Control design using Jordan controllable canonical form

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Abstract

In this paper, we present a new control design strategy for a class of single-input nonlinear dynamical systems. The design consists in transforming the system into a new controllable canonical form which we call the *Jordan controllable canonical form (JCCF).* In fact, the Brunowski controllable canonical form is an special case of the JCCF. We first show that any controllable pair can be transformed into the JCCF. We next, extend the result to a controllable pair which is state dependent. Using this extended Jordan controllable canonical form we propose a control design strategy for a class of single-input control affine systems. The design is simple and systematic and provides two degrees of freedom to fix the convergence of the closed-loop system. An example is given to illustrate the control design.

1 Introduction

During the last few decades, a considerable number of research efforts has been dedicated to control design for nonlinear systems. Consequently, several techniques has appeared on the scene, namely, feedback linearization, passivity, Lyapunov design and fairly recently the back-stepping (see list of references herein). Most of these techniques relies on purely differential geometric approaches since the latter is well suited to deal with nonlinear systems ([5], [6], [8], [9]). However, some of these techniques, namely the feedback linearisation, may lead to highly complicated control laws and their implementation is sometimes questionable or simply unthinkable. On the other hand, the back-stepping approach attempts to reduce such complexity. Roughly speaking, its basic idea is to keep the "good" terms of the system while cancelling the "bad" terms in an attempt to keep the control law as simple as possible (see [8]). The good terms are those which helps in the stabilsation of the system while the bad terms are those which are susceptible to lead to instability.

In this paper, we present a new controller design strategy for a class of nonlinear systems. The design is an attempt to employ linear design techniques to nonlinear control design. It consists in transforming the system into a new controllable canonical form which we call the *Jordan controllable canonical form (JCCF).* In fact, the Brunowski controllable canonical form [3], is an special case of the JCCF. We first show that any controllable pair can be transformed into the JCCF. An interesting feature of the JCCF is that it provides additional degrees of freedom to tune the controller. We next, extend the result to a controllable pair which is state dependent. Using this extended JCCF we propose a controller for a class of control affine systems. The design is simple and systematic and requires only some matrix computations and, as a result, avoids the use of differential geometric tools such as Lie derivatives. The price to be paid while using such linear techniques is that the resulting control law is of high gain type in general, except for few special cases.

The paper is organised as follows: In the next section, we present some preliminary results concerning controllability pairs and their equivalence. In Section 3, we show the construction of a linear transformation which permit to transform a controllable pair into the JCCF. In Section 4, the JCCF is used in the special case of linear controllable systems to illustrate the design strategy adopted in the paper. In Section 5, we extend the design strategy to a class of nonlinear systems. Finally, an example is given to illustrate the proposed control design.

2 SOME PRELIMINARIES

In this section, we are going to recall some preliminary results, on the equivalence of controllable systems, which are used in the subsequent sections.

Definition 1 *The pair of matrices* (F, G) *where* $F \in \mathcal{R}^{n\times n}$ *and* $G \in \mathcal{R}^{n\times 1}$ *is said to be controllable if the rank of the matrix*

$$U_c = \left[G,\ FG,\ \ldots,\ F^{n-1}G\right]$$

is equal to n.

Definition 2 *Let (F, G) be a controllable pair. Then, the pair $(\bar{F}, \bar{G})$ is said to be an equivalent controllable pair of (F, G) if there exists a nonsingular matrix P_c such that $\bar{F} = P_c F P_c^{-1}$ and $\bar{G} = P_c G$.*

An important result on controllable equivalent pairs is given in [3]:

Lemma 1 [3] *Let (F, G) and $(\bar{F}, \bar{G})$, with $F, \bar{F} \in \mathcal{R}^{n\times n}$ and $G, \bar{G} \in \mathcal{R}^{n\times 1}$, be two equivalent controllable pairs. Then, the matrix P_c such that $\bar{F} = P_c F P_c^{-1}$ and $\bar{G} = P_c G$ is given by*

$$P_c = Y_c U_c^{-1} \tag{1}$$

where $U_c = \left[G, FG, \ldots, F^{n-1}G\right]$ and $Y_c = \left[\bar{G}, \bar{F}\bar{G}, \ldots, \bar{F}^{n-1}\bar{G}\right]$.

Remark 1.

1. If two controllable pairs (F, G) and $(\bar{F}, \bar{G})$ are equivalent then necessarily the characteristic equation of F and $\bar{F}$ are equal; i.e. $\det\left[\lambda I_n - F\right] = \det\left[\lambda I_n - \bar{F}\right]$ where I_n is the $n \times n$ identity matrix.
2. For the formula (1) to be applicable the pairs (F, G) and $(\bar{F}, \bar{G})$ must be *known*.

3 The Jordan controllable canonical form

It is well-known that any controllable pair (F, G); $F \in \mathcal{R}^{n\times n}$ and $G \in \mathcal{R}^{n\times 1}$ is equivalent to the Brunowski controllable pair $(A + BL,\ B)$ where

$$\begin{gathered} A = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ 0 & & \ldots & 0 \end{pmatrix} \\ L = [-a_n, -a_{n-1}, \ldots, -a_1] \\ B = col\left(\begin{array}{cccc} 0, & \cdots, & 0, & 1 \end{array}\right) \end{gathered} \tag{2}$$

and the a_i's are the coefficients of the characteristic equation

$$\det\left[\lambda I_n - F\right] = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n.$$

This comes from the fact that for any $L = \left[\begin{array}{cccc} l_n, & l_{n-1}, & \cdots, & l_1 \end{array}\right] \in \mathcal{R}^n$, we have

$$\det\left[\lambda I_n - (A + BL)\right] = \lambda^n - l_1\lambda^{n-1} - \cdots - l_{n-1}\lambda - l_n.$$

Consequently, if (F, G) is equivalent to $(A + BL,\ B)$, then (see Remark 1)

$$l_i = -a_i. \tag{3}$$

We can generalise the above result to a more general controllable pair defined by $(A_{\delta\gamma} + BL,\ B)$ where

$$A_{\delta\gamma} = \begin{pmatrix} \delta & \gamma & 0 & \cdots & 0 \\ 0 & \delta & \gamma & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \delta & \gamma \\ 0 & \cdots & \cdots & 0 & \delta \end{pmatrix} \tag{4}$$

and δ and $\gamma \neq 0$ are arbitrary constants, B is as in (2) and L is some $1 \times n$ vector. We call such a pair a *Jordan controllable canonical pair.*

Lemma 2. *Let (F, H) ; $F \in \mathcal{R}^{n\times n}$, $G \in \mathcal{R}^{n\times 1}$ be a controllable pair and U_c its corresponding controllability matrix*

$$U_c = \left[G, FG, \ldots, F^{n-1}G\right] \tag{5}$$

Let $(A_{\delta\gamma} + BL,\ B)$ be a Jordan controllable canonical pair, with $L = [l_n, l_{n-1}, ..., l_1]$ and let W_J be its corresponding controllability matrix

$$W_J = \left[B,\ (A_{\delta\gamma} + BL)B,\ \ldots,\ (A_{\delta\gamma} + BL)^{n-1}B\right]. \tag{6}$$

Then, the matrix defined by

$$M_{\delta\gamma} = W_J U_c^{-1} \tag{7}$$

satisfies

$$M_{\delta\gamma} F M_{\delta\gamma}^{-1} = A_{\delta\gamma} + BL \ \text{ and } M_{\delta\gamma} G = B \tag{8}$$

if

$$l_k = -\frac{1}{\gamma^{k-1}}\left[C_n^k \delta^k + \sum_{i=1}^{k} a_i C_{n-i}^{k-i} \delta^{k-i}\right] \tag{9}$$

with $C_n^k = \frac{n!}{(n-k)!k!}$ and the a_i's are the coefficients the characteristic equation of F

$$\det\left[\lambda I_n - F\right] = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n.$$

In other words, the pair (F, G) is equivalent to the pair $(A_{\delta\gamma} + BL,\ B)$ if the entries of L is as in (9).

Proof

First of all, it is easy to check that the pair $(A_{\delta\gamma} + BL,\ B)$ is controllable for every vector $L = \left[\begin{array}{cccc} l_n & l_{n-1} & \cdots & l_1 \end{array}\right]$. Indeed, the controllability matrix W_J is lower triangular and its diagonal entries $W_J(i, i) = \gamma^{i-1}$. Therefore, the determinant of W_J is independent of L and is nonzero since $\gamma \neq 0$, *i.e.* W_J is nonsingular. Notice that $A_{\delta\gamma} = \delta I_n + \gamma A$ where A is as in (2). It can be checked that for any $L = \left[\begin{array}{cccc} l_n & l_{n-1} & \cdots & l_1 \end{array}\right] \in \mathcal{R}^n$, $\det\left[\lambda I_n - (\gamma A + BL)\right] = \lambda^n - l_1\lambda^{n-1} - \cdots - \gamma^{n-2} l_{n-1}\lambda - \gamma^{n-1} l_n$.

Consequently, since $\det\left[\lambda I_n - (A_{\delta\gamma} + BL)\right] = \det\left[(\lambda - \delta) I_n - (\gamma A + BL)\right]$ we have

$\det[\lambda I_n - (A_{\delta\gamma}+BL)] = \bar{\lambda}^n - l_1\bar{\lambda}^{n-1} - \cdots - \gamma^{n-2}l_{n-1}\bar{\lambda} - \gamma^{n-1}l_n$ where $\bar{\lambda} = \lambda - \delta$.

Now,

$$\begin{aligned}\det[\lambda I_n - F] &= \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n \\ &= (\bar{\lambda}+\delta)^n + a_1(\bar{\lambda}+\delta)^{n-1} + \\ &\quad \cdots + a_{n-1}(\bar{\lambda}+\delta) + a_n\end{aligned}$$

Using the binomial expansion of $(\bar{\lambda}+\delta)^{n-i}$ one can show that

$$\det[\lambda I_n - F] = \bar{\lambda}^n + r_1\bar{\lambda}^{n-1} + \cdots + r_{n-1}\bar{\lambda} + r_n$$

where

$$\begin{aligned}r_k &= C_n^k\delta^k + a_1C_{n-1}^{k-1}\delta^{k-1} + a_2C_{n-2}^{k-2}\delta^{k-2} + \\ &\quad \cdots + a_iC_{n-i}^{k-i}\delta^{k-i} + \cdots + a_k \\ &= C_n^k\delta^k + \sum_{i=1}^{k} a_iC_{n-i}^{k-i}\delta^{k-i}.\end{aligned}$$

As noted in Remark 1, if (F,G) is equivalent to $A_{\delta\gamma} + BL,\ B)$ then, the characteristics equations of F and $A_{\delta\gamma} + BL$ are the same. It is then easy to see that $\det[\lambda I_n - F] = \det[\lambda I_n - (A_{\delta\gamma}+BL)]$ if

$$l_k = -\frac{r_k}{\gamma^{k-1}} = -\frac{1}{\gamma^{k-1}}\left[C_n^k\delta^k + \sum_{i=1}^{k} a_iC_{n-i}^{k-i}\delta^{k-i}\right].$$

□

Remark 2.

1. In the above lemma, the Brunowski canonical controllable form consists of the particular case where $\delta = 0$ and $\gamma = 1$. In such a case $l_k = -a_k$ which is consistent to (3).
2. We have deliberately denoted the elements of the vector $L = [l_n, l_{n-1}, \ldots, l_1]$ in the reverse order so as to keep the formula (9) as simple as possible.

Now, since the determinant of the controllability matrix W_J defined in (6) is independent of the vector L, this implies that we can allows L to be state dependent, *i.e.* $L := L(x(t))$ for $x \in \mathcal{R}^n$. This in turn implies that we can allow F and G to be state dependent and the formula (7) still holds provided that the matrix U_c is of full rank for every state $x \in \mathcal{R}^n$.

The lemma below give the condition under which a state dependent pair $(F(x), G(x))$; $F \in \mathcal{R}^{n\times n}$, $G \in \mathcal{R}^{1\times n}$, $x \in \mathcal{R}^n$ can be transformed into the *extended Jordan controllable canonical pair* $(A_{\delta\gamma} + BL(x),\ B)$ where $A_{\delta\gamma}$ is as in (4), B is as in (2) and $L(x)$ is some state dependent $1 \times n$ vector.

Lemma 3: *Consider the pair* $(F(x), G(x))$; $F \in \mathcal{R}^{n\times n}$, $G \in \mathcal{R}^{n\times 1}$ *and* $x \in \mathcal{R}^n$. *Assume that the matrix*

$$U_c(x) = \left[G(x),\ F(x)G(x),\ \ldots,\ F^{n-1}(x)G(x)\right]^T \tag{10}$$

is of full rank for all $x \in \mathcal{R}^n$. *Let* $(A_{\delta\gamma} + BL(x),\ B)$ *be an extended Jordan controllable canonical pair with* $L(x) = [\ l_n(x),\ \ldots\ ,\ l_1(x)\ ]$, *and let*

$$\begin{aligned}W_J(x) &= [B,\ (A_{\delta\gamma}+BL(x))B, \\ &\quad \ldots,\ (A_{\delta\gamma}+BL(x))^{n-1}B.]\end{aligned} \tag{11}$$

Then, the matrix defined by

$$M_{\delta\gamma}(x) = W_J(x)U_c^{-1}(x) \tag{12}$$

satisfies

$$\begin{aligned}A_{\delta\gamma} + BL(x) &= M_{\delta\gamma}(x)F(x)M_{\delta\gamma}^{-1}(x) \\ \text{and } B &= M_{\delta\gamma}(x)G(x).\end{aligned}$$

if

$$l_k(x) = -\frac{1}{\gamma^{k-1}}\left[C_n^k\delta^k + \sum_{i=1}^{k} a_i(x)C_{n-i}^{k-i}\delta^{k-i}\right] \tag{13}$$

where the functions $a_i(x)$ *are the "coefficients" of characteristic equation of* $F(x)$

$$\begin{aligned}\det[\lambda I_n - F(x)] &= \lambda^n + a_1(x)\lambda^{n-1} + \\ &\quad \cdots + a_{n-1}(x)\lambda + a_n(x).\end{aligned}$$

That is, the pair $(F(x), G(x))$ *is equivalent to the pair* $(A_{\delta\gamma} + BL(x),\ B)$ *if the entries of* $L(x)$ *is as above.*

The proof of this lemma can be done in a similar fashion as for the proof of Lemma 2.

In the subsequent sections, this last result will be used to synthesize an controller for a class of control affine systems. First of all, we treat the simple case of linear controllable systems in order to clarify the design strategy.

4 Control design for single-input linear systems using JCCF

Consider the single-input linear system

$$\dot{x} = Fx + Gu \tag{14}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}$. F and G are constant matrices of appropriate dimensions.

Consider the feedback law defined by

$$u(x) = -LM_{\delta\gamma}x + \gamma KM_{\delta\gamma}x \tag{15}$$

where

- $M_{\delta\gamma}$ is defined as in (7),
- $L = \left[\begin{array}{ccc} l_n, & \dots, & l_1 \end{array}\right]$ where the l_k's are defined as in (9),
- K is a vector such that the matrix $(A + BK)$ is Hurwitz and A and B are as in (2).

Theorem 1. *The origin of the closed-loop system*

$$\dot{x} = Fx + Gu(x) \tag{16}$$

where $u(x)$ is as in (15), is globally asymptotically stable for all $\gamma > 0$ and $\delta \leq 0$.

Proof:
Consider closed-loop system (16) and let $\bar{x} = M_{\delta\gamma}x$. Then,

$$\begin{aligned} \dot{\bar{x}} &= M_{\delta\gamma}FM_{\delta\gamma}^{-1}\bar{x} + M_{\delta\gamma}Gu(x) \\ &= (A_{\delta\gamma} + BL)\bar{x} + Bu(x) \\ &= (\delta I_n + \gamma A + BL)\bar{x} + Bu(x) \\ &= \delta\bar{x} + \gamma A\bar{x} + BL\bar{x} + Bu(x) \end{aligned}$$

Since $u(x) = -L\bar{x} + \gamma K\bar{x}$ we have

$$\begin{aligned} \dot{\bar{x}} &= \delta\bar{x} + \gamma A\bar{x} + \gamma BK\bar{x} \\ &= \delta\bar{x} + \gamma\,(A + BK)\,\bar{x} \end{aligned}$$

Now, since $(A + BK)$ is Hurwitz, there exists a symmetric positive definite matrix P such that:

$$(A + BK)^T P + P(A + BK) = -I_n.$$

Consider the following candidate Lyapunov function $V(\bar{x}) = \bar{x}^T P\bar{x}$. Then,

$$\begin{aligned} \dot{V} &= 2\bar{x}^T P\,\dot{\bar{x}} \\ &= 2\delta\bar{x}^T P\bar{x} + 2\gamma\bar{x}^T P(A + BK)\bar{x} \\ &\leq 2\delta\bar{x}^T P\bar{x} - 2\gamma\bar{x}^T\bar{x} < 0 \end{aligned}$$

if $\delta \leq 0$ and $\gamma > 0$. This completes the proof of Theorem 1.

□

5 Controller design for a class of single-input control affine systems

In this section, we extend the above controller design to a class of control affine systems. We consider the single-input control affine systems described by

$$\dot{x} = F(x)x + G(x)u \tag{17}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}$ and $F(x)$ and $G(x)$ are of the following form $F(x) =$
$$\left[\begin{array}{ccccc} f_{11}(\underline{x}_1) & f_{12}(\underline{x}_1) & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ f_{n-1,1}(\underline{x}_{n-1}) & & & & f_{n-1,n}(\underline{x}_{n-1}) \\ f_{n1}(\underline{x}_n) & \cdots & & \cdots & f_{nn}(\underline{x}_n) \end{array}\right]$$
$G(x) = col\left[\begin{array}{cccc} 0, & \cdots, & 0, & g(\underline{x}_n) \end{array}\right]$
where $\underline{x}_i := (x_1, x_2, \dots, x_i)$.
We assume that

A1) the functions $f_{i,i+1}(\underline{x}_i); i = 1, ..., n-1$ and $g(\underline{x}_n)$ are *not* identically zero for all $x \in \mathcal{R}^n$

A2) the entries of the matrix $F(x)$ are smooth and bounded.

A3) the state of the system is bounded.

Note that if A1) is satisfied then the matrix

$$U_c(x) = \left[\begin{array}{cccc} G(x), & F(x)G(x), & \cdots & ,F^{n-1}(x)G(x) \end{array}\right] \tag{18}$$

is of full rank for all $x \in \mathcal{R}^n$. Consequently, the pair $(F(x), G(x))$ can be steered into a Jordan controllable pair $(A_{\delta\gamma} + BL(x),\ B)$ by applying Lemma 3, when the vector $L(x)$ is appropriately defined.

Based on these remarks and the previous design strategy, we consider the control law given by

$$u(x) = -L(x)M_{\delta\gamma}(x)x + \gamma KM_{\delta\gamma}(x)x \tag{19}$$

where

- $M_{\delta\gamma}(x)$ is as in (12) with $\gamma > 0$ and $\delta \leq 0$,
- $L(x) = \left[\begin{array}{ccc} l_n(x), & \dots, & l_1(x) \end{array}\right]$ where the $l_k(x)$'s are as in (13) and
- K is a vector chosen such that the matrix $(A+BK)$ is Hurwitz with A and B are as in (2).

We then state the following

Theorem 2. *Assume that system (17) satisfies Assumptions A1) to A3). Then, there exists $\delta_0 \leq 0$ such that for all $\delta < \delta_0$, there exists $\gamma_0 \geq 0$ such that for all $\gamma \geq \gamma_0$ the origin of the closed-loop system*

$$\dot{x} = F(x)x + G(x)u(x) \tag{20}$$

where $u(x)$ is as in (19), is globally asymptotically stable.

Proof:

Consider the closed-loop system (20) and let $\bar{x} = M_{\delta\gamma}(x)x$. Then,

$$\begin{array}{rcl} \dot{\bar{x}} & = & M_{\delta\gamma} F M_{\delta\gamma}^{-1} + M_{\delta\gamma} G(x) u(x) + \dot{M}_{\delta\gamma} M_{\delta\gamma}^{-1} \bar{x} \\ & = & (A_{\delta\gamma} + BL)\bar{x} + Bu(x) + \dot{M}_{\delta\gamma} M_{\delta\gamma}^{-1} \bar{x} \\ & = & (\delta I_n + \gamma A + BL)\bar{x} + Bu(x) + \dot{M}_{\delta\gamma} M_{\delta\gamma}^{-1} \bar{x} \end{array}$$

Since $u(x) = -L\bar{x} + \gamma K \bar{x}$ we have

$$\dot{\bar{x}} = \delta \bar{x} + \gamma \left(A + BK\right) \bar{x} + \dot{M}_{\delta\gamma} M_{\delta\gamma}^{-1} \bar{x}$$

Now, since $(A + BK)$ is Hurwitz, there exists a symmetric positive definite matrix P such that:

$$(A + BK)^T P + P(A + BK) = -I_n.$$

Consider the following candidate Lyapunov function $V(\bar{x}) = \bar{x}^T P \bar{x}$. Then,

$$\begin{array}{rcl} \dot{V} & = & 2\bar{x}^T P \, \dot{\bar{x}} \\ & = & 2\delta \bar{x}^T P \bar{x} + 2\gamma \bar{x}^T P (A + BK)\bar{x} \\ & & + 2\bar{x}^T P \dot{M}_{\delta\gamma} M_{\delta\gamma}^{-1} \bar{x} \\ & \leq & 2\delta \bar{x}^T P \bar{x} - 2\gamma \bar{x}^T \bar{x} + 2\bar{x}^T P \dot{M}_{\delta\gamma} M_{\delta\gamma}^{-1} \bar{x} \end{array}$$

Using the fact that $2x^T P y \leq x^T P x + y^T P y$ for all $x, y \in \mathcal{R}^n$, we get

$$\begin{array}{rcl} \dot{V} & \leq & 2\delta \bar{x}^T P \bar{x} - 2\gamma \bar{x}^T \bar{x} + \bar{x}^T P \bar{x} \\ & & + \bar{x}^T (\dot{M}_{\delta\gamma} M_{\delta\gamma}^{-1}) P \dot{M}_{\delta\gamma} M_{\delta\gamma}^{-1} \bar{x} \\ & \leq & 2\delta \bar{x}^T P \bar{x} - 2\gamma \bar{x}^T \bar{x} + \bar{x}^T P \bar{x} \\ & & + \lambda_{\max}[P] \left\| \dot{M}_{\delta\gamma} M_{\delta\gamma}^{-1} \right\|^2 \|\bar{x}\|^2 \end{array}$$

Now, to bound the term $\Gamma_{\delta\gamma} = \dot{M}_{\delta\gamma} M_{\delta\gamma}^{-1}$ it is important to point out some particular features of the matrix $M_{\delta\gamma}$. In effect, due to the particular structure of $F(x)$ and $G(x)$, it can be checked that the matrix $M_{\delta\gamma}$ is lower triangular. This comes from the fact that, in this case, both controllability matrices (10) and (11) are lower triangular. Next, it can be checked that the entries of $\Gamma_{\delta\gamma}$ are polynomials in the argument $\frac{1}{\gamma}$. This implies that if we choose $\gamma \geq 1$, then the entries of $\Gamma_{\delta\gamma}$ are all bounded by some terms which are independent on γ since $\frac{1}{\gamma} \leq 1$. Consequently, since the state is bounded and entries of $M_{\delta\gamma}$ are smooth and bounded as well, there exists a constant $c_\delta > 0$ independent of γ such that $\left\| \dot{M}_{\delta\gamma} M_{\delta\gamma}^{-1} \right\|^2 \leq c_\delta$. It is in fact for this particular reason that such structure on $F(x)$ and $G(x)$ were imposed. In effect, such particular properties on the matrix $M_{\delta\gamma}(x)$ does not hold for general state dependent matrices $F(x)$ and $G(x)$.

Therefore, assuming that $\gamma \geq 1$, we get

$$\dot{V} \leq 2\delta \bar{x}^T P \bar{x} - 2\gamma \|\bar{x}\|^2 + \bar{x}^T P \bar{x} + c_\delta \lambda_{\max}[P] \|\bar{x}\|^2$$

where $\lambda_{\max}[P]$ is the largest eigenvalue of P.

Now, choosing $\delta < -\frac{1}{2} = \delta_0$ and $\gamma > \frac{c_\delta \lambda_{\max}[P]}{2} = \gamma_0$, we get $\dot{V} < 0$.

This completes the proof of Theorem 2.

□

6 Example

Consider the following example

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 + x_2 e^{-x_1} \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u \end{array} \right. \qquad (21)$$

which is of the form (17) with $F(x) = \begin{pmatrix} 0 & 1 + e^{-x_1} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $G(x) = B = col \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$.

It can readily be checked that assumption A1) is satisfied. The entries of $F(x)$ are obviously smooth and bounded so that A2) is also satisfied. We can thus transform the pair $(F(x), G(x))$ into a Jordan controllable canonical pair. First, let us calculate

$$\begin{array}{rcl} U_c(x) & = & \left[\begin{array}{ccc} G(x), & F(x)G(x), & F^2(x)G(x) \end{array} \right] \\ & = & \begin{pmatrix} 0 & 0 & 1 + e^{-x_1} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{array}$$

Next,

$$\det(\lambda I_3 - F(x)) = \lambda^3$$

consequently,

$$L = \left[\begin{array}{ccc} -\frac{\delta^3}{\gamma^2}, & -\frac{3\delta^2}{\gamma}, & -3\delta \end{array} \right]$$

and

$$A_{\delta\gamma} + BL = \begin{pmatrix} \delta & \gamma & 0 \\ 0 & \delta & \gamma \\ -\frac{\delta^3}{\gamma^2} & -\frac{3\delta^2}{\gamma} & -2\delta \end{pmatrix}.$$

Therefore,

$$\begin{array}{rcl} W_J & = & \left[B, \ (A_{\delta\gamma} + BL(x))B, \ (A_{\delta\gamma} + BL(x))^2 B\right] \\ & = & \begin{pmatrix} 0 & 0 & \gamma^2 \\ 0 & \gamma & -\delta\gamma \\ 1 & -2\delta & \delta^2 \end{pmatrix} \end{array}$$

and hence

$$\begin{array}{rcl} M_{\delta\gamma}(x) & = & W_J(x) U_c^{-1}(x) \\ & = & \begin{pmatrix} \frac{\gamma^2}{1+e^{-x_1}} & 0 & 0 \\ -\frac{\delta\gamma}{1+e^{-x_1}} & \gamma & 0 \\ \frac{\delta^2}{1+e^{-x_1}} & -2\delta & 1 \end{pmatrix}. \end{array}$$

Notice that

$$\begin{aligned}\Gamma_{\delta\gamma}(x) &= \dot{M}_{\delta\gamma}(x)M_{\delta\gamma}^{-1}(x)\\ &= \frac{\dot{x}_1 e^{-x_1}}{1+e^{-x_1}}\begin{pmatrix} 1 & 0 & 0\\ -\frac{\delta}{\gamma} & 0 & 0\\ \frac{\delta^2}{\gamma^2} & 0 & 0\end{pmatrix}\end{aligned}$$

which show that the entries of $\Gamma_{\delta\gamma}(x)$ are polynomials in the arguments $\frac{1}{\gamma}$. It can also be verified that

$$\|\Gamma_{\delta\gamma}(x)\|_2 = \left|x_2 e^{-x_1}\right|\sqrt{1+\frac{\delta^2}{\gamma^2}+\frac{\delta^4}{\gamma^4}}.$$

Consequently, if $\gamma \geq 1$ and $|x_2 e^{-x_1}| \leq c_0$ (bounded state), then

$$\begin{aligned}\|\Gamma_{\delta\gamma}(x)\|_2 &\leq \left|x_2 e^{-x_1}\right|\sqrt{1+\delta^2+\delta^4}\\ &\leq c_0\sqrt{1+\delta^2+\delta^4} = c_\delta.\end{aligned}$$

Finally, the control law which stabilise (21) is given by

$$\begin{aligned}u(x) &= -L(x)M_{\delta\gamma}(x)x + \gamma K M_{\delta\gamma}(x)x\\ &= \frac{\delta^2 x_1}{1+e^{-x_1}} - 3\delta^2 x_2 + 3\delta x_3\\ &\quad + \frac{x_1 k_1 \gamma^3 - x_1 k_2 \delta\gamma^2 + x_1 k_3 \gamma\delta^2}{1+e^{-x_1}}\\ &\quad + x_2 k_2 \gamma^2 - 2x_2 k_3 \delta\gamma + k_3 \gamma x_3\end{aligned}$$

where $K = \begin{bmatrix} k_1 & k_2 & k_3\end{bmatrix}$ is chosen such that the matrix $A+BK$ is stable where

$$A = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}.$$

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