Observers design for linear time-varying systems

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Abstract : In this note we give some results on the convergence of the Kalman Filter (K.F.) when used as an observer for linear time-varying systems. Based on the block-input / block-output state model, we prove that the state observer given in [7] is equivalent to the K.F. algorithm. One of the main features, however, is that no assumption on the invertibility of the state matrix, namely A_k in the paper, is needed and the computational requirements are reduced. Furthermore, the obtained result can be extended, by duality, to resolve the state feedback control problem.

1. Introduction

In the last decade, many research activities were focused on the theory of linear time-varying systems. One of the main reasons is that most of the works on non-linear systems studied first linearized models. This may be important to establish local properties and to set up analysis and synthesis [2]-[3]-[5]-[8]-[9]-[10]-[11]. In this note we deal with state observers design for linear discrete-time time-varying systems. In 1988, Baras et al. [2] have shown that under some strong observability and controlability assumptions the Kalman filter ensures global convergence when used as an observer for linear time varying systems. Similar conditions were already established in the work of Deyst et al. [5]. More recently, Song et al. 11] have shown that, under invertibility of the state matrix and the observability conditions, the Kalman filter is a global and uniform asymptotic observer. In [8], authors use the reconstructibility condition, which also needs the invertibility of the state matrix, to guarantee global convergence of the Kalman filter.

A second approach to build observers for linear time-varying systems consists in using a block-input / block-output model from the state space represenation [7] (see also [1] and [6] for control design). The main result of this technique is that only the observability condition is needed. The strong invertibility of the state matrix assumption is not required. However to prove global convergence of the proposed observer, we have to compute inversion of an pq.pq matrix build upon square of the observability matrix.

To avoid such computational requirements, in particular, for large scale-systems or when q is large, we prove by induction that the state observer given in [7] may be written as the K.F. algorithm when used as an observer for linear time-varying systems. On the other hand, no assumption on the invertibility of the state matrix is needed.

2. Problem formulation

Consider the linear time-varying system :

$$x_{k+1} = A_k x_k + B_k u_k \tag{1}$$

$$y_k = C_k x_k \tag{2}$$

where $u_k \in R^m$, $x_k \in R^n$ and $y_k \in R^p$ represent the input, the state and output vectors at time instant k respectively. A_k, B_k and C_k are known matrices of appropriate dimensions.

Following the notations in [7], (1)-(2) can be written in terms of the q-block input / block output state model :

$$x_{q+k} = \Phi_{q+k} x_k + \Gamma_{q+k} U_k \quad (3)$$

$$Y_{q+k} = \Omega_{q+k} x_k + E_{q+k} U_k \quad (4)$$

where $\Phi_{q+k} \in R^{n.n}$, $\Gamma_{q+k} \in R^{n.qm}$ and $\Omega_{q+k} \in R^{qp.n}$ denote the q-step transition matrix, the q-step reachability matrix and q-step observability matrix respectively, defined by :

$$\Phi_{q+k} = A_{q+k-1} A_{q+k-2} \ldots A_k \quad (5)$$

$$\Gamma_{q+k} = \left[B_{q+k-1} \ \ A_{q+k-1} B_{q+k-2} \ \ldots \ A_{q+k-1} A_{q+k-2} \ldots A_{k+1} B_k \right] \quad (6)$$

$$\Omega_{q+k} = \begin{bmatrix} C_k \\ C_{k+1} A_k \\ . \\ C_{q+k-1} A_{q+k-2} \ldots A_k \end{bmatrix} \text{ and}$$

$$E_{q+k} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & C_{k+1} B_k \\ 0 & 0 & \ldots & C_{k+2} B_{k+1} & C_{k+2} A_{k+1} B_k \\ : & : & : & : & : \\ 0 & C_{q+k-1} B_{q+k-2} & \ldots & x & x \end{bmatrix} \quad (7)$$

U_{q+k} and Y_{q+k} are the q - block input / block output vectors :

$$U_{q+k} = \left[u_{q+k-1}^T \ \ u_{q+k-2}^T \ \ldots \ u_k^T \right]^T$$

$$Y_{q+k} = \left[y_k^T \ \ y_{k+1}^T \ \ldots \ y_{q+k-1}^T \right]^T \quad (8)$$

Now, let us summarize the main result developed in [7] through the following theorem :

Theorem 1

If we assume that A_k and C_k are bounded and system (1)-(2) is q-step observable, that is Ω_{q+k} has a left inverse, then the error dynamics when the state observer is :

$$\hat{x}_{q+k} = \Phi_{q+k} \hat{x}_k + \Gamma_{q+k} U_{q+k} + L_{q+k} (Y_{q+k} - \Omega_{q+k} \hat{x}_k - E_{q+k} U_{q+k}) \quad (9)$$

$$L_{q+k} = \Phi_{q+k} \Omega_{q+k}^T \Psi_{q+k}^{-1} \quad (10)$$

with $\quad \Psi_{q+k} = \Omega_{q+k} \Omega_{q+k}^T + \sigma I \quad (11)$

converges to zero.

I is the pq.pq identity matrix and σ is a small positive real number.

We notice that the general form of (10)-(11) may be obtained by duality to the optimal state feedback control solution given in [6], the gain matrix for the state estimation is therefore:

$$\hat{x}_{q+k} = \Phi_{q+k} \hat{x}_k + \Gamma_{q+k} U_{q+k} + L_{q+k} (Y_{q+k} - \Omega_{q+k} \hat{x}_k - E_{q+k} U_{q+k}) \quad (12)$$

$$L_{q+k} = \Phi_{q+k} P_k \Omega_{q+k}^T \Psi_{q+k}^{-1} \quad (13)$$

with $\quad \Psi_{q+k} = \Omega_{q+k} P_k \Omega_{q+k}^T + R_k \quad (14)$

P_k is a n.n positive semidefinite matrix and R_k is a pq.pq positive definite matrix. (10)-(11) is obtained for $P_k = I_n$ and $R_k = \sigma I_{pq}$.

In this paper we will show that the Kalman filter (the correct term is in fact the Kalman predictor) is equivalent to the state observer (12)-(14). Therefore we obtain a one-step recursive observer without inversion of the pq.pq matrix Ψ_{q+k}. On the other hand the strong assumption on invertibility of the state matrix A_k, frequently used in the literature, is not required.

The following theorem summarizes then the main result of this paper :

Theorem 2

If we assume that A_k and C_k are bounded, and system (1)-(2) is q-step observable then the Kalman filter algorithm :

$$\hat{x}_{q+k} = A_{q+k-1} \hat{x}_{q+k-1} + B_{q+k-1} u_{q+k-1} + K_{q+k-1} (y_{q+k-1} - C_{q+k-1} \hat{x}_{q+k-1}) \quad (15)$$

$$K_{q+k-1} = A_{q+k-1} P_{q+k-1} C_{q+k-1}^T \Sigma_{q+k-1}^{-1} \quad (16)$$

and

$$P_{q+k} = A_{q+k-1} P_{q+k-1} (A_{q+k-1}^T - C_{q+k-1}^T K_{q+k-1}^T) \quad (17)$$

with $\Sigma_{q+k-1} = C_{q+k-1}P_{q+k-1}C^T_{q+k-1} + R_{q+k-1}$ (18)

R_{q+k-1} is a positive definite matrix.

is equivalent to the optimal state observer (12)-(14).

Proof

The proof that we detail here, is by induction. For simplicity and without loss of generality, we set $u_k = 0$.

First, suppose that q = 1, we have :

$$\Omega_{q+k} = \Omega_{1+k} = C_k$$

$$\Psi_{q+k} = \Psi_{1+k} = \Sigma_k$$

and $\Phi_{q+k} = \Phi_{1+k} = A_k$

the 1-step observability condition means that :

$$\mathrm{rank}\,(C_k) = n$$

it is clear that (12)-(14) is equivalent to (15)-(18).

For q = 2 we have :

$$\hat{x}_{k+2} = A_{k+1}\hat{x}_{k+1} + K_{k+1}(y_{k+1} - C_{k+1}\hat{x}_{k+1}) \tag{19}$$

$$K_{k+1} = A_{k+1}P_{k+1}C^T_{k+1}\Sigma^{-1}_{k+1} \tag{20}$$

and

$$P_{k+2} = (A_{k+1} - K_{k+1}C_{k+1})P_{k+1}A^T_{k+1} \tag{21}$$

(19) may be written in the extended form as :

$$\hat{x}_{k+2} = A_{k+1}(A_k\hat{x}_k + K_k(y_k - C_k\hat{x}_k)) + K_{k+1}(y_{k+1} - C_{k+1}A_k\hat{x}_k - C_{k+1}K_k(y_k - C_k\hat{x}_k)) \tag{22}$$

or

$$\hat{x}_{k+2} = A_{k+1}A_k\hat{x}_k + \begin{bmatrix} A_{k+1}K_k - K_{k+1}C_{k+1}K_k & K_{k+1} \end{bmatrix} \begin{bmatrix} y_k - C_k\hat{x}_k \\ y_{k+1} - C_{k+1}A_k\hat{x}_k \end{bmatrix} \tag{23}$$

We notice that K_{k+1} is in the form :

$$K_{k+1} = A_{k+1}A_kP_k(A^T_k - C^T_kK^T_k)C^T_{k+1}\Sigma^{-1}_{k+1} \tag{24}$$

or

$$K_{k+1} = A_{k+1}A_kP_k \begin{bmatrix} C^T_k & A^T_kC^T_{k+1} \end{bmatrix} \begin{bmatrix} -K^T_kC^T_{k+1}\Sigma^{-1}_{k+1} \\ \Sigma^{-1}_{k+1} \end{bmatrix} \tag{25}$$

On the other hand, we have :

$$A_{k+1}K_k - K_{k+1}C_{k+1}K_k = A_{k+1}A_kP_k \times \left(C^T_k\Sigma^{-1}_k - \begin{bmatrix} C^T_k & A^T_kC^T_{k+1} \end{bmatrix} \begin{bmatrix} -K^T_kC^T_{k+1}\Sigma^{-1}_{k+1}C_{k+1}K_k \\ \Sigma^{-1}_{k+1}C_{k+1}K_k \end{bmatrix} \right) \tag{26}$$

or

$$A_{k+1}K_k - K_{k+1}C_{k+1}K_k = A_{k+1}A_kP_k \times \begin{bmatrix} C^T_k & A^T_kC^T_{k+1} \end{bmatrix} \begin{bmatrix} \Sigma^{-1}_k + K^T_kC^T_{k+1}\Sigma^{-1}_{k+1}C_{k+1}K_k \\ -\Sigma^{-1}_{k+1}C_{k+1}K_k \end{bmatrix} \tag{27}$$

the gain matrix in (23) is then written as :

$$\begin{bmatrix} A_{k+1}K_k - K_{k+1}C_{k+1}K_k & K_{k+1} \end{bmatrix} = A_{k+1}A_kP_k \begin{bmatrix} C^T_k & A^T_kC^T_{k+1} \end{bmatrix} \times \begin{bmatrix} \Sigma^{-1}_k + K^T_kC^T_{k+1}\Sigma^{-1}_{k+1}C_{k+1}K_k & -K^T_kC^T_{k+1}\Sigma^{-1}_{k+1} \\ -\Sigma^{-1}_{k+1}C_{k+1}K_k & \Sigma^{-1}_{k+1} \end{bmatrix} \tag{28}$$

with

$$\Sigma_{k+1} = C_{k+1}A_kP_kA^T_kC^T_{k+1} + R_{k+1} - C_{k+1}A_kP_kC^T_k\Sigma^{-1}_kC_kP_kA^T_kC^T_{k+1}$$

Now if we set :

$$a_{11} = \Sigma_k \ (= \Psi_{1+k})$$

$$a_{22} = C_{k+1}A_kP_kA^T_kC^T_{k+1} + R_{k+1}$$

$$a_{12} = C_kP_kA^T_kC^T_{k+1} = \Omega_{1+k}P_k\Phi^T_{1+k}C^T_{k+1}$$

and by the use of the inversion lemma of a partitioned matrix, the right hand side matrix of (28) may be written as :

$$\begin{bmatrix} \Sigma^{-1}_k + K^T_kC^T_{k+1}\Sigma^{-1}_{k+1}C_{k+1}K_k & -K^T_kC^T_{k+1}\Sigma^{-1}_{k+1} \\ -\Sigma^{-1}_{k+1}C_{k+1}K_k & \Sigma^{-1}_{k+1} \end{bmatrix} = \begin{bmatrix} (a_{11} - a_{12}a^{-1}_{22}a^T_{12})^{-1} & -a^{-1}_{11}a_{12}(a_{22} - a^T_{12}a^{-1}_{11}a_{12})^{-1} \\ -(a_{22} - a^T_{12}a^{-1}_{11}a_{12})^{-1}a^T_{12}a^{-1}_{11} & (a_{22} - a^T_{12}a^{-1}_{11}a_{12})^{-1} \end{bmatrix}$$

or equivalently :

$$\begin{aligned} \begin{bmatrix} \Sigma^{-1}_k + K^T_kC^T_{k+1}\Sigma^{-1}_{k+1}C_{k+1}K_k & -K^T_kC^T_{k+1}\Sigma^{-1}_{k+1} \\ -\Sigma^{-1}_{k+1}C_{k+1}K_k & \Sigma^{-1}_{k+1} \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a^T_{12} & a_{22} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \Omega_{2+k}P_k\Omega^T_{2+k} + \mathrm{diag}(R_k, R_{k+1}) \end{bmatrix}^{-1} \\ &= \Psi^{-1}_{2+k} \end{aligned}$$

The final form of (23) is then :

$$\hat{x}_{k+2} = \Phi_{2+k}\hat{x}_k + L_{2+k}\left(Y_{2+k} - \Omega_{2+k}\hat{x}_k\right) \quad (29)$$

with

$$L_{2+k} = \Phi_{2+k}P_k\Omega_{2+k}^T\left[\Omega_{2+k}P_k\Omega_{2+k}^T + \mathrm{diag}(R_k, R_{k+1})\right]^{1} \quad (30)$$

Now assume that :

$$\hat{x}_{q+k} = \Phi_{q+k}\hat{x}_k + L_{q+k}\left(Y_{q+k} - \Omega_{q+k}\hat{x}_k\right) \quad (31)$$

with

$$L_{q+k} = \Phi_{q+k}P_k\Omega_{q+k}^T \times \left[\Omega_{q+k}P_k\Omega_{q+k}^T + \mathrm{diag}(R_k, ..., R_{q+k-1})\right]^{1} \quad (32)$$

From (15) and (31) at time instant q+k+1, we have :

$$\hat{x}_{q+k+1} = A_{q+k}\left(\Phi_{q+k}\hat{x}_k + L_{q+k}\left(Y_{q+k} - \Omega_{q+k}\hat{x}_k\right)\right) + K_{q+k}\left(y_{q+k} - C_{q+k}\left(\Phi_{q+k}\hat{x}_k + L_{q+k}\left(Y_{q+k} - \Omega_{q+k}\hat{x}_k\right)\right)\right) \quad (33)$$

or

$$\hat{x}_{q+k+1} = \Phi_{q+k+1}\hat{x}_k + \left[A_{q+k}L_{q+k} - K_{q+k}C_{q+k}L_{q+k} \quad K_{q+k}\right] \times \begin{bmatrix} Y_{q+k} - \Omega_{q+k}\hat{x}_k \\ y_{q+k} - C_{q+k}\Phi_{q+k}\hat{x}_k \end{bmatrix} \quad (34)$$

where

$$\begin{bmatrix} Y_{q+k} - \Omega_{q+k}\hat{x}_k \\ y_{q+k} - C_{q+k}\Phi_{q+k}\hat{x}_k \end{bmatrix} = Y_{q+k+1} - \Omega_{q+k+1}\hat{x}_k \quad (35)$$

Hereafter we will show that the gain matrix in (34) is in the form :

$$\left[A_{q+k}L_{q+k} - K_{q+k}C_{q+k}L_{q+k} \quad K_{q+k}\right] = \Phi_{q+k+1}P_k\Omega_{q+k+1}^T\left[\Omega_{q+k+1}P_k\Omega_{q+k+1}^T + \mathrm{diag}(R_k, ..., R_{q+k})\right]^{1} = \Phi_{q+k+1}P_k\Omega_{q+k+1}^T\Psi_{q+k+1}^{-1}$$

First of all, by the use of the recursive form (17), K_{q+k} is written as :

$$K_{q+k} = A_{q+k}P_{q+k}C_{q+k}^T\Sigma_{q+k}^{-1} \quad (36)$$

or

$$K_{q+k} = A_{q+k}A_{q+k-1}...A_kP_k\left(A_k^T - C_k^TK_k^T\right)..\times\left(A_{q+k-1}^T - C_{q+k-1}^TK_{q+k-1}^T\right)C_{q+k}^T\Sigma_{q+k}^{-1} \quad (37)$$

The extended form of (37), and by the use of (31)-(32), leads to :

$$K_{q+k} = \Phi_{q+k+1}P_k\Omega_{q+k+1}^T\begin{bmatrix} -L_{q+k}^TC_{q+k}^T\Sigma_{q+k}^{-1} \\ \Sigma_{q+k}^{-1} \end{bmatrix} \quad (38)$$

on the other hand we have :

$$A_{q+k}L_{q+k} - K_{q+k}C_{q+k}L_{q+k} = A_{q+k}\Phi_{q+k}P_k\Omega_{q+k}^T\Psi_{q+k}^{-1} - \Phi_{q+k+1}P_k\Omega_{q+k+1}^T\begin{bmatrix} -L_{q+k}^TC_{q+k}^T\Sigma_{q+k}^{-1} \\ \Sigma_{q+k}^{-1} \end{bmatrix}C_{q+k}L_{q+k} \quad (39)$$

or

$$A_{q+k}L_{q+k} - K_{q+k}C_{q+k}L_{q+k} = \Phi_{q+k+1}P_k\Omega_{q+k+1}^T \times \begin{bmatrix} \Psi_{q+k}^{-1} + L_{q+k}^TC_{q+k}^T\Sigma_{q+k}^{-1}C_{q+k}L_{q+k} \\ -\Sigma_{q+k}^{-1}C_{q+k}L_{q+k} \end{bmatrix} \quad (40)$$

The gain matrix in (34) may be then written as :

$$\left[A_{q+k}L_{q+k} - K_{q+k}C_{q+k}L_{q+k} \quad K_{q+k}\right] = \Phi_{q+k+1}P_k\Omega_{q+k+1}^T \times \begin{bmatrix} \Psi_{q+k}^{-1} + L_{q+k}^TC_{q+k}^T\Sigma_{q+k}^{-1}C_{q+k}L_{q+k} & -L_{q+k}^TC_{q+k}^T\Sigma_{q+k}^{-1} \\ -\Sigma_{q+k}^{-1}C_{q+k}L_{q+k} & \Sigma_{q+k}^{-1} \end{bmatrix} \quad (41)$$

Now if we set : $a_{11} = \Psi_{q+k}$

$a_{22} = C_{q+k}\Phi_{q+k}P_k\Phi_{q+k}^TC_{k+1}^T + R_{q+k}$

$a_{12} = \Omega_{q+k}P_k\Phi_{q+k}^TC_{k+1}^T$

and by the use of the inversion lemma of a partitioned matrix as above, we obtain :

$$\begin{bmatrix} \Psi_{q+k}^{-1} + L_{q+k}^TC_{q+k}^T\Sigma_{q+k}^{-1}C_{q+k}L_{q+k} & -L_{q+k}^TC_{q+k}^T\Sigma_{q+k}^{-1} \\ -\Sigma_{q+k}^{-1}C_{q+k}L_{q+k} & \Sigma_{q+k}^{-1} \end{bmatrix} = \left[\Omega_{q+k+1}P_k\Omega_{q+k+1}^T + \mathrm{diag}(R_k, ..., R_{q+k})\right] \quad (42)$$

and finally the gain matrix in (34) is equivalent to (13) :

$$\left[A_{q+k}L_{q+k} - K_{q+k}C_{q+k}L_{q+k} \quad K_{q+k}\right] = \Phi_{q+k+1}P_k\Omega_{q+k+1}^T\Psi_{q+k+1}^{-1} \quad (43)$$

■

3. Conclusion

In this paper we have shown that the state observer given in [7] is equivalent to the Kalman filter when used as an observer for linear time-varying systems. One of the main features of this technique is that the strong invertibility condition on the state matrix, as usually used in the literature, and the computation of the q-step matrix Ψ_{q+k}^{-1} are not required.

References

[1] P. Albertos, "Block multirate Input-Output model for sampled-Data control systems," *IEEE Transactions on Automatic Control*, Vol. 35, pp.1085-1088, 1990.

[2] J. S. Baras, A. Bensoussan and M. R. James, "Dynamic observers as asymptotic limits of recursive filters : special cases", *SIAM Journal on Applied Mathematics*, Vol. 48, No. 5, pp. 1147-1158, 1988.

[3] M. Boutayeb and D. Aubry "A strong tracking Extended Kalman observer for non-linear systems," *IEEE Transactions on Automatic Control*, Vol.44, No. 8, pp. 1550-1556, 1999.

[4] G. Ciccarella, M. Dalla Mora and A. Germani, "A robust observer for discrete time nonlinear systems," *Systems & Control Letters*, Vol. 24, pp. 291-300, 1995.

[5] J. J. Deyst and C.F. Price, "Conditions for asymptotic stability of the discrete minimum-variance linear estimator," *IEEE Transactions on Automatic Control*, pp. 702-705, 1968.

[6] E. W. Kamen, "Study of linear time-varying discrete-time systems in terms of time-compressed models," *Proc. IEEE Conference on Decision and Control*, Tucson, USA, pp. 3070-3075, 1992.

[7] E. W. Kamen, "Block form observers for linear time-varying discrete-time systems," *Proc. IEEE Conference on Decision and Control*, San Antonio, USA, pp. 355-356, 1993.

[8] P. E. Moraal and J. W. Grizzle, "Observer design for nonlinear systems with discrete-time measurements," *IEEE Transactions on Automatic Control*, Vol. 40, No. 3, pp. 395-404, 1995.

[9] P. E. Moraal and J. W. Grizzle, "Asymptotic observers for detectable and poorly observable systems," *Proc. IEEE Conference on Decision and Control*, New Orleans, USA, pp. 108-114, 1995.

[10] M. S. Chen and Yen J. Y., "Application of the least squares algorithm to the observer design for linear time-varying systems," *IEEE Transactions on Automatic Control*, pp.1742-1745, 1999.

[11] Y. Song and J. W. Grizzle, "The extended Kalman Filter as a local asymptotic observer for discrete-time nonlinear systems," *Journal of Mathematical systems, Estimation and Control*, Vol. 5, No. 1, pp. 59-78, 1995.