This research was supported in part by NASA under Grant NGT 4-52405, NSF under Grant ECS-9496249, and AFOSR under Grant F49620-96-1-0125.

this criterion, we define the sets $\mathcal{D}$ and $\mathcal{N}$ of Hermitian rational scaling matrix functions by

$$\begin{aligned}\mathcal{D} &\triangleq \{D : \mathbb{C} \to \mathbb{C}^{p\times p} \,:\, D(j\omega) \geq 0,\ D(j\omega)\Delta = \Delta D(j\omega),\\ &\qquad \omega \in \mathbb{R},\ \Delta \in \boldsymbol{\Delta}_{\rm bs}\},\\ \mathcal{N} &\triangleq \{N : \mathbb{C} \to \mathbb{C}^{p\times p} \,:\, N(j\omega) = N^*(j\omega),\\ &\qquad N(j\omega)\Delta = \Delta^* N(j\omega),\ \omega \in \mathbb{R},\ \Delta \in \boldsymbol{\Delta}_{\rm bs}\}.\end{aligned}$$

Furthermore, define the set $\mathcal{Z}$ of rational multiplier functions by

$$\begin{aligned}\mathcal{Z} &\triangleq \{Z : \mathbb{C} \to \mathbb{C}^{q\times q} \,:\, Z(j\omega) = D(j\omega) - j\omega N(j\omega),\\ &\qquad D(\cdot) \in \mathcal{D},\ N(\cdot) \in \mathcal{N}\}.\end{aligned}$$

Note that if $Z(\cdot) \in \mathcal{Z}$, $D(\cdot) \in \mathcal{D}$, and $N(\cdot) \in \mathcal{N}$, then $Z(j\omega) = D(j\omega) - j\omega N(j\omega)$ if and only if $D(j\omega) = \text{He}\ Z(j\omega)$ and $N(j\omega) = \frac{-1}{j\omega}\text{Sh}\ Z(j\omega)$, $\omega \neq 0$, where He and Sh denote the Hermitian and skew-Hermitian part of a complex matrix. Hence, since $D(j\omega) \geq 0$, $\omega \in \mathbb{R} \cup \infty$, $Z(\cdot) \in \mathcal{Z}$ consists of generalized positive real functions.

We now state our main robust stability result.

Theorem 2.1 [4]. Suppose $(I + G(s)M_1)^{-1}G(s)$ is asymptotically stable. If there exists $Z(\cdot) \in \mathcal{Z}$ such that

$$\text{He}\,[Z(s)(M^{-1} + (I + G(s)M_1)^{-1}G(s))] > 0, \qquad (1)$$

for all $s = j\omega$, $\omega \in \mathbb{R} \cup \infty$, then the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \boldsymbol{\Delta}$.

3. Stability Multiplier Structure

To ensure the commutability of the rational scaling matrix functions $D(s)$ and $N(s)$ with the uncertainty set $\boldsymbol{\Delta}$, we must structure $Z(s) = C_{\rm m}(sI - A_{\rm m})^{-1}B_{\rm m} + D_{\rm m}$ so that $Z(s) \in \mathcal{Z}$. To assure this, we construct the multiplier $Z(s)$ from the constituent multipliers $D(s)$ and $N(s)$. Furthermore, instead of obtaining a realization for $N(s)$, we obtain a realization for $sN(s)$ directly since $Z(s) = D(s) - sN(s)$. We therefore choose multiplier realizations $D(s) \sim \left[\begin{array}{c|c} A_{\rm d} & B_{\rm d} \\ \hline C_{\rm d} & D_{\rm d}\end{array}\right]$ and $N(s) \sim \left[\begin{array}{c|c} A_{\rm n} & B_{\rm n} \\ \hline C_{\rm n} & 0\end{array}\right]$, where $A_{\rm d} \in \mathcal{S}_{A_{\rm d}}$, $A_{\rm n} \in \mathcal{S}_{A_{\rm n}}$, $B_{\rm d} \in \mathcal{S}_{B_{\rm d}}$, $B_{\rm n} \in \mathcal{S}_{B_{\rm n}}$, $C_{\rm d} \in \mathcal{S}_{C_{\rm d}}$, $C_{\rm n} \in \mathcal{S}_{C_{\rm n}}$, and $D_{\rm d} \in \mathcal{S}_{D_{\rm d}}$. The sets $\mathcal{S}_{A_{\rm d}}$, $\mathcal{S}_{A_{\rm n}}$, $\mathcal{S}_{B_{\rm d}}$, $\mathcal{S}_{B_{\rm n}}$, $\mathcal{S}_{C_{\rm d}}$, $\mathcal{S}_{C_{\rm n}}$, and $\mathcal{S}_{D_{\rm d}}$ are chosen to enforce the diagonal structure of $D(s)$ and $N(s)$ given by

$$\begin{aligned}D(s) &= C_{\rm d}(sI - A_{\rm d})^{-1}B_{\rm d} + D_{\rm d}\\ &= \text{block-diag}\,[D_1(s) \otimes I_{p_1}, \ldots, D_v(s) \otimes I_{p_v}],\\ N(s) &= C_{\rm n}(sI - A_{\rm d})^{-1}B_{\rm n}\\ &= \text{block-diag}\,[N_1(s) \otimes I_{p_1}, \ldots, N_v(s) \otimes I_{p_v}],\end{aligned}$$

where $0 \leq D_i(s) \in \mathbb{C}^{\psi_i \times \psi_i}$, $i = 1, \ldots, v$, and $N_i(s) = N_i^*(s) \in \mathbb{C}^{\psi_i \times \psi_i}$, $N_i(s)\Delta_i = N_i(s)\Delta_i^*$, $i = 1, \ldots, v$. The structure chosen for the rational functions representing $D(s)$ and $N(s)$ in this paper is similar to the one for the curve-fitting operation in standard μ-synthesis [5]. In particular, we define $\mathcal{S}_{A_{\rm d}}$, $\mathcal{S}_{A_{\rm n}}$, $\mathcal{S}_{B_{\rm d}}$, $\mathcal{S}_{B_{\rm n}}$, $\mathcal{S}_{C_{\rm d}}$, $\mathcal{S}_{C_{\rm n}}$, and $\mathcal{S}_{D_{\rm d}}$ as follows:

$$\begin{aligned}\mathcal{S}_{A_{\rm d}} \triangleq \{&A_{\rm d} = \text{block-diag}\,[I_{\psi_1 p_1} \otimes A_{{\rm d}_1}, \ldots, I_{\psi_v p_v} \otimes A_{{\rm d}_v}] :\\ &A_{{\rm d}_i} \in \mathbb{R}^{n_{{\rm d}_i} \times n_{{\rm d}_i}}; A_{{\rm d}_i} \text{ is in controllable}\\ &\text{canonical companion form with the}\\ &1^{\rm st} \text{ row} = \begin{bmatrix}0 & a_{{\rm d}_{i,2}} & 0 & a_{{\rm d}_{i,4}} & \cdots & 0 & a_{{\rm d}_{i,n_{{\rm d}_i}}}\end{bmatrix};\\ &\text{sgn}\ a_{{\rm d}_{i,j}} = (-1)^{(j/2)+1}; i = 1, \ldots, v; j = 1, \ldots, n_{{\rm d}_i}\}, \qquad (2)\\ \mathcal{S}_{A_{\rm n}} \triangleq \{&A_{\rm n} = \text{block-diag}\,[I_{\psi_1 p_1} \otimes A_{{\rm n}_1}, \ldots, I_{\psi_v p_v} \otimes A_{{\rm n}_v}] :\\ &A_{{\rm n}_i} \in \mathbb{R}^{n_{{\rm n}_i} \times n_{{\rm n}_i}}; A_{{\rm n}_i} \text{ is in controllable}\\ &\text{canonical companion form with the}\\ &1^{\rm st} \text{ row} = \begin{bmatrix}0 & a_{{\rm n}_{i,2}} & 0 & a_{{\rm n}_{i,4}} & \cdots & 0 & a_{{\rm n}_{i,n_{{\rm n}_i}}}\end{bmatrix};\\ &i = 1, \ldots, v\}, \qquad (3)\\ \mathcal{S}_{B_{\rm d}} \triangleq \{&B_{\rm d} = \text{block-diag}\,[I_{\psi_1 p_1} \otimes B_{{\rm d}_1}, \ldots, I_{\psi_v p_v} \otimes B_{{\rm d}_s}] :\\ &B_{{\rm d}_i} \in \mathbb{R}^{n_{{\rm d}_i} \times 1}; B_{{\rm d}_i}^{\rm T} = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}; i = 1, \ldots, v\}, \qquad (4)\\ \mathcal{S}_{B_{\rm n}} \triangleq \{&B_{\rm n} = \text{block-diag}\,[I_{\psi_1 p_1} \otimes B_{{\rm n}_1}, \ldots, I_{\psi_v p_v} \otimes B_{{\rm n}_v}] :\\ &B_{{\rm n}_i} \in \mathbb{R}^{n_{{\rm n}_i} \times 1}; B_{{\rm n}_i}^{\rm T} = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}; i = 1, \ldots, v\}, \qquad (5)\\ \mathcal{S}_{C_{\rm d}} \triangleq \{&C_{\rm d} = \text{block-diag}\,[I_{\psi_1 p_1} \otimes C_{{\rm d}_1}, \ldots, I_{\psi_v p_v} \otimes C_{{\rm d}_v}] :\\ &C_{{\rm d}_i} \in \mathbb{R}^{1 \times n_{{\rm d}_i}}; C_{{\rm d}_i} = \begin{bmatrix}0 & c_{{\rm d}_{i,2}} & 0 & c_{{\rm d}_{i,4}} & \cdots & 0 & c_{{\rm d}_{i,n_{{\rm d}_i}}}\end{bmatrix};\\ &\text{sgn}\ c_{{\rm d}_{i,j}} = (-1)^{j/2}; i = 1, \ldots, v; j = 1, \ldots, n_{{\rm d}_i}\}, \qquad (6)\\ \mathcal{S}_{C_{\rm n}} \triangleq \{&C_{\rm n} = [\text{block-diag}\,[I_{\psi_1 p_1} \otimes C_{{\rm n}_1}, \ldots, I_{\psi_v p_v} \otimes C_{{\rm n}_v}]] :\\ &C_{{\rm n}_i} \in \mathbb{R}^{1 \times n_{{\rm n}_i}}; C_{{\rm n}_i} = \begin{bmatrix}c_{{\rm n}_{i,1}} & 0 & c_{{\rm n}_{i,3}} & \cdots & 0 & c_{{\rm n}_{i,n_{{\rm n}_i - 1}}} & 0\end{bmatrix};\\ &i = 1, \ldots, v\}, \qquad (7)\\ \mathcal{S}_{D_{\rm d}} \triangleq \{&D_{\rm d} = \text{block-diag}\,[d_{{\rm d}_1} I_{\psi_1 p_1}, \ldots, d_{{\rm d}_v} I_{\psi_v p_v}] :\\ &d_{{\rm d}_i} \in \mathbb{R}; d_{{\rm d}_i} > 0; i = 1, \ldots, v\}, \qquad (8)\end{aligned}$$

where $n_{{\rm d}_i}$ and $n_{{\rm n}_i}$ are the *a priori* fixed even orders of the rational functions representing the $i^{\rm th}$ diagonal element of the multipliers $D(s)$ and $N(s)$, respectively. Thus we see that $n_{\rm d} \triangleq \sum_{i=1}^{v} n_{{\rm d}_i}\psi_i$ and $n_{\rm n} \triangleq \sum_{i=1}^{v} n_{{\rm n}_i}\psi_i$ are the total number of states describing $D(s)$ and $N(s)$, respectively.

To construct $Z(s) = D(s) - sN(s)$, we obtain the augmented realizations

$$A_{\rm m} = \text{block-diag}\,[A_{\rm d}, A_{\rm n}], \quad B_{\rm m} = \begin{bmatrix} B_{\rm d} \\ B_{\rm n}\end{bmatrix}, \qquad (9)$$

$$C_{\rm m} = \begin{bmatrix} C_{\rm d} & -C_{\rm n}\end{bmatrix}, \qquad D_{\rm m} = D_{\rm d}, \qquad (10)$$

where $A_{\rm d} \in \mathcal{S}_{A_{\rm d}}$, $A_{\rm n} \in \mathcal{S}_{A_{\rm n}}$, $B_{\rm d} \in \mathcal{S}_{B_{\rm d}}$, $B_{\rm n} \in \mathcal{S}_{B_{\rm n}}$, $C_{\rm d} \in \mathcal{S}_{C_{\rm d}}$, $C_{\rm n} \in \mathcal{S}_{C_{\rm n}}$, and $D_{\rm d} \in \mathcal{S}_{D_{\rm d}}$. Note that $A_{\rm m} \in \mathbb{R}^{n_{\rm m} \times n_{\rm m}}$, where $n_{\rm m} = n_{\rm d} + n_{\rm n}$. Furthermore note that there is no contribution to $D_{\rm m}$ from the $sN(s)$ term. This is due to the fact that the rational function $N(s)$ is strictly proper and has only even powers of s. Thus $N(s)$ necessarily has a relative degree of two and hence $sN(s)$ is strictly proper.

Next, we note that with $Z(s) \sim \left[\begin{array}{c|c} A_{\rm m} & B_{\rm m} \\ \hline C_{\rm m} & D_{\rm m}\end{array}\right]$, as defined in (9)–(10), we obtain the necessary commutability properties, as well as ensuring that $D(s) > 0$ and

$N(s) = N^*(s)$. However, to satisfy the condition $N(s)\Delta = \Delta^* N(s)$, we require that the i^{th} block of $N(s)$ be zero whenever the i^{th} uncertainty block Δ_i is complex. Thus, in the case where $\Delta_i \in \mathbb{C}^{p_i \times p_i}$, we require that the realization of $N_i(s)$ be given by $A_{n_i} = [\ \ ]_{0\times 0}$, $B_{n_i} = [\ ]_{0\times 1}$, $C_{n_i} = [\ ]_{1\times 0}$, where $[\ ]_{0\times j}$ is the $0 \times j$ empty matrix [6]. Thus, in this case, the i^{th} block of $N(s)$ is given by $N_i(s) = C_{\mathrm{n}_i}(sI - A_{\mathrm{n}_i})^{-1}B_{\mathrm{n}_i} = 0$.

Finally, we note that the stability multipliers cannot have arbitrary realizations and still be elements of their appropriate sets $\mathcal{S}_{A_{\mathrm{d}}}$, $\mathcal{S}_{A_{\mathrm{n}}}$, $\mathcal{S}_{B_{\mathrm{d}}}$, $\mathcal{S}_{B_{\mathrm{n}}}$, $\mathcal{S}_{C_{\mathrm{d}}}$, $\mathcal{S}_{C_{\mathrm{n}}}$, and $\mathcal{S}_{D_{\mathrm{d}}}$. Thus we recast the stability multiplier matrices so that the free parameters appear along the diagonal of a separate matrix, $\mathcal{K}_{\mathrm{m}}$. The stability multipliers can then be constructed as

$$\begin{aligned} A_{\mathrm{m}} &= A_{\mathrm{mC}} + A_{\mathrm{mL}}\mathcal{K}_{\mathrm{m}}A_{\mathrm{mR}}, & B_{\mathrm{m}} &= B_{\mathrm{mC}}, \\ C_{\mathrm{m}} &= C_{\mathrm{mL}}\mathcal{K}_{\mathrm{m}}C_{\mathrm{mR}}, & D_{\mathrm{m}} &= D_{\mathrm{mL}}\mathcal{K}_{\mathrm{m}}D_{\mathrm{mR}}, \end{aligned}$$

where the matrices A_{mC}, A_{mL}, A_{mR}, B_{mC}, C_{mL}, C_{mR}, D_{mL}, and D_{mR} are structured appropriately. To illustrate the structure of these matrices, consider the scalar multiplier

$$Z(s) = d_{d_1} + \frac{c_{d_{1,2}}}{s^2 - a_{d_{1,2}}} - \frac{c_{n_{1,1}}s}{s^2 - a_{n_{1,2}}}.$$

The gain and structure matrices for the stability multiplier with $n_{\mathrm{n}} = n_{\mathrm{d}} = 2$ are then given by

$$\mathcal{K}_{\mathrm{m}} = \begin{bmatrix} a_{d_{1,2}} & 0 & 0 & 0 & 0 \\ 0 & c_{d_{1,2}} & 0 & 0 & 0 \\ 0 & 0 & d_{d_1} & 0 & 0 \\ 0 & 0 & 0 & a_{n_{1,2}} & 0 \\ 0 & 0 & 0 & 0 & c_{n_{1,1}} \end{bmatrix},$$

$$A_{\mathrm{mC}} = \begin{bmatrix} 0\,0\,0\,0 \\ 1\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,1\,0 \end{bmatrix},\ B_{\mathrm{mC}} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ A_{\mathrm{mL}} = \begin{bmatrix} 1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0 \end{bmatrix},$$

$$C_{\mathrm{mL}} = \begin{bmatrix} 0\ 1\ 0\ 0\ -1 \end{bmatrix},\quad D_{\mathrm{mL}} = \begin{bmatrix} 0\ 0\ 1\ 0\ 0 \end{bmatrix},$$

$$A_{\mathrm{mR}} = \begin{bmatrix} 0\,1\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,0\,0 \end{bmatrix},\ C_{\mathrm{mR}} = \begin{bmatrix} 0\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,1\,0 \end{bmatrix},\ D_{\mathrm{mR}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

With these definitions, we see that

$$A_{\mathrm{m}} = \begin{bmatrix} 0 & a_{d_{1,2}} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{n_{1,2}} \\ 0 & 0 & 1 & 0 \end{bmatrix},\quad B_{\mathrm{m}} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C_{\mathrm{m}} = \begin{bmatrix} 0 & c_{d_{1,2}} & -c_{n_{1,1}} & 0 \end{bmatrix},\quad D_{\mathrm{m}} = \begin{bmatrix} d_{d_1} \end{bmatrix},$$

and thus

$$\begin{aligned} Z(s) &= C_{\mathrm{m}}(sI - A_{\mathrm{m}})^{-1}B_{\mathrm{m}} + D_{\mathrm{m}} \\ &= d_{d_1} + \frac{c_{d_{1,2}}}{s^2 - a_{d_{1,2}}} - \frac{c_{n_{1,1}}s}{s^2 - a_{n_{1,2}}}. \end{aligned}$$

4. Decentralized Static Output Feedback Formulation

In this section we review the decentralized static output feedback problem formulation for fixed-structure controller synthesis [7]. Consider the $(m+q+1)$-vector-input, $(m+q+1)$-vector-output decentralized system shown in Figure 1, where e_i and d_i, $i = 1,\ldots,q$, are used to account for model uncertainty, w is the exogenous disturbance input, z is the performance variable, and the signals y_i and u_i, $i = 1,\ldots,m$, are measurement and control signals, respectively. The decentralized static output feedback multi vector-input, multi vector-output system shown in Figure 1 is characterized by the dynamics

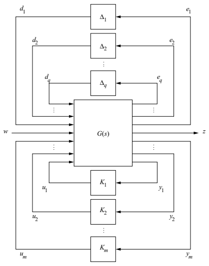

Figure 1: Decentralized Static Output Feedback Framework

$$\dot{\tilde{x}}(t) = \mathcal{A}\tilde{x}(t) + \sum_{j=1}^{m}\mathcal{B}_{u_j}u_j(t) + \sum_{k=1}^{q}\mathcal{B}_{d_k}d_k(t) + \mathcal{B}_w w(t), \tag{11}$$

$$y_i(t) = \mathcal{C}_{y_i}\tilde{x}(t) + \sum_{j=1}^{m}\mathcal{D}_{yu_{ij}}u_j(t) + \sum_{k=1}^{q}\mathcal{D}_{yd_{ik}}d_k(t) + \mathcal{D}_{yw_i}w(t),$$
$$i = 1,2,\ldots,m, \tag{12}$$

$$e_i(t) = \mathcal{C}_{e_i}\tilde{x}(t) + \sum_{j=1}^{m}\mathcal{D}_{eu_{ij}}u_j(t) + \sum_{k=1}^{q}\mathcal{D}_{ed_{ik}}d_k(t) + \mathcal{D}_{ew_i}w(t),$$
$$i = 1,2,\ldots,q, \tag{13}$$

$$z(t) = \mathcal{C}_z\tilde{x}(t) + \sum_{j=1}^{m}\mathcal{D}_{zu_j}u_j(t) + \sum_{k=1}^{q}\mathcal{D}_{zd_k}d_k(t) + \mathcal{D}_{zw}w(t). \tag{14}$$

In the above formulation, model uncertainty is represented by the decentralized static output feedback map

$$d_i(t) = \Delta_i e_i(t), \quad i = 1,\ldots,q, \tag{15}$$

where the uncertain matrices Δ_i are not necessarily distinct. To represent decentralized static output feedback control with possibly repeated gains, we consider

$$u_i(t) = \mathcal{K}_i y_i(t), \quad i = 1,\ldots,m, \tag{16}$$

where the matrices $\mathcal{K}_i$ are not necessarily distinct. Reordering the variables in (15) and (16) if necessary and defining

$$\hat{u}(t) = \left[u_1^{\mathrm{T}}(t)\ \cdots\ u_m^{\mathrm{T}}(t)\right]^{\mathrm{T}},\ \hat{y}(t) = \left[y_1^{\mathrm{T}}(t)\ \cdots\ y_m^{\mathrm{T}}(t)\right]^{\mathrm{T}}, \tag{17}$$

$$\hat{d}(t) = \left[d_1^{\mathrm{T}}(t)\ \cdots\ d_q^{\mathrm{T}}(t)\right]^{\mathrm{T}},\ \hat{e}(t) = \left[e_1^{\mathrm{T}}(t)\ \cdots\ e_q^{\mathrm{T}}(t)\right]^{\mathrm{T}}, \tag{18}$$

$$\mathcal{B}_u \triangleq \left[\mathcal{B}_{u_1}\ \cdots\ \mathcal{B}_{u_m}\right],\quad \mathcal{B}_d \triangleq \left[\mathcal{B}_{d_1}\ \cdots\ \mathcal{B}_{d_q}\right], \tag{19}$$

$$\mathcal{D}_{zu} \triangleq \left[\mathcal{D}_{zu_1}\ \cdots\ \mathcal{D}_{zu_m}\right],\ \mathcal{D}_{zd} \triangleq \left[\mathcal{D}_{zd_1}\ \cdots\ \mathcal{D}_{zd_q}\right], \tag{20}$$

$$\mathcal{C}_y \triangleq \begin{bmatrix} \mathcal{C}_{y_1} \\ \vdots \\ \mathcal{C}_{y_m} \end{bmatrix}, \ \mathcal{D}_{yu} \triangleq \begin{bmatrix} \mathcal{D}_{yu_{11}} & \cdots & \mathcal{D}_{yu_{1m}} \\ \vdots & \ddots & \vdots \\ \mathcal{D}_{yu_{m1}} & \cdots & \mathcal{D}_{yu_{mm}} \end{bmatrix}, \tag{21}$$

$$\mathcal{C}_e \triangleq \begin{bmatrix} \mathcal{C}_{e_1} \\ \vdots \\ \mathcal{C}_{e_q} \end{bmatrix}, \quad \mathcal{D}_{eu} \triangleq \begin{bmatrix} \mathcal{D}_{eu_{11}} & \cdots & \mathcal{D}_{eu_{1m}} \\ \vdots & \ddots & \vdots \\ \mathcal{D}_{eu_{q1}} & \cdots & \mathcal{D}_{eu_{qm}} \end{bmatrix}, \tag{22}$$

$$\mathcal{D}_{yd} \triangleq \begin{bmatrix} \mathcal{D}_{yd_{11}} & \cdots & \mathcal{D}_{yd_{1q}} \\ \vdots & \ddots & \vdots \\ \mathcal{D}_{yd_{m1}} & \cdots & \mathcal{D}_{yd_{mq}} \end{bmatrix}, \quad \mathcal{D}_{yw} \triangleq \begin{bmatrix} \mathcal{D}_{yw_1} \\ \vdots \\ \mathcal{D}_{yw_m} \end{bmatrix}, \tag{23}$$

$$\mathcal{D}_{ed} \triangleq \begin{bmatrix} \mathcal{D}_{ed_{11}} & \cdots & \mathcal{D}_{ed_{1q}} \\ \vdots & \ddots & \vdots \\ \mathcal{D}_{ed_{q1}} & \cdots & \mathcal{D}_{ed_{qq}} \end{bmatrix}, \quad \mathcal{D}_{ew} \triangleq \begin{bmatrix} \mathcal{D}_{ew_1} \\ \vdots \\ \mathcal{D}_{ew_q} \end{bmatrix}, \tag{24}$$

(15) and (16) can be rewritten

$$\hat{d}(t) = \Delta \hat{e}(t), \quad \hat{u}(t) = \mathcal{K}\hat{y}(t), \tag{25}$$

where Δ and $\mathcal{K}$ have the form

$$\Delta \triangleq \text{block-diag}\left[I_{\psi_1} \otimes \Delta_1, \ldots, I_{\psi_v} \otimes \Delta_v\right], \tag{26}$$

$$\mathcal{K} \triangleq \text{block-diag}\left[I_{\phi_1} \otimes \mathcal{K}_1, \ldots, I_{\phi_g} \otimes \mathcal{K}_g\right], \tag{27}$$

where v is the number of *distinct* uncertainties $\Delta_i \in \mathbb{C}^{p_i \times p_i}$ or $\mathbb{R}^{p_i \times p_i}$, ψ_i is the number of repetitions of uncertainty Δ_i, g is the number of *distinct* gains $\mathcal{K}_i \in \mathbb{R}^{r_i \times c_i}$ and ϕ_i is the number of repetitions of gain $\mathcal{K}_i$. Note that $\mathcal{K}_1, \ldots, \mathcal{K}_g$ are not necessarily square matrices, and $\sum_{i=1}^{v} \psi_i = q$ and $\sum_{i=1}^{g} \phi_i = m$.

With the definitions in (17)–(24), the transfer function $G(s)$ from $[\hat{u}^{\mathrm{T}}, \hat{d}^{\mathrm{T}}, w^{\mathrm{T}}]^{\mathrm{T}}$ to $[\hat{y}^{\mathrm{T}}, \hat{e}^{\mathrm{T}}, z^{\mathrm{T}}]^{\mathrm{T}}$ of the decentralized system has the realization

$$G(s) \sim \left[\begin{array}{c|c:c:c} \mathcal{A} & \mathcal{B}_u & \mathcal{B}_d & \mathcal{B}_w \\ \hline \mathcal{C}_y & \mathcal{D}_{yu} & \mathcal{D}_{yd} & \mathcal{D}_{yw} \\ \hdashline \mathcal{C}_e & \mathcal{D}_{eu} & \mathcal{D}_{ed} & \mathcal{D}_{ew} \\ \hdashline \mathcal{C}_z & \mathcal{D}_{zu} & \mathcal{D}_{zd} & \mathcal{D}_{zw} \end{array}\right], \tag{28}$$

which represents the linear, time-invariant dynamic system

$$\dot{\tilde{x}}(t) = \mathcal{A}\tilde{x}(t) + \mathcal{B}_u\hat{u}(t) + \mathcal{B}_d\hat{d}(t) + \mathcal{B}_w w(t), \tag{29}$$

$$\hat{y}(t) = \mathcal{C}_y\tilde{x}(t) + \mathcal{D}_{yu}\hat{u}(t) + \mathcal{D}_{yd}\hat{d}(t) + \mathcal{D}_{yw}w(t), \tag{30}$$

$$\hat{e}(t) = \mathcal{C}_e\tilde{x}(t) + \mathcal{D}_{eu}\hat{u}(t) + \mathcal{D}_{ed}\hat{d}(t) + \mathcal{D}_{ew}w(t), \tag{31}$$

$$z(t) = \mathcal{C}_z\tilde{x}(t) + \mathcal{D}_{zu}\hat{u}(t) + \mathcal{D}_{zd}\hat{d}(t) + \mathcal{D}_{zw}w(t), \tag{32}$$

which is equivalent to (11)–(14). Furthermore, by rewriting the decentralized control signals (16) in the compact form given by (25), the closed-loop system realization from $[\hat{d}^{\mathrm{T}}, w^{\mathrm{T}}]^{\mathrm{T}}$ to $[\hat{e}^{\mathrm{T}}, z^{\mathrm{T}}]^{\mathrm{T}}$ is given by

$$\tilde{G}(s) \sim \left[\begin{array}{c|c:c} \tilde{A} & \tilde{B}_0 & \tilde{D} \\ \hline \tilde{C}_0 & \tilde{D}_0 & \tilde{D}_1 \\ \hdashline \tilde{E} & \tilde{E}_1 & \tilde{E}_0 \end{array}\right], \tag{33}$$

where

$$\begin{aligned} \tilde{A} &\triangleq \mathcal{A} + \mathcal{B}_u\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{C}_y, & \tilde{B}_0 &\triangleq \mathcal{B}_d + \mathcal{B}_u\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yd}, \\ \tilde{D} &\triangleq \mathcal{B}_w + \mathcal{B}_u\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yw}, & \tilde{C}_0 &\triangleq \mathcal{C}_e + \mathcal{D}_{eu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{C}_y, \\ \tilde{D}_0 &\triangleq \mathcal{D}_{ed} + \mathcal{D}_{eu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yd}, & \tilde{D}_1 &\triangleq \mathcal{D}_{ew} + \mathcal{D}_{eu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yw}, \\ \tilde{E} &\triangleq \mathcal{C}_z + \mathcal{D}_{zu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{C}_y, & \tilde{E}_1 &\triangleq \mathcal{D}_{zd} + \mathcal{D}_{zu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yd}, \\ \tilde{E}_0 &\triangleq \mathcal{D}_{zw} + \mathcal{D}_{zu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yw}, \end{aligned}$$

and where $L_{\mathcal{K}} \triangleq I - \mathcal{D}_{yu}\mathcal{K}$. Note that we assume $\det(L_{\mathcal{K}}) \neq 0$ for all $\mathcal{K}$ given by (27) to ensure the well-posedness of the feedback interconnection.

Finally, given the closed-loop system realization given by (33) with $\tilde{D}_0 \equiv 0$, $\tilde{D}_1 \equiv 0$, $\tilde{E}_1 \equiv 0$, and $\tilde{E}_0 \equiv 0$, and the multiplier realization given by $Z(s) \sim \left[\begin{array}{c|c} A_{\mathrm{m}} & B_{\mathrm{m}} \\ \hline C_{\mathrm{m}} & D_{\mathrm{m}} \end{array}\right]$, the realization of $\hat{G}(s) \triangleq Z(s)(M^{-1} + (I + \tilde{G}_{ed}(s)M_1)^{-1}\tilde{G}_{ed}(s))$, where $\tilde{G}_{ed}(s)$ is the closed-loop transfer function from uncertainty inputs $\hat{d}(t)$ to uncertainty signals $\hat{e}(t)$, is given by $\hat{G}(s) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B}_0 \\ \hline \hat{C}_0 & \hat{D}_0 \end{array}\right]$, where

$$\begin{aligned} \hat{A} &\triangleq \begin{bmatrix} \tilde{A} - \tilde{B}_0 M_1 \tilde{C}_0 & 0 \\ B_{\mathrm{m}}\tilde{C}_0 & A_{\mathrm{m}} \end{bmatrix}, & \hat{B}_0 &\triangleq \begin{bmatrix} \tilde{B}_0 \\ B_{\mathrm{m}}M^{-1} \end{bmatrix}, \\ \hat{C}_0 &\triangleq \begin{bmatrix} D_{\mathrm{m}}\tilde{C}_0 & C_{\mathrm{m}} \end{bmatrix}, & \hat{D}_0 &\triangleq \begin{bmatrix} D_{\mathrm{m}}M^{-1} \end{bmatrix}. \end{aligned}$$

5. Specialization to Centralized Strictly Proper Dynamic Compensation

Consider the uncertain dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) + B_0 d(t) + D_1 w(t), \tag{34}$$

$$y(t) = Cx(t) + Du(t) + F_1 d(t) + D_2 w(t), \tag{35}$$

$$e(t) = C_0 x(t) + F_2 u(t), \tag{36}$$

$$z(t) = E_1 x(t) + E_2 u(t), \tag{37}$$

with uncertain plant perturbations $\Delta A = B_0 \Delta C_0$, $\Delta B = B_0 \Delta F_2$, $\Delta C = F_1 \Delta C_0$, $\Delta D = F_1 \Delta F_2$, of the nominal system matrices (A, B, C, D).

The dynamics of the centralized, strictly proper controller are given by

$$\dot{x}_{\mathrm{c}}(t) = A_{\mathrm{c}} x_{\mathrm{c}}(t) + B_{\mathrm{c}} y(t), \tag{38}$$

$$u(t) = C_{\mathrm{c}} x_{\mathrm{c}}(t), \tag{39}$$

so that the closed-loop system can be written as

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}_0 d(t) + \tilde{D}w(t), \tag{40}$$

$$e(t) = \tilde{C}_0 \tilde{x}(t), \tag{41}$$

$$z(t) = \tilde{E}\tilde{x}(t), \tag{42}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_{\mathrm{c}}(t) \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A & BC_{\mathrm{c}} \\ B_{\mathrm{c}}C & A_{\mathrm{c}} + B_{\mathrm{c}}DC_{\mathrm{c}} \end{bmatrix},$$

$$\tilde{B}_0 \triangleq \begin{bmatrix} B_0 \\ B_{\mathrm{c}}F_1 \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_{\mathrm{c}}D_2 \end{bmatrix},$$

$$\tilde{C}_0 \triangleq \begin{bmatrix} C_0 & F_2C_{\mathrm{c}} \end{bmatrix}, \quad \tilde{E} \triangleq \begin{bmatrix} E_1 & E_2C_{\mathrm{c}} \end{bmatrix}.$$

Writing this system in the decentralized static output feedback framework, we obtain

$$\dot{\tilde{x}}(t) = \mathcal{A}\tilde{x}(t) + \sum_{j=1}^{3} \mathcal{B}_{u_j} u_j(t) + \mathcal{B}_d d(t) + \mathcal{B}_w w(t), \tag{43}$$

$$y_i(t) = \mathcal{C}_{y_i}\tilde{x}(t) + \sum_{j=1}^{3} \mathcal{D}_{yu_{ij}} u_j(t) + \mathcal{D}_{yd_i} d(t) + \mathcal{D}_{yw_i} w(t), \quad i = 1, 2, 3, \tag{44}$$

$$e(t) = \mathcal{C}_e\tilde{x}(t) + \sum_{j=1}^{3} \mathcal{D}_{eu_j} u_j(t), \tag{45}$$

$$z(t) = \mathcal{C}_z\tilde{x}(t) + \sum_{j=1}^{3} \mathcal{D}_{zu_j} u_j(t), \tag{46}$$

and $u_1(t) = A_{\mathrm{c}} y_1(t)$, $u_2(t) = B_{\mathrm{c}} y_2(t)$, $u_3(t) = C_{\mathrm{c}} y_3(t)$, where

$$\mathcal{A} \triangleq \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_{u1} \triangleq \begin{bmatrix} 0 \\ I_{n_{\mathrm{c}}} \end{bmatrix}, \mathcal{B}_{u2} \triangleq \begin{bmatrix} 0 \\ I_{n_{\mathrm{c}}} \end{bmatrix}, \mathcal{B}_{u3} \triangleq \begin{bmatrix} B \\ 0 \end{bmatrix},$$
$$\mathcal{B}_d \triangleq \begin{bmatrix} B_0 \\ 0 \end{bmatrix}, \quad \mathcal{B}_w \triangleq \begin{bmatrix} D_1 \\ 0 \end{bmatrix},$$
$$\mathcal{C}_{y_1} \triangleq \begin{bmatrix} 0 & I_{n_{\mathrm{c}}} \end{bmatrix}, \quad \mathcal{D}_{yu_{11}} \triangleq 0, \quad \mathcal{D}_{yu_{12}} \triangleq 0, \quad \mathcal{D}_{yu_{13}} \triangleq 0,$$
$$\mathcal{D}_{yd_1} \triangleq 0, \quad \mathcal{D}_{yw_1} \triangleq 0,$$
$$\mathcal{C}_{y_2} \triangleq \begin{bmatrix} C & 0 \end{bmatrix}, \quad \mathcal{D}_{yu_{21}} \triangleq 0, \quad \mathcal{D}_{yu_{22}} \triangleq 0, \quad \mathcal{D}_{yu_{23}} \triangleq D,$$
$$\mathcal{D}_{yd_2} \triangleq F_1, \quad \mathcal{D}_{yw_2} \triangleq D_2,$$
$$\mathcal{C}_{y_3} \triangleq \begin{bmatrix} 0 & I_{n_{\mathrm{c}}} \end{bmatrix}, \quad \mathcal{D}_{yu_{31}} \triangleq 0, \quad \mathcal{D}_{yu_{32}} \triangleq 0, \quad \mathcal{D}_{yu_{33}} \triangleq 0,$$
$$\mathcal{D}_{yd_3} \triangleq 0, \quad \mathcal{D}_{yw_3} \triangleq 0,$$
$$\mathcal{C}_e \triangleq \begin{bmatrix} C_0 & 0 \end{bmatrix}, \quad \mathcal{D}_{eu1} \triangleq 0, \quad \mathcal{D}_{eu2} \triangleq 0, \quad \mathcal{D}_{eu3} \triangleq F_2,$$
$$\mathcal{C}_z \triangleq \begin{bmatrix} E_1 & 0 \end{bmatrix}, \quad \mathcal{D}_{zu1} \triangleq 0, \quad \mathcal{D}_{zu2} \triangleq 0, \quad \mathcal{D}_{zu3} \triangleq E_2.$$

Next, defining

$$\hat{u}(t) \triangleq \begin{bmatrix} u_1^{\mathrm{T}}(t) & u_2^{\mathrm{T}}(t) & u_3^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}, \quad \hat{y}(t) \triangleq \begin{bmatrix} y_1^{\mathrm{T}}(t) & y_2^{\mathrm{T}}(t) & y_3^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}},$$

$$\mathcal{B}_u \triangleq \begin{bmatrix} \mathcal{B}_{u1} & \mathcal{B}_{u2} & \mathcal{B}_{u3} \end{bmatrix}, \quad \mathcal{D}_{eu} \triangleq \begin{bmatrix} \mathcal{D}_{eu1} & \mathcal{D}_{eu2} & \mathcal{D}_{eu3} \end{bmatrix},$$
$$\mathcal{D}_{zu} \triangleq \begin{bmatrix} \mathcal{D}_{zu1} & \mathcal{D}_{zu2} & \mathcal{D}_{zu3} \end{bmatrix},$$
$$\mathcal{C}_y \triangleq \begin{bmatrix} \mathcal{C}_{y_1} \\ \mathcal{C}_{y_2} \\ \mathcal{C}_{y_3} \end{bmatrix}, \quad \mathcal{D}_{yu} \triangleq \begin{bmatrix} \mathcal{D}_{yu_{11}} & \mathcal{D}_{yu_{12}} & \mathcal{D}_{yu_{13}} \\ \mathcal{D}_{yu_{21}} & \mathcal{D}_{yu_{22}} & \mathcal{D}_{yu_{23}} \\ \mathcal{D}_{yu_{31}} & \mathcal{D}_{yu_{32}} & \mathcal{D}_{yu_{33}} \end{bmatrix},$$
$$\mathcal{D}_{yd} \triangleq \begin{bmatrix} \mathcal{D}_{yd_1} \\ \mathcal{D}_{yd_2} \\ \mathcal{D}_{yd_3} \end{bmatrix}, \quad \mathcal{D}_{yw} \triangleq \begin{bmatrix} \mathcal{D}_{yw_1} \\ \mathcal{D}_{yw_2} \\ \mathcal{D}_{yw_3} \end{bmatrix},$$

and rewriting the decentralized control signals in the compact form $\hat{u}(t) = \mathcal{K}\hat{y}(t)$, where

$$\mathcal{K} \triangleq \begin{bmatrix} A_{\mathrm{c}} & 0 & 0 \\ 0 & B_{\mathrm{c}} & 0 \\ 0 & 0 & C_{\mathrm{c}} \end{bmatrix},$$

the system matrices $\tilde{A}$, $\tilde{B}_0$, $\tilde{D}$, $\tilde{C}_0$, and $\tilde{E}$ in the closed-loop dynamics

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}_0 d(t) + \tilde{D}w(t), \tag{47}$$
$$e(t) = \tilde{C}_0\tilde{x}(t), \tag{48}$$
$$z(t) = \tilde{E}\tilde{x}(t), \tag{49}$$

can now be written as

$$\tilde{A} = \mathcal{A} + \mathcal{B}_u\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{C}_y = \begin{bmatrix} A & BC_{\mathrm{c}} \\ B_{\mathrm{c}}C & A_{\mathrm{c}} + B_{\mathrm{c}}DC_{\mathrm{c}} \end{bmatrix},$$
$$\tilde{B}_0 = \mathcal{B}_w + \mathcal{B}_u\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yw} = \begin{bmatrix} B_0 \\ B_{\mathrm{c}}F_1 \end{bmatrix},$$
$$\tilde{D} = \mathcal{B}_w + \mathcal{B}_u\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yw} = \begin{bmatrix} D_1 \\ B_{\mathrm{c}}D_2 \end{bmatrix},$$
$$\tilde{C}_0 = \mathcal{C}_z + \mathcal{D}_{zu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{C}_y = \begin{bmatrix} C_0 & F_2C_{\mathrm{c}} \end{bmatrix},$$
$$\tilde{E} = \mathcal{C}_z + \mathcal{D}_{zu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{C}_y = \begin{bmatrix} E_1 & E_2C_{\mathrm{c}} \end{bmatrix}.$$

Furthermore, closing the uncertainty loop from $d(t)$ to $e(t)$ yields the closed-loop realization

$$\tilde{G}_{\Delta}(s) \sim \left[\begin{array}{c|c} \tilde{A} + \tilde{B}_0\Delta\tilde{C}_0 & \tilde{D} \\ \hline \tilde{E} & 0 \end{array}\right].$$

6. Robust Stability and Performance

In this section we present sufficient conditions for robust stability of $\tilde{G}(s)$ and a robust $\mathcal{H}_2$ performance bound for $\|\tilde{G}_{zw}(s)\|_2^2$. For the statement of the next theorem define the notation $\hat{E} \triangleq \begin{bmatrix} \tilde{E} & 0 \end{bmatrix}$, $\hat{D} \triangleq \begin{bmatrix} \tilde{D} \\ 0 \end{bmatrix}$, and recall the definition for a strongly positive real transfer function.

Theorem 6.1. Suppose there exists an $\tilde{n}+n_{\mathrm{m}} \times \tilde{n}+n_{\mathrm{m}}$ nonnegative definite matrix $\hat{P}$ satisfying

$$0 = \hat{A}^{\mathrm{T}}\hat{P} + \hat{P}\hat{A} + (\hat{B}_0^{\mathrm{T}}\hat{P} - \hat{C}_0)(\hat{D}_0 + \hat{D}_0^{\mathrm{T}})^{-1}(\hat{B}_0^{\mathrm{T}}\hat{P} - \hat{C}_0)^{\mathrm{T}} + \hat{E}^{\mathrm{T}}\hat{E}. \tag{50}$$

Then $Z(s)[M^{-1} + [I + \tilde{G}_{ed}(s)M_1]^{-1}\tilde{G}_{ed}(s)]$ is strongly positive real. Consequently, the feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \boldsymbol{\Delta}$. Furthermore, $\|\tilde{G}_{zw}(s)\|_2^2 \leq \operatorname{tr} \hat{P}\hat{D}\hat{D}^{\mathrm{T}}$.

To apply Theorem 6.1 to controller synthesis, we use the modified Riccati equation (50) to guarantee that the closed-loop system is robustly stable. This leads to the following optimization problem.

Optimization Problem. Determine gain matrices $\mathcal{K} \in \mathbb{R}^{\sum_{i=1}^{g} r_i \times \sum_{i=1}^{g} c_i}$ and $\mathcal{K}_{\mathrm{m}} \in \mathbb{R}^{(n_{\mathrm{m}}+v)\times(n_{\mathrm{m}}+v)}$ that minimize $\mathcal{J}(\mathcal{K}, \mathcal{K}_{\mathrm{m}}) \triangleq \operatorname{tr} \hat{P}\hat{D}\hat{D}^{\mathrm{T}}$, where $\hat{P}$ is an $\hat{n} \times \hat{n}$ nonnegative definite matrix satisfying (50), and where $\hat{n} \triangleq \tilde{n} + n_{\mathrm{m}}$.

7. Sufficient Conditions for Fixed-Order Robust Compensation with Dynamic Multipliers

In this section we state sufficient conditions for characterizing dynamic output feedback controllers and dynamic stability multipliers guaranteeing robust stability and robust $\mathcal{H}_2$ performance. For the statement of the next theorem, partition the matrices $\hat{P}$ and $\hat{Q}$ as

$$\hat{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix}, \qquad \hat{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{bmatrix},$$

where P_{11}, Q_{11} are $\tilde{n} \times \tilde{n}$ nonnegative definite matrices and P_{22}, Q_{22} are $n_{\mathrm{m}} \times n_{\mathrm{m}}$ nonnegative definite matrices, and define

$$Q_{\mathrm{L}_{ij}} \triangleq \begin{bmatrix} 0_{r_1\phi_1 \times r_i} \\ 0_{r_2\phi_2 \times r_i} \\ \vdots \\ 0_{r_{i-1}\phi_{i-1} \times r_i} \\ 0_{r_i(j-1) \times r_i} \\ I_{r_i} \\ 0_{r_i(\phi_i - j) \times r_i} \\ 0_{r_{i+1}\phi_{i+1} \times r_i} \\ \vdots \\ 0_{r_v\phi_v \times r_i} \end{bmatrix}, \quad Q_{\mathrm{R}_{ij}} \triangleq \begin{bmatrix} 0_{c_1\phi_1 \times c_i} \\ 0_{c_2\phi_2 \times c_i} \\ \vdots \\ 0_{c_{i-1}\phi_{i-1} \times c_i} \\ 0_{c_i(j-1) \times c_i} \\ I_{c_i} \\ 0_{c_i(\phi_i - j) \times c_i} \\ 0_{c_{i+1}\phi_{i+1} \times c_i} \\ \vdots \\ 0_{c_v\phi_v \times c_i} \end{bmatrix}^{\mathrm{T}}, \qquad (51)$$

where r_i and c_i are the dimensions of the i^{th} controller gain, $\mathcal{K}_i \in \mathbb{R}^{r_i \times c_i}$, $i = 1, \ldots, v$, and $j = 1, \ldots, \phi_i$.

Theorem 7.1. Suppose there exists $\hat{n} \times \hat{n}$ nonnegative definite matrices $\hat{P}$ and $\hat{Q}$ satisfying

$$\begin{aligned} 0 = {} & \hat{A}^{\mathrm{T}}\hat{P} + \hat{P}\hat{A} + (\hat{B}_0^{\mathrm{T}}\hat{P} - \hat{C}_0)(\hat{D}_0 + \hat{D}_0^{\mathrm{T}})^{-1}(\hat{B}_0^{\mathrm{T}}\hat{P} - \hat{C}_0)^{\mathrm{T}} \\ & + \hat{E}^{\mathrm{T}}\hat{E}, \qquad (52) \\ 0 = {} & \left(\hat{A} - \hat{B}_0(\hat{D}_0 + \hat{D}_0^{\mathrm{T}})^{-1}(\hat{B}_0^{\mathrm{T}}\hat{P} - \hat{C}_0)\right)\hat{Q} \\ & + \hat{Q}\left(\hat{A} - \hat{B}_0(\hat{D}_0 + \hat{D}_0^{\mathrm{T}})^{-1}(\hat{B}_0^{\mathrm{T}}\hat{P} - \hat{C}_0)\right)^{\mathrm{T}} + \hat{D}\hat{D}^{\mathrm{T}}, \qquad (53) \end{aligned}$$

and let $\mathcal{K}_i$ satisfy

$$\begin{aligned} 0 = {} & 2\sum_{j=1}^{\phi_i} Q_{\mathrm{L}_{ij}}^{\mathrm{T}} \left(I + \mathcal{D}_{yu}^{\mathrm{T}} L_{\mathcal{K}}^{-\mathrm{T}} \mathcal{K}^{\mathrm{T}}\right) \Big[\mathcal{B}_u^{\mathrm{T}}(P_1 Q_1 + P_{12} Q_{12}^{\mathrm{T}}) \\ & \times (\mathcal{C}_y^{\mathrm{T}} - \tilde{C}_0^{\mathrm{T}} M_1 \mathcal{D}_{yd}^{\mathrm{T}}) + \mathcal{B}_u^{\mathrm{T}} P_1 \tilde{B}_w \mathcal{D}_{yw}^{\mathrm{T}} + \mathcal{D}_{zu}^{\mathrm{T}} \tilde{C}_z Q_1 \mathcal{C}_y^{\mathrm{T}} \\ & + \mathcal{D}_{eu}^{\mathrm{T}}[B_{\mathrm{m}}^{\mathrm{T}}(P_{12}^{\mathrm{T}} Q_1 + P_2 Q_{12}^{\mathrm{T}}) - M_1 \tilde{B}_d^{\mathrm{T}}(P_1 Q_1 + P_{12} Q_{12}^{\mathrm{T}})]\mathcal{C}_y^{\mathrm{T}} \\ & + \mathcal{B}_u^{\mathrm{T}}[(P_1 Q_1 + P_{12} Q_{12}^{\mathrm{T}})(P_1 \tilde{B}_d + P_{12} B_{\mathrm{m}} M^{-1} - \tilde{C}_e^{\mathrm{T}} D_{\mathrm{m}}) \\ & + (P_1 Q_{12} + P_{12} Q_2)(P_{12}^{\mathrm{T}} \tilde{B}_d + P_2 B_{\mathrm{m}} M^{-1} - C_{\mathrm{m}}^{\mathrm{T}})] \\ & \times (D_{\mathrm{m}} M^{-1} + M^{-1} D_{\mathrm{m}})^{-1} \mathcal{D}_{yd}^{\mathrm{T}} \\ & - \mathcal{D}_{eu}^{\mathrm{T}} D_{\mathrm{m}} (D_{\mathrm{m}} M^{-1} + M^{-1} D_{\mathrm{m}})^{-1} \\ & \times [(P_1 \tilde{B}_d + P_{12} B_{\mathrm{m}} M^{-1} - \tilde{C}_e^{\mathrm{T}} D_{\mathrm{m}})^{\mathrm{T}} Q_1 \\ & + (P_{12}^{\mathrm{T}} \tilde{B}_d + P_2 B_{\mathrm{m}} M^{-1} - C_{\mathrm{m}}^{\mathrm{T}})^{\mathrm{T}} Q_{12}^{\mathrm{T}}]\mathcal{C}_y^{\mathrm{T}}\Big] L_{\mathcal{K}}^{-\mathrm{T}} Q_{\mathrm{R}_{ij}}^{\mathrm{T}}. \qquad (54) \end{aligned}$$

Furthermore, let $\mathcal{K}_{\mathrm{m}}$ satisfy

$$\begin{aligned} 0 = {} & \mathrm{diag}\{2A_{\mathrm{mL}}^{\mathrm{T}}(P_{12}^{\mathrm{T}} Q_{12} + P_2 Q_2) A_{\mathrm{mR}}^{\mathrm{T}} \\ & - 2C_{\mathrm{mL}}^{\mathrm{T}}(D_{\mathrm{m}} M^{-1} + M^{-1} D_{\mathrm{m}})^{-1}(T_1^{\mathrm{T}} Q_{12} + T_2^{\mathrm{T}} Q_2) C_{\mathrm{mR}}^{\mathrm{T}} \\ & - 2D_{\mathrm{mL}}^{\mathrm{T}}(D_{\mathrm{m}} M^{-1} + M^{-1} D_{\mathrm{m}})^{-1}(T_1^{\mathrm{T}} Q_1 + T_2^{\mathrm{T}} Q_{12}^{\mathrm{T}}) \tilde{C}_e^{\mathrm{T}} D_{\mathrm{mR}}^{\mathrm{T}} \\ & - D_{\mathrm{mL}}^{\mathrm{T}}(D_{\mathrm{m}} M^{-1} + M^{-1} D_{\mathrm{m}})^{-1}(M^{-1} T_1^{\mathrm{T}} Q_1 T_1 \\ & \quad + T_1^{\mathrm{T}} Q_1 T_1 M^{-1} + M^{-1} T_2^{\mathrm{T}} Q_2 T_2 + T_2^{\mathrm{T}} Q_2 T_2 M^{-1} \\ & \quad + 2M^{-1} T_1^{\mathrm{T}} Q_{12} T_2 + 2T_2^{\mathrm{T}} Q_{12}^{\mathrm{T}} T_1 M^{-1}) \\ & \times (D_{\mathrm{m}} M^{-1} + M^{-1} D_{\mathrm{m}})^{-1} D_{\mathrm{mR}}^{\mathrm{T}}\}, \qquad (55) \end{aligned}$$

where $T_1 \triangleq P_1 \tilde{B}_d + P_{12} B_{\mathrm{m}} M^{-1} - \tilde{C}_e^{\mathrm{T}} D_{\mathrm{m}}$, and $T_2 \triangleq P_{12}^{\mathrm{T}} \tilde{B}_d + P_2 B_{\mathrm{m}} M^{-1} - C_{\mathrm{m}}^{\mathrm{T}}$. Then $Z(s)[M^{-1} + [I + \tilde{G}_{ed}(s) M_1]^{-1} \tilde{G}_{ed}(s)]$ is strongly positive real. Thus the closed-loop system from $w(t)$ to $z(t)$ is asymptotically stable for all $\Delta \in \boldsymbol{\Delta}$. Furthermore, the worst-case $\mathcal{H}_2$ performance of the closed-loop system satisfies the bound $\|\tilde{G}_{zw}(s)\|_2^2 \leq \mathrm{tr}\, \hat{P}\hat{D}\hat{D}^{\mathrm{T}}$.

Equations (52)–(55) provide constructive sufficient conditions that yield dynamic controllers for robust fixed-order (i.e., full- and reduced-order) output feedback compensation. By using these equations within a numerical optimization algorithm, the optimal robust fixed-order controllers and stability multipliers can be determined *simultaneously*, thus avoiding D, $N - K$ iterations.

8. Quasi-Newton Optimization Algorithm

To solve the Robust Stability and Performance Problem posed in Section 6, a general-purpose BFGS quasi-Newton algorithm is used. The line-search portions of the algorithm were modified to include a constraint-checking subroutine which decreases the length of the search direction vector until it lies entirely within the set of parameters that yield a stable closed-loop system. This modification ensures that the cost function $\mathcal{J}$ remains defined at every point in the line-search process. Numerical experience indicates that this subroutine is usually invoked only during the first few iterations of a synthesis problem, though this number increases as the uncertainty bounds M_1 and M_2 increase.

References

[1] A. Packard and J. Doyle, "The Complex Structured Singular Value," *Automatica*, vol. 29, pp. 71–109, 1993.

[2] L. Lee and A. L. Tits, "Linear Fractional Transformations for the Approximation of Various Uncertainty Sets," in *Control of Uncertain Dynamic Systems,* edited by S. Bhattacharyya and L. Keel, pp. 53–62, CRC Press, 1991.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory,* vol. 15 of *Studies in Applied Mathematics.* SIAM, 1994.

[4] W. M. Haddad, D. S. Bernstein, and V. Chellaboina, "Generalized Mixed-μ Bounds for Real and Complex Multiple-Block Uncertainty with Internal Matrix Structure," *Int. J. Contr.*, vol. 64, pp. 789–806, 1996.

[5] G. J. Balas, J. C. Doyle, K. Glover, A. Packard, and R. Smith, *μ-Analysis and Synthesis Toolbox: Matlab Functions for the Analysis and Design of Robust Control Systems.* Natick, MA, The MathWorks, 1991.

[6] C. N. Nett and W. M. Haddad, "A System-Theoretic Appropriate Realization of the Empty Matrix Concept," *IEEE Trans. Autom. Contr.*, vol. 38, pp. 771–775, 1993.

[7] D. S. Bernstein, W. M. Haddad, and C. N. Nett, "Minimal Complexity Control Law Synthesis, Part 2: Problem Solution via $\mathcal{H}_2/\mathcal{H}_\infty$ Optimal Static Output Feedback," in *Proc. Amer. Contr. Conf.*, Pittsburgh, PA, pp. 2506–2511, June 1989.