On the Equivalence Between Dissipativity and Optimality of Nonlinear Hybrid Controllers

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Abstract

In this paper we derive guaranteed hybrid gain, sector, and disk margins for nonlinear optimal and inverse optimal hybrid regulators that minimize a nonlinear-nonquadratic hybrid performance functional. Furthermore, we develop a hybrid return difference inequality to provide connections between dissipativity and optimality of nonlinear hybrid controllers. Specifically, we show that optimal hybrid controllers imply dissipativity with respect to a quadratic supply rate.

1. Introduction

Modern complex engineering systems typically possess a multiechelon hierarchical architecture characterized by continuous-time dynamics at the lower levels of the hierarchy and discrete-time dynamics at the higher levels of the hierarchy. Hence, it is not surprising that hybrid systems have been the subject of intensive research over the past recent years (see [1, 2] and the numerous references therein). The mathematical descriptions of some of these systems can be characterized by impulsive differential equations [3–5]. Impulsive dynamical systems can be viewed as a subclass of hybrid systems and consist of three elements; namely, a continuous-time differential equation, which governs the motion of the dynamical system between impulsive or resetting events; a difference equation, which governs the way the system states are instantaneously changed when a resetting event occurs; and a criterion for determining when the states of the system are to be reset.

In a recent two-part paper the authors in [6, 7] developed a general framework for hybrid feedback systems by addressing stability, dissipativity, optimality, and inverse optimality of impulsive dynamical systems. In particular, [7] considers a hybrid feedback optimal control problem over an infinite horizon involving a hybrid nonlinear-nonquadratic performance functional. The performance functional involves a continuous-time cost for addressing performance of the continuous-time system dynamics and a discrete-time cost for addressing performance at the resetting instants. Furthermore, the hybrid cost functional can be evaluated in the closed-form as long as the nonlinear-nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop impulsive system. This Lyapunov function is shown to be a solution of a steady-state, hybrid Hamilton-Jacobi-Bellman equation and thus guaranteeing both optimality and stability of the feedback controlled impulsive system. The overall framework provides the foundation for extending linear-quadratic feedback control methods to nonlinear impulsive dynamical systems.

For continuous-time nonlinear systems, the problem of guaranteed stability margins for inverse optimal regulators is well known [8–10]. Specifically, nonlinear inverse optimal controllers that minimize a meaningful nonlinear-nonquadratic performance criterion involving a nonlinear-nonquadratic, nonnegative-definite function of the state and a quadratic positive definite function of the control are known to possess sector margin guarantees to component decoupled input nonlinearities lying in the conic sector $(\frac{1}{2}, \infty)$. These results also hold for disk margin guarantees where asymptotic stability of the closed loop system is guaranteed in the face of a dissipative dynamic input operator. In addition, an equivalence between dissipativity with respect to a quadratic supply rate and optimality of a nonlinear regulator also holds [8].

In this paper we use the results of [6, 7] to develop sufficient conditions for hybrid gain, sector, and disk margins guarantees for nonlinear hybrid dynamical systems controlled by optimal and inverse optimal hybrid regulators. Furthermore, we develop a hybrid counterpart of the return difference inequality for continuous-time systems [8, 11] to provide connections between dissipativity and optimality of nonlinear hybrid controllers. In particular, we show that unlike the case for continuous-time systems, the equivalence between dissipativity and optimality of hybrid controllers breaks down. However, we do show that optimal hybrid controllers imply dissipativity with respect to a quadratic supply rate.

2. Mathematical Preliminaries

In this section we establish definitions, notation, and several key results used later in the paper. Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{S}^n$ denote the set of $n \times n$ symmetric matrices, let $\mathbb{N}^n$ denote the set of $n \times n$ nonnegative definite matrices, let $(\,)^{\mathrm{T}}$ denote transpose, let $\mathcal{N}$ denote the set of nonnegative integers, and let I_n or I denote the $n \times n$ identity matrix. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm, $\sigma_{\max}(\cdot)$ for the maximum singular value, $\sigma_{\min}(\cdot)$ for the minimum singular value, and $M \geq 0$ (resp., $M > 0$) to denote the fact that the Hermitian matrix M is nonnegative (resp., positive) definite.

In this paper we consider state-dependent impulsive dynamical systems $\mathcal{G}$ of the form

$$\begin{aligned} \dot{x}(t) &= f_{\mathrm{c}}(x(t)) + G_{\mathrm{c}}(x(t))u_{\mathrm{c}}(t),\ x(0) = x_0, \\ &\qquad x(t) \notin \mathcal{Z}, \quad (1) \\ \Delta x(t) &= f_{\mathrm{d}}(x(t)) + G_{\mathrm{d}}(x(t))u_{\mathrm{d}}(t),\ x(t) \in \mathcal{Z}, \quad (2) \\ y_{\mathrm{c}}(t) &= h_{\mathrm{c}}(x(t)) + J_{\mathrm{c}}(x(t))u_{\mathrm{c}}(t),\ x(t) \notin \mathcal{Z}, \quad (3) \\ y_{\mathrm{d}}(t) &= h_{\mathrm{d}}(x(t)) + J_{\mathrm{d}}(x(t))u_{\mathrm{d}}(t),\ x(t) \in \mathcal{Z}. \quad (4) \end{aligned}$$

where $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u_{\mathrm{c}}(t) \in \mathbb{R}^{m_{\mathrm{c}}}$, $u_{\mathrm{d}}(t_k) \in \mathbb{R}^{m_{\mathrm{d}}}$, t_k denotes the k^{th} instant of time at which $x(t)$ intersects

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$\mathcal{Z}$ for a particular trajectory $x(t)$, $y_{\mathrm{c}}(t) \in \mathbb{R}^{l_{\mathrm{c}}}$, $y_{\mathrm{d}}(t_k) \in \mathbb{R}^{l_{\mathrm{d}}}$, $f_{\mathrm{c}} : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $f_{\mathrm{c}}(0) = 0$, $G_{\mathrm{c}} : \mathbb{R}^n \to \mathbb{R}^{n \times m_{\mathrm{c}}}$, $f_{\mathrm{d}} : \mathcal{Z} \to \mathbb{R}^n$ is continuous, $G_{\mathrm{d}} : \mathcal{Z} \to \mathbb{R}^{n \times m_{\mathrm{d}}}$, $h_{\mathrm{c}} : \mathbb{R}^n \to \mathbb{R}^{l_{\mathrm{c}}}$ and satisfies $h_{\mathrm{c}}(0) = 0$, $J_{\mathrm{c}} : \mathbb{R}^n \to \mathbb{R}^{l_{\mathrm{c}} \times m_{\mathrm{c}}}$, $h_{\mathrm{d}} : \mathcal{Z} \to \mathbb{R}^{l_{\mathrm{d}}}$, $J_{\mathrm{d}} : \mathcal{Z} \to \mathbb{R}^{l_{\mathrm{d}} \times m_{\mathrm{d}}}$, and $\mathcal{Z} \subset \mathbb{R}^n$ is the *resetting set.* Here we assume that if $x \in \mathcal{Z}$ then $x + f_{\mathrm{d}}(x) + G_{\mathrm{d}}(x)u_{\mathrm{d}} \notin \mathcal{Z}$, $u_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}}}$. In addition, we assume that if at time t the trajectory $x(t) \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$, then there exists $\varepsilon > 0$ such that for $0 < \delta < \varepsilon$, $x(t+\delta) \notin \mathcal{Z}$ so that Assumptions A2 and A3 of [6] are satisfied.

Definition 2.1 [6]. An impulsive dynamical system $\mathcal{G}$ given by (1)–(4) is *zero-state observable* if $(u_{\mathrm{c}}(t), u_{\mathrm{d}}(t)) \equiv (0,0)$, $(y_{\mathrm{c}}(t), y_{\mathrm{d}}(t)) \equiv (0,0)$ implies $x(t) \equiv 0$. An impulsive dynamical system $\mathcal{G}$ is *completely reachable* if for all $x_{\mathrm{i}} \in \mathbb{R}^n$, there exist a finite time $t_{\mathrm{i}} \leq 0$, square integrable inputs $u_{\mathrm{c}}(t)$ defined on $[t_{\mathrm{i}}, 0]$, and inputs $u_{\mathrm{d}}(t)$ defined on $[t_{\mathrm{i}}, 0)$, such that the state $x(t)$, $t \geq t_{\mathrm{i}}$, can be driven from $x(t_{\mathrm{i}}) = x_{\mathrm{i}}$ to $x(0) = 0$.

For the impulsive dynamical system $\mathcal{G}$ given by (1)–(4) a function $(r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), r_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$, where $r_{\mathrm{c}} : \mathbb{R}^{m_{\mathrm{c}}} \times \mathbb{R}^{l_{\mathrm{c}}} \to \mathbb{R}$ and $r_{\mathrm{d}} : \mathbb{R}^{m_{\mathrm{d}}} \times \mathbb{R}^{l_{\mathrm{d}}} \to \mathbb{R}$ are such that $r_{\mathrm{c}}(0,0) = 0$ and $r_{\mathrm{d}}(0,0) = 0$, is called a *hybrid supply rate* [6] if $r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}})$ is locally integrable; that is, for all input-output pairs $u_{\mathrm{c}}(t) \in \mathbb{R}^{m_{\mathrm{c}}}$, $y_{\mathrm{c}}(t) \in \mathbb{R}^{l_{\mathrm{c}}}$, $r_{\mathrm{c}}(\cdot,\cdot)$ satisfies $\int_t^{\hat{t}} |r_{\mathrm{c}}(u_{\mathrm{c}}(s), y_{\mathrm{c}}(s))|\,\mathrm{d}s < \infty, t, \hat{t} \geq 0$. Note that since all input-output pairs $u_{\mathrm{d}}(t_k) \in \mathbb{R}^{m_{\mathrm{d}}}$, $y_{\mathrm{d}}(t_k) \in \mathbb{R}^{l_{\mathrm{d}}}$, are defined for discrete instants, $r_{\mathrm{d}}(\cdot,\cdot)$ satisfies $\sum_{k \in \mathcal{N}_{[t,\hat{t})}} |r_{\mathrm{d}}(u_{\mathrm{d}}(i), y_{\mathrm{d}}(i))| < \infty$, where $k \in \mathcal{N}_{[t,\hat{t})} \triangleq \{k : t \leq t_k < \hat{t}\}$.

Definition 2.2 [6]. An impulsive dynamical system $\mathcal{G}$ of the form (1)–(4) is *dissipative with respect to the hybrid supply rate* $(r_{\mathrm{c}}, r_{\mathrm{d}})$ if there exists a C^0 nonnegative-definite function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$ called a *storage function*, such that the *hybrid dissipation inequality*

$$\begin{aligned} V_{\mathrm{s}}(x(t_2)) \ \leq \ V_{\mathrm{s}}(x(t_1)) &+ \int_{t_1}^{t_2} r_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\mathrm{d}t \\ &+ \sum_{k \in \mathcal{N}_{[t_1,t_2)}} r_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)), \end{aligned} \tag{5}$$

is satisfied for all $0 \leq t_1 < t_2 < \infty$, and where $x(t)$, $t \geq t_0$, is a solution to (1)–(4) with $(u_{\mathrm{c}}, u_{\mathrm{d}}) \in \mathbb{R}^{m_{\mathrm{c}}} \times \mathbb{R}^{m_{\mathrm{d}}}$ and $x(t_0) = x_0$.

The following result proven in [6] gives necessary and sufficient conditions for dissipativity over an interval $t \in (t_k, t_{k+1}]$ involving the consecutive resetting times t_k and t_{k+1}.

Theorem 2.1 [6]. $\mathcal{G}$ is dissipative with respect to the supply rate $(r_{\mathrm{c}}, r_{\mathrm{d}})$ if and only if there exists a C^0 nonnegative-definite function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$ such that, for all $k \in \mathcal{N}$,

$$V_{\mathrm{s}}(x(\hat{t})) - V_{\mathrm{s}}(x(t)) \leq \int_t^{\hat{t}} r_{\mathrm{c}}(u_{\mathrm{c}}(s), y_{\mathrm{c}}(s))\mathrm{d}s, \qquad t_k < t \leq \hat{t} \leq t_{k+1}, \tag{6}$$

$$\begin{aligned} V_{\mathrm{s}}(x(t_k) + f_{\mathrm{d}}(x(t_k)) \ + \ G_{\mathrm{d}}(x(t_k))u_{\mathrm{d}}(t_k)) - V_{\mathrm{s}}(x(t_k)) \\ \leq r_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)). \end{aligned} \tag{7}$$

If in Theorem 2.1 $V_{\mathrm{s}}(x(\cdot))$ is C^1 a.e. on $[t_0, \infty)$ except on an unbounded closed discrete set $\mathcal{T} = \{t_1, t_2, \ldots\}$, where $\mathcal{T}$ is the set of times when jumps occur for $x(t)$, then an equivalent statement for dissipativeness of the impulsive dynamical system $\mathcal{G}$ with respect to the hybrid supply rate $(r_{\mathrm{c}}, r_{\mathrm{d}})$ is

$$\dot{V}_{\mathrm{s}}(x(t)) \ \leq \ r_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t)), \qquad t_k < t \leq t_{k+1}, \tag{8}$$

$$\Delta V_{\mathrm{s}}(x(t_k)) \ \leq \ r_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)), \qquad k \in \mathcal{N}, \tag{9}$$

where $\dot{V}_{\mathrm{s}}(\cdot)$ denotes the total derivative of $V_{\mathrm{s}}(x(t))$ along the state trajectories $x(t)$, $t \in (t_k, t_{k+1}]$, of the impulsive dynamical system (1)–(4) and $\Delta V_{\mathrm{s}}(x(t_k)) \triangleq V_{\mathrm{s}}(x(t_k^+)) - V_{\mathrm{s}}(x(t_k)) = V_{\mathrm{s}}(x(t_k) + f_{\mathrm{d}}(x(t_k)) + G_{\mathrm{d}}(x(t_k))u_{\mathrm{d}}(t_k)) - V_{\mathrm{s}}(x(t_k))$, $k \in \mathcal{N}$, denotes the difference of the storage function $V_{\mathrm{s}}(x)$ at the resetting times t_k, $k \in \mathcal{N}$, of the impulsive dynamical system (1)–(4).

Next, we consider feedback interconnections of dissipative impulsive dynamical systems. Specifically, consider the impulsive dynamical system $\mathcal{G}$ given by (1)–(4) with the nonlinear feedback system $\mathcal{G}_{\mathrm{c}}$ given by

$$\dot{x}_{\mathrm{c}}(t) = f_{\mathrm{cc}}(x_{\mathrm{c}}(t)) + G_{\mathrm{cc}}(u_{\mathrm{cc}}(t), x_{\mathrm{c}}(t))u_{\mathrm{cc}}(t), \ x_{\mathrm{c}}(t) \notin \mathcal{Z}_{\mathrm{c}}, \tag{10}$$

$$\Delta x_{\mathrm{c}}(t) = f_{\mathrm{dc}}(x_{\mathrm{c}}(t)) + G_{\mathrm{dc}}(u_{\mathrm{dc}}(t), x_{\mathrm{c}}(t))u_{\mathrm{dc}}(t), \ x_{\mathrm{c}}(t) \in \mathcal{Z}_{\mathrm{c}}, \tag{11}$$

$$y_{\mathrm{cc}}(t) = h_{\mathrm{cc}}(x_{\mathrm{c}}(t)) + J_{\mathrm{cc}}(u_{\mathrm{cc}}(t), x_{\mathrm{c}}(t))u_{\mathrm{cc}}(t), \ x_{\mathrm{c}}(t) \notin \mathcal{Z}_{\mathrm{c}}, \tag{12}$$

$$y_{\mathrm{dc}}(t) = h_{\mathrm{dc}}(x_{\mathrm{c}}(t)) + J_{\mathrm{dc}}(u_{\mathrm{dc}}(t), x_{\mathrm{c}}(t))u_{\mathrm{dc}}(t), \ x_{\mathrm{c}}(t) \in \mathcal{Z}_{\mathrm{c}}, \tag{13}$$

where $t \geq 0$, $x_{\mathrm{c}}(t) \in \mathbb{R}^{n_{\mathrm{c}}}$, $u_{\mathrm{cc}}(t) \in \mathcal{U}_{\mathrm{cc}} \subseteq \mathbb{R}^{m_{\mathrm{cc}}}$, $u_{\mathrm{dc}}(t_k) \in \mathcal{U}_{\mathrm{dc}} \subseteq \mathbb{R}^{m_{\mathrm{dc}}}$, $y_{\mathrm{cc}}(t) \in \mathbb{R}^{l_{\mathrm{cc}}}$, $y_{\mathrm{dc}}(t_k) \in \mathbb{R}^{l_{\mathrm{dc}}}$, $f_{\mathrm{cc}} : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}}}$ is Lipschitz continuous and satisfies $f_{\mathrm{cc}}(0) = 0$, $G_{\mathrm{cc}} : \mathbb{R}^{m_{\mathrm{cc}}} \times \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}}} \times \mathbb{R}^{m_{\mathrm{cc}}}$, $f_{\mathrm{dc}} : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}}}$ is continuous and satisfies $f_{\mathrm{dc}}(0) = 0$, $G_{\mathrm{dc}} : \mathbb{R}^{m_{\mathrm{dc}}} \times \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}}} \times \mathbb{R}^{m_{\mathrm{dc}}}$, $J_{\mathrm{cc}} : \mathbb{R}^{m_{\mathrm{cc}}} \times \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{l_{\mathrm{cc}}} \times \mathbb{R}^{m_{\mathrm{cc}}}$, $h_{\mathrm{cc}} : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{l_{\mathrm{cc}}}$ and satisfies $h_{\mathrm{cc}}(0) = 0$, $J_{\mathrm{dc}} : \mathbb{R}^{m_{\mathrm{dc}}} \times \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{l_{\mathrm{dc}}} \times \mathbb{R}^{m_{\mathrm{dc}}}$, $h_{\mathrm{dc}} : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{l_{\mathrm{dc}}}$ and satisfies $h_{\mathrm{dc}}(0) = 0$, $m_{\mathrm{cc}} = l_{\mathrm{c}}$, $m_{\mathrm{dc}} = l_{\mathrm{d}}$, $l_{\mathrm{cc}} = m_{\mathrm{c}}$, $l_{\mathrm{dc}} = m_{\mathrm{d}}$, and $\mathcal{Z}_{\mathrm{c}} \subset \mathbb{R}^{n_{\mathrm{c}}}$ is such that Assumptions A2 and A3 of [6] hold. Note that with the feedback interconnection given by Figure 1, $(u_{\mathrm{cc}}, u_{\mathrm{dc}}) = (y_{\mathrm{c}}, y_{\mathrm{d}})$ and $(y_{\mathrm{cc}}, y_{\mathrm{dc}}) = (-u_{\mathrm{c}}, -u_{\mathrm{d}})$. Furthermore, even though the input-output pairs of the feedback interconnection shown on Figure 1 consist of two-vector inputs/two-vector outputs, at any given instant of time a single-vector input/single-vector output is active. Here, we assume that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is well posed; that is, $\det[I_{m_{\mathrm{c}}} + J_{\mathrm{cc}}(y_{\mathrm{c}}, x_{\mathrm{c}})J_{\mathrm{c}}(x)] \neq 0$ and $\det[I_{m_{\mathrm{d}}} + J_{\mathrm{dc}}(y_{\mathrm{d}}, x_{\mathrm{c}})J_{\mathrm{d}}(x)] \neq 0$ for all $y_{\mathrm{c}}, y_{\mathrm{d}}, x$, and x_{c}. The following result gives sufficient conditions for asymptotic stability of the negative feedback interconnection given by Figure 1.

Theorem 2.2 [7]. Let $Q_{\mathrm{c}} \in \mathbb{S}^{l_{\mathrm{c}}}$, $S_{\mathrm{c}} \in \mathbb{R}^{l_{\mathrm{c}} \times m_{\mathrm{c}}}$, $R_{\mathrm{c}} \in \mathbb{S}^{m_{\mathrm{c}}}$, $Q_{\mathrm{d}} \in \mathbb{S}^{l_{\mathrm{d}}}$, $S_{\mathrm{d}} \in \mathbb{R}^{l_{\mathrm{d}} \times m_{\mathrm{d}}}$, $R_{\mathrm{d}} \in \mathbb{S}^{m_{\mathrm{d}}}$, $Q_{\mathrm{cc}} \in \mathbb{S}^{l_{\mathrm{cc}}}$, $S_{\mathrm{cc}} \in \mathbb{R}^{l_{\mathrm{cc}} \times m_{\mathrm{cc}}}$, $R_{\mathrm{cc}} \in \mathbb{S}^{m_{\mathrm{cc}}}$, $Q_{\mathrm{dc}} \in \mathbb{S}^{l_{\mathrm{dc}}}$, $S_{\mathrm{dc}} \in \mathbb{R}^{l_{\mathrm{dc}} \times m_{\mathrm{dc}}}$, and $R_{\mathrm{dc}} \in \mathbb{S}^{m_{\mathrm{dc}}}$. Consider the closed-loop

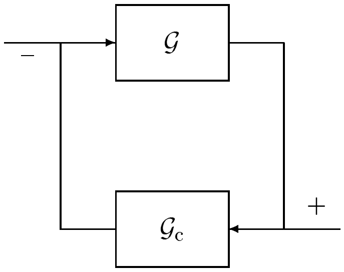

Figure 1: Feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$

system consisting of the nonlinear impulsive dynamical systems $\mathcal{G}$ given by (1)–(4) and $\mathcal{G}_{\mathrm{c}}$ given by (10)–(13) and assume $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are zero-state observable. Furthermore, assume $\mathcal{G}$ is dissipative with respect to the quadratic supply rate $(r_{\mathrm{c}}(u_{\mathrm{c}},y_{\mathrm{c}}), r_{\mathrm{d}}(u_{\mathrm{d}},y_{\mathrm{d}})) = (y_{\mathrm{c}}^{\mathrm{T}} Q_{\mathrm{c}} y_{\mathrm{c}} + 2y_{\mathrm{c}}^{\mathrm{T}} S_{\mathrm{c}} u_{\mathrm{c}} + u_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{c}} u_{\mathrm{c}},\, y_{\mathrm{d}}^{\mathrm{T}} Q_{\mathrm{d}} y_{\mathrm{d}} + 2y_{\mathrm{d}}^{\mathrm{T}} S_{\mathrm{d}} u_{\mathrm{d}} + u_{\mathrm{d}}^{\mathrm{T}} R_{\mathrm{d}} u_{\mathrm{d}})$ and has a radially unbounded storage function $V_{\mathrm{s}}(\cdot)$, and $\mathcal{G}_{\mathrm{c}}$ is dissipative with respect to the quadratic supply rate $(r_{\mathrm{cc}}(u_{\mathrm{cc}},y_{\mathrm{cc}}), r_{\mathrm{dc}}(u_{\mathrm{dc}},y_{\mathrm{dc}})) = (y_{\mathrm{cc}}^{\mathrm{T}} Q_{\mathrm{cc}} y_{\mathrm{cc}} + 2y_{\mathrm{cc}}^{\mathrm{T}} S_{\mathrm{cc}} u_{\mathrm{cc}} + u_{\mathrm{cc}}^{\mathrm{T}} R_{\mathrm{cc}} u_{\mathrm{cc}},\, y_{\mathrm{dc}}^{\mathrm{T}} Q_{\mathrm{dc}} y_{\mathrm{dc}} + 2y_{\mathrm{dc}}^{\mathrm{T}} S_{\mathrm{dc}} u_{\mathrm{dc}} + u_{\mathrm{dc}}^{\mathrm{T}} R_{\mathrm{dc}} u_{\mathrm{dc}})$ and has a radially unbounded storage function $V_{\mathrm{sc}}(\cdot)$. Finally, assume there exists a scalar $\sigma > 0$ such that

$$\hat{Q}_{\mathrm{c}} \triangleq \begin{bmatrix} Q_{\mathrm{c}} + \sigma R_{\mathrm{cc}} & -S_{\mathrm{c}} + \sigma S_{\mathrm{cc}}^{\mathrm{T}} \\ -S_{\mathrm{c}}^{\mathrm{T}} + \sigma S_{\mathrm{cc}} & R_{\mathrm{c}} + \sigma Q_{\mathrm{cc}} \end{bmatrix} < 0, \quad (14)$$

$$\hat{Q}_{\mathrm{d}} \triangleq \begin{bmatrix} Q_{\mathrm{d}} + \sigma R_{\mathrm{dc}} & -S_{\mathrm{d}} + \sigma S_{\mathrm{dc}}^{\mathrm{T}} \\ -S_{\mathrm{d}}^{\mathrm{T}} + \sigma S_{\mathrm{dc}} & R_{\mathrm{d}} + \sigma Q_{\mathrm{dc}} \end{bmatrix} < 0. \quad (15)$$

Then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is globally asymptotically stable.

Corollary 2.1. Consider the closed-loop system consisting of the nonlinear impulsive dynamical systems $\mathcal{G}$ given by (1)–(4) and $\mathcal{G}_{\mathrm{c}}$ given by (10)–(13) and assume $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are zero-state observable. Let $a_{\mathrm{c}}, b_{\mathrm{c}}, a_{\mathrm{cc}}, b_{\mathrm{cc}}, \delta_{\mathrm{c}}, a_{\mathrm{d}}, b_{\mathrm{d}}, a_{\mathrm{dc}}, b_{\mathrm{dc}}, \delta_{\mathrm{d}} \in \mathbb{R}$ be such that $b_{\mathrm{c}}, b_{\mathrm{d}} > 0$, $0 < a_{\mathrm{c}} + b_{\mathrm{c}}$, $0 < a_{\mathrm{d}} + b_{\mathrm{d}}$, $0 < 2\delta_{\mathrm{c}} < b_{\mathrm{c}} - a_{\mathrm{c}}$, $0 < 2\delta_{\mathrm{d}} < b_{\mathrm{d}} - a_{\mathrm{d}}$, $a_{\mathrm{cc}} = a_{\mathrm{c}} + \delta_{\mathrm{c}}$, $a_{\mathrm{dc}} = a_{\mathrm{d}} + \delta_{\mathrm{d}}$, $b_{\mathrm{cc}} = b_{\mathrm{c}} - \delta_{\mathrm{c}}$, $b_{\mathrm{dc}} = b_{\mathrm{d}} - \delta_{\mathrm{d}}$, and let $M_{\mathrm{c}} \in \mathbb{R}^{m_{\mathrm{c}} \times m_{\mathrm{c}}}$ and $M_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}} \times m_{\mathrm{d}}}$ be positive definite. If $\mathcal{G}$ is dissipative with respect to the quadratic supply rate $(r_{\mathrm{c}}(u_{\mathrm{c}},y_{\mathrm{c}}), r_{\mathrm{d}}(u_{\mathrm{d}},y_{\mathrm{d}})) = (\frac{a_{\mathrm{c}}b_{\mathrm{c}}}{a_{\mathrm{c}}+b_{\mathrm{c}}} y_{\mathrm{c}}^{\mathrm{T}} M_{\mathrm{c}} y_{\mathrm{c}} + u_{\mathrm{c}}^{\mathrm{T}} M_{\mathrm{c}} y_{\mathrm{c}} + \frac{1}{a_{\mathrm{c}}+b_{\mathrm{c}}} u_{\mathrm{c}}^{\mathrm{T}} M_{\mathrm{c}} u_{\mathrm{c}},\, \frac{a_{\mathrm{d}}b_{\mathrm{d}}}{a_{\mathrm{d}}+b_{\mathrm{d}}} y_{\mathrm{d}}^{\mathrm{T}} M_{\mathrm{d}} y_{\mathrm{d}} + u_{\mathrm{d}}^{\mathrm{T}} M_{\mathrm{d}} y_{\mathrm{d}} + \frac{1}{a_{\mathrm{d}}+b_{\mathrm{d}}} u_{\mathrm{d}}^{\mathrm{T}} M_{\mathrm{d}} u_{\mathrm{d}})$ and has a radially unbounded storage function and $\mathcal{G}_{\mathrm{c}}$ is dissipative with respect to the quadratic supply rate $(r_{\mathrm{cc}}(u_{\mathrm{cc}},y_{\mathrm{cc}}), r_{\mathrm{dc}}(u_{\mathrm{dc}},y_{\mathrm{dc}})) = (u_{\mathrm{cc}}^{\mathrm{T}} M_{\mathrm{c}} y_{\mathrm{cc}} - \frac{1}{a_{\mathrm{cc}}+b_{\mathrm{cc}}} y_{\mathrm{cc}}^{\mathrm{T}} M_{\mathrm{c}} y_{\mathrm{cc}} - \frac{a_{\mathrm{cc}}b_{\mathrm{cc}}}{a_{\mathrm{cc}}+b_{\mathrm{cc}}} u_{\mathrm{cc}}^{\mathrm{T}} M_{\mathrm{c}} u_{\mathrm{cc}},\, u_{\mathrm{dc}}^{\mathrm{T}} M_{\mathrm{d}} y_{\mathrm{dc}} - \frac{1}{a_{\mathrm{dc}}+b_{\mathrm{dc}}} y_{\mathrm{dc}}^{\mathrm{T}} M_{\mathrm{d}} y_{\mathrm{dc}} - \frac{a_{\mathrm{dc}}b_{\mathrm{dc}}}{a_{\mathrm{dc}}+b_{\mathrm{dc}}} u_{\mathrm{dc}}^{\mathrm{T}} M_{\mathrm{d}} u_{\mathrm{dc}})$ and has a radially unbounded storage function, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is globally asymptotically stable.

Proof. The proof is a direct consequence of Theorem 2.2 with $Q_{\mathrm{c}} = \frac{a_{\mathrm{c}}b_{\mathrm{c}}}{a_{\mathrm{c}}+b_{\mathrm{c}}} M_{\mathrm{c}}$, $S_{\mathrm{c}} = \frac{1}{2} M_{\mathrm{c}}$, $R_{\mathrm{c}} = \frac{1}{a_{\mathrm{c}}+b_{\mathrm{c}}} M_{\mathrm{c}}$, $Q_{\mathrm{cc}} = \frac{1}{a_{\mathrm{cc}}+b_{\mathrm{cc}}} M_{\mathrm{c}}$, $S_{\mathrm{cc}} = \frac{1}{2} M_{\mathrm{c}}$, $R_{\mathrm{cc}} = \frac{a_{\mathrm{cc}}b_{\mathrm{cc}}}{a_{\mathrm{cc}}+b_{\mathrm{cc}}} M_{\mathrm{c}}$, $Q_{\mathrm{d}} = \frac{a_{\mathrm{d}}b_{\mathrm{d}}}{a_{\mathrm{d}}+b_{\mathrm{d}}} M_{\mathrm{d}}$, $S_{\mathrm{d}} = \frac{1}{2} M_{\mathrm{d}}$, $R_{\mathrm{d}} = \frac{1}{a_{\mathrm{d}}+b_{\mathrm{d}}} M_{\mathrm{d}}$, $Q_{\mathrm{dc}} = \frac{1}{a_{\mathrm{dc}}+b_{\mathrm{dc}}} M_{\mathrm{d}}$, $S_{\mathrm{dc}} = \frac{1}{2} M_{\mathrm{d}}$, and $R_{\mathrm{dc}} = \frac{a_{\mathrm{dc}}b_{\mathrm{dc}}}{a_{\mathrm{dc}}+b_{\mathrm{dc}}} M_{\mathrm{d}}$. Specifically, let $\sigma > 0$ be such that

$$\sigma \left(\frac{\delta_{\mathrm{c}}^2}{(a_{\mathrm{c}} + b_{\mathrm{c}})^2} - \frac{1}{4} \right) + \frac{1}{4} > 0,\ \sigma \left(\frac{\delta_{\mathrm{d}}^2}{(a_{\mathrm{d}} + b_{\mathrm{d}})^2} - \frac{1}{4} \right) + \frac{1}{4} > 0.$$

In this case, $\hat{Q}_{\mathrm{c}}$ and $\hat{Q}_{\mathrm{d}}$ given by (14) and (15), respectively, satisfy $\hat{Q}_{\mathrm{c}} < 0$ and $\hat{Q}_{\mathrm{d}} < 0$ so that all the conditions of Theorem 2.2 are satisfied. □

Now, we consider impulsive nonlinear systems $\mathcal{G}$ of the form given by (1)–(4) with $l_{\mathrm{c}} = m_{\mathrm{c}}$, $l_{\mathrm{d}} = m_{\mathrm{d}}$, $J_{\mathrm{c}}(x) \equiv 0$, $J_{\mathrm{d}}(x) \equiv 0$, $h_{\mathrm{c}}(x) = -\phi_{\mathrm{c}}(x)$, and $h_{\mathrm{d}}(x) = -\phi_{\mathrm{d}}(x)$, where $\phi_{\mathrm{c}} : \mathbb{R}^n \to \mathbb{R}^{m_{\mathrm{c}}}$ and $\phi_{\mathrm{d}} : \mathbb{R}^n \to \mathbb{R}^{m_{\mathrm{d}}}$ are such that $\mathcal{G}$ is asymptotically stable with $(u_{\mathrm{c}}, u_{\mathrm{d}}) = (-y_{\mathrm{c}}, -y_{\mathrm{d}})$. Furthermore, we assume that the system $\mathcal{G}$ is zero-state observable. In this case, $\mathcal{G}$ becomes

$$\begin{aligned} \dot{x}(t) &= f_{\mathrm{c}}(x(t)) + G_{\mathrm{c}}(x(t)) u_{\mathrm{c}}(t),\ x(0) = x_0, \\ &\qquad\qquad x(t) \notin \mathcal{Z}, & (16) \\ \Delta x(t) &= f_{\mathrm{d}}(x(t)) + G_{\mathrm{d}}(x(t)) u_{\mathrm{d}}(t),\ x(t) \in \mathcal{Z}, & (17) \\ y_{\mathrm{c}}(t) &= -\phi_{\mathrm{c}}(x(t)),\ x(t) \notin \mathcal{Z}, & (18) \\ y_{\mathrm{d}}(t) &= -\phi_{\mathrm{d}}(x(t)),\ x(t) \in \mathcal{Z}. & (19) \end{aligned}$$

Next, we define the hybrid robustness margins for $\mathcal{G}$ given by (16)–(19). Specifically, consider the negative feedback interconnection of $\mathcal{G}$ and $\Delta(\cdot,\cdot)$ given in Figure 2, where $\Delta : \mathbb{R}^{m_{\mathrm{c}}} \times \mathbb{R}^{m_{\mathrm{d}}} \to \mathbb{R}^{m_{\mathrm{c}}} \times \mathbb{R}^{m_{\mathrm{d}}}$ is either a linear operator $\Delta(y_{\mathrm{c}}, y_{\mathrm{d}}) = (\Delta_{\mathrm{c}} y_{\mathrm{c}}, \Delta_{\mathrm{d}} y_{\mathrm{d}})$, a nonlinear static operator $\Delta(y_{\mathrm{c}}, y_{\mathrm{d}}) = (\sigma_{\mathrm{c}}(y_{\mathrm{c}}), \sigma_{\mathrm{d}}(y_{\mathrm{d}}))$, or a dynamic operator $\Delta(\cdot,\cdot)$. In this case, $(u_{\mathrm{c}}, u_{\mathrm{d}}) = (-\Delta_{\mathrm{c}}(y_{\mathrm{c}}), -\Delta_{\mathrm{d}}(y_{\mathrm{d}}))$. Furthermore, we assume that in the nominal case $\Delta(\cdot,\cdot)$ is such that $(u_{\mathrm{c}}, u_{\mathrm{d}}) = (-\Delta_{\mathrm{c}}(y_{\mathrm{c}}), -\Delta_{\mathrm{d}}(y_{\mathrm{d}})) = (-y_{\mathrm{c}}, -y_{\mathrm{d}})$ so that the nominal closed-loop system is asymptotically stable.

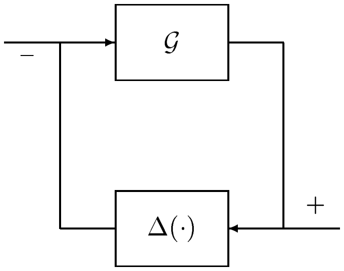

Figure 2: Feedback interconnection of $\mathcal{G}$ and $\Delta(\cdot)$

Definition 2.3. Let $\alpha_{\mathrm{c}}, \beta_{\mathrm{c}}, \alpha_{\mathrm{d}}, \beta_{\mathrm{d}} \in \mathbb{R}$ be such that $0 \leq \alpha_{\mathrm{c}} < 1 < \beta_{\mathrm{c}} < \infty$ and $0 \leq \alpha_{\mathrm{d}} < 1 < \beta_{\mathrm{d}} < \infty$. Then the nonlinear system $\mathcal{G}$ given by (16), (19) is said to have a *hybrid gain margin* $((\alpha_{\mathrm{c}}, \beta_{\mathrm{c}}), (\alpha_{\mathrm{d}}, \beta_{\mathrm{d}}))$ if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(y_{\mathrm{c}}, y_{\mathrm{d}}) = (\Delta_{\mathrm{c}} y_{\mathrm{c}}, \Delta_{\mathrm{d}} y_{\mathrm{d}})$ is globally asymptotically stable for all $\Delta_{\mathrm{c}} = \mathrm{diag}(k_{1\mathrm{c}}, \cdots, k_{m_{\mathrm{c}}\mathrm{c}})$, where $k_{i\mathrm{c}} \in (\alpha_{\mathrm{c}}, \beta_{\mathrm{c}})$, $i = 1, \ldots, m_{\mathrm{c}}$ and $\Delta_{\mathrm{d}} = \mathrm{diag}(k_{1\mathrm{d}}, \cdots, k_{m_{\mathrm{d}}\mathrm{d}})$, where $k_{i\mathrm{d}} \in (\alpha_{\mathrm{d}}, \beta_{\mathrm{d}})$, $i = 1, \ldots, m_{\mathrm{d}}$.

Definition 2.4. Let $\alpha_{\mathrm{c}}, \beta_{\mathrm{c}}, \alpha_{\mathrm{d}}, \beta_{\mathrm{d}} \in \mathbb{R}$ be such that $0 \leq \alpha_{\mathrm{c}} < 1 < \beta_{\mathrm{c}} < \infty$ and $0 \leq \alpha_{\mathrm{d}} < 1 < \beta_{\mathrm{d}} < \infty$. Then the nonlinear system $\mathcal{G}$ given by (16), (19) is said to have a *hybrid sector margin* $((\alpha_{\rm c},\beta_{\rm c}),(\alpha_{\rm d},\beta_{\rm d}))$ if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(y_{\rm c},y_{\rm d}) = (\sigma_{\rm c}(y_{\rm c}),\sigma_{\rm d}(y_{\rm d}))$ is globally asymptotically stable for all static nonlinearities $\sigma_{\rm c} : \mathbb{R}^{m_{\rm c}} \to \mathbb{R}^{m_{\rm c}}$ and $\sigma_{\rm d} : \mathbb{R}^{m_{\rm d}} \to \mathbb{R}^{m_{\rm d}}$ such that $\sigma_{\rm c}(0) = 0$, $\sigma_{\rm d}(0) = 0$, $\sigma_{\rm c}(y_{\rm c}) = [\sigma_{1\rm c}(y_{1\rm c}),\cdots,\sigma_{m_{\rm c}\rm c}(y_{m_{\rm c}\rm c})]$, $\sigma_{\rm d}(y_{\rm d}) = [\sigma_{1\rm d}(y_{1\rm d}),\cdots,\sigma_{m_{\rm d}\rm d}(y_{m_{\rm d}\rm d})]$, $\alpha_{\rm c}y_{i\rm c}^2 < \sigma_{i\rm c}(y_{i\rm c})y_{i\rm c} < \beta_{\rm c}y_{i\rm c}^2$, for all $y_{i\rm c} \neq 0$, $i = 1,\ldots,m_{\rm c}$, and $\alpha_{\rm d}y_{i\rm d}^2 < \sigma_{i\rm d}(y_{i\rm d})y_{i\rm d} < \beta_{\rm d}y_{i\rm d}^2$, for all $y_{i\rm d} \neq 0$, $i = 1,\ldots,m_{\rm d}$.

Definition 2.5. Let $\alpha_{\rm c},\beta_{\rm c},\alpha_{\rm d},\beta_{\rm d} \in \mathbb{R}$ be such that $0 \leq \alpha_{\rm c} < 1 < \beta_{\rm c} < \infty$ and $0 \leq \alpha_{\rm d} < 1 < \beta_{\rm d} < \infty$. Then the nonlinear system $\mathcal{G}$ given by (16), (19) is said to have a *hybrid disk margin* $((\alpha_{\rm c},\beta_{\rm c}),(\alpha_{\rm d},\beta_{\rm d}))$ if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(y_{\rm c},y_{\rm d}) = (\Delta_{\rm c}(y_{\rm c}),\Delta_{\rm d}(y_{\rm d}))$ is globally asymptotically stable for all dynamic operators $\Delta(\cdot,\cdot)$ such that $\Delta(\cdot,\cdot)$ is zero-state observable and dissipative with respect to the hybrid supply rate $(r_{\rm c},r_{\rm d})$ where $r_{\rm c}(u_{\rm c},u_{\rm c}) = u_{\rm c}^{\rm T}y_{\rm c} - \frac{1}{\hat\alpha_{\rm c}+\hat\beta_{\rm c}}y_{\rm c}^{\rm T}y_{\rm c} - \frac{\hat\alpha_{\rm c}\hat\beta_{\rm c}}{\hat\alpha_{\rm c}+\hat\beta_{\rm c}}u_{\rm c}^{\rm T}u_{\rm c}$, $r_{\rm d}(u_{\rm d},u_{\rm d}) = u_{\rm d}^{\rm T}y_{\rm d} - \frac{1}{\hat\alpha_{\rm d}+\hat\beta_{\rm d}}y_{\rm d}^{\rm T}y_{\rm d} - \frac{\hat\alpha_{\rm d}\hat\beta_{\rm d}}{\hat\alpha_{\rm d}+\hat\beta_{\rm d}}u_{\rm d}^{\rm T}u_{\rm d}$, and where $\hat\alpha_{\rm c} = \alpha_{\rm c} + \delta$, $\hat\beta_{\rm c} = \beta_{\rm c} - \delta$, and $\delta \in \mathbb{R}$ such that $0 < 2\delta < \min\{(\beta_{\rm c} - \alpha_{\rm c}),(\beta_{\rm d} - \alpha_{\rm d})\}$.

Remark 2.1. Note that if $\mathcal{G}$ has a hybrid disk margin $((\alpha_{\rm c},\beta_{\rm c}),(\alpha_{\rm d},\beta_{\rm d}))$ then $\mathcal{G}$ has hybrid gain and sector margins $((\alpha_{\rm c},\beta_{\rm c}),(\alpha_{\rm d},\beta_{\rm d}))$.

3. Gain, Sector, and Disk Margins of Optimal Hybrid Regulators

In this section we derive the robustness margins for an optimal hybrid regulator that minimizes a hybrid performance functional involving a continuous-time cost for addressing performance of the continuous-time system dynamics and a discrete-time cost for addressing performance at the resetting instants. Specifically, we consider the impulsive nonlinear system given by (16), (17) with a nonlinear-nonquadratic performance criterion

$$\begin{aligned} &J(x_0,u_{\rm c}(\cdot),u_{\rm d}(\cdot)) \\ &\quad = \int_0^\infty [L_{1\rm c}(x(t)) + u_{\rm c}^{\rm T}(t)R_{2\rm c}(x(t))u_{\rm c}(t)]{\rm d}t \\ &\qquad + \sum_{k\in\mathcal{N}_{[0,\infty)}} [L_{1\rm d}(x(t_k)) + u_{\rm d}^{\rm T}(t_k)R_{2\rm d}(x(t_k))u_{\rm d}(t_k)], \end{aligned} \tag{20}$$

where $L_{1\rm c} : \mathbb{R}^n \to \mathbb{R}$ and satisfies $L_{1\rm c}(x) \geq 0$, $x \in \mathbb{R}^n$, $R_{2\rm c} : \mathbb{R}^n \to \mathbb{P}^{m_{\rm c}}$, $L_{1\rm d} : \mathbb{R}^n \to \mathbb{R}$ and satisfies $L_{1\rm d}(x) \geq 0$, $x \in \mathbb{R}^n$, and $R_{2\rm d} : \mathbb{R}^n \to \mathbb{P}^{m_{\rm d}}$. The optimal nonlinear hybrid feedback controller $(u_{\rm c},u_{\rm d}) = (\phi_{\rm c}(x),\phi_{\rm d}(x))$ that minimizes the nonlinear-nonquadratic performance criterion (20) is given in [7]. For the statement of the following result define the set of asymptotically stabilizing hybrid controllers by

$$\begin{aligned} \mathcal{C}(x_0) \overset{\triangle}{=} \{&(u_{\rm c}(\cdot),u_{\rm d}(\cdot)) : (u_{\rm c}(\cdot),u_{\rm d}(\cdot)) \text{ is admissible and} \\ &\text{the zero solution } x(t) \equiv 0 \text{ to (16), (17)} \\ &\text{is asymptotically stable}\}. \end{aligned}$$

Theorem 3.1 [7]. Consider the nonlinear impulsive controlled system (16) and (17) with performance functional (20). Assume there exists a $\rm C^1$ function $V : \mathbb{R}^n \to \mathbb{R}$, and functions $P_{12} : \mathbb{R}^n \to \mathbb{R}^{1\times m_{\rm d}}$ and $P_2 : \mathbb{R}^n \to \mathbb{N}^{m_{\rm d}}$ such that $V(0) = 0$, $V(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$,

$$V'(x)[f_{\rm c}(x) - \tfrac{1}{2}G_{\rm c}(x)R_{2\rm c}^{-1}(x)G_{\rm c}^{\rm T}(x)V'^{\rm T}(x)] < 0, \quad x \notin \mathcal{Z},\ x \neq 0, \tag{21}$$

$$\begin{aligned} &V\Big(\ x + f_{\rm d}(x) \\ &\quad -\tfrac{1}{2}G_{\rm d}(x)(R_{2\rm d}(x) + P_2(x))^{-1}P_{12}^{\rm T}(x)\Big) - V(x) \leq 0, \\ &\qquad x \in \mathcal{Z}, \end{aligned} \tag{22}$$

$$\begin{aligned} &V(x + f_{\rm d}(x) + G_{\rm d}(x)u_{\rm d}) \\ &= V(x + f_{\rm d}(x)) + P_{12}(x)u_{\rm d} + u_{\rm d}^{\rm T}P_2(x)u_{\rm d}, \end{aligned} \tag{23}$$

where $u_{\rm d}$ is admissible, and

$$V(x) \to \infty \ \text{ as } \ \|x\| \to \infty. \tag{24}$$

Then the zero solution $x(t) \equiv 0$ to the closed-loop system

$$\begin{aligned} \dot x(t) &= f_{\rm c}(x(t)) + G_{\rm c}(x(t))\phi_{\rm c}(x(t)), \\ &\qquad x(0) = x_0,\ x(t) \notin \mathcal{Z}, \quad (25) \\ \Delta x(t) &= f_{\rm d}(x(t)) + G_{\rm d}(x(t))\phi_{\rm d}(x(t)),\ x(t) \in \mathcal{Z} \quad (26) \end{aligned}$$

is globally asymptotically stable with the hybrid feedback control law

$$\begin{aligned} \phi_{\rm c}(x) &= -\tfrac{1}{2}R_{2\rm c}^{-1}(x)G_{\rm c}^{\rm T}(x)V'^{\rm T}(x), \qquad x \notin \mathcal{Z}, \quad (27) \\ \phi_{\rm d}(x) &= -\tfrac{1}{2}(R_{2\rm d}(x) + P_2(x))^{-1}P_{12}^{\rm T}(x),\ x \in \mathcal{Z}, \quad (28) \end{aligned}$$

and performance functional (20), with

$$\begin{aligned} L_{1\rm c}(x) &= \phi_{\rm c}^{\rm T}(x)R_{2\rm c}(x)\phi_{\rm c}(x) - V'(x)f_{\rm c}(x), \quad (29) \\ L_{1\rm d}(x) &= \phi_{\rm d}^{\rm T}(x)(R_{2\rm d}(x) + P_2(x))\phi_{\rm d}(x) \\ &\quad -V(x + f_{\rm d}(x)) + V(x), \quad (30) \end{aligned}$$

is minimized in the sense that

$$\begin{aligned} &J(x_0,\phi_{\rm c}(x(\cdot)),\phi_{\rm d}(x(\cdot))) \\ &= \min_{(u_{\rm c}(\cdot),u_{\rm d}(\cdot))\in\mathcal{C}(x_0)} J(x_0,u_{\rm c}(\cdot),u_{\rm d}(\cdot)),\ x_0 \in \mathbb{R}^n. \end{aligned} \tag{31}$$

Finally,

$$J(x_0,\phi_{\rm c}(x(\cdot)),\phi_{\rm d}(x(\cdot))) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \tag{32}$$

The following key lemma is needed for developing the main result of this paper.

Lemma 3.1. Consider the impulsive nonlinear dynamical system $\mathcal{G}$ given by (16)–(19) where $(\phi_{\rm c}(x),\ \phi_{\rm d}(x))$ is a stabilizing optimal hybrid control law given by (27), (28) and where $V(x)$, $P_{12}(x)$, and $P_2(x)$ are such that $V(0) = 0$, $V(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, and satisfy (21)–(24). Then for all $u_{\rm c}(\cdot) \in \mathbb{R}^{m_{\rm c}}$, $u_{\rm d}(\cdot) \in \mathbb{R}^{m_{\rm d}}$, the solution $x(t)$, $t \geq 0$, to (16), (17) satisfies

$$\begin{aligned} &V(x(\hat t)) - V(x(t)) \\ &\quad \leq \int_t^{\hat t} \{[u_{\rm c}(t) + y_{\rm c}(t)]^{\rm T}R_{2\rm c}(x(t))[u_{\rm c}(t) + y_{\rm c}(t)] \\ &\quad -u_{\rm c}^{\rm T}(t)R_{2\rm c}(x(t))u_{\rm c}(t)\}{\rm d}t,\ t_k \leq t < \hat t < t_{k+1}, \quad (33) \\ &V(x(t_k^+)) - V(x(t_k)) \\ &\quad \leq \{[u_{\rm d}(t_k) + y_{\rm d}(t_k)]^{\rm T}\Big(\ R_{2\rm d}(x(t_k)) \\ &\quad +P_{2u}(x(t_k))\ \Big)[u_{\rm d}(t_k) + y_{\rm d}(t_k)] \\ &\quad -u_{\rm d}^{\rm T}(x(t_k))R_{2\rm d}(x(t_k))u_{\rm d}(t_k)\},\ k \in \mathcal{N}. \quad (34) \end{aligned}$$

Proof. Note that it follows from (29) that for all $x \in \mathbb{R}^n$ and $u_{\rm c} \in \mathbb{R}^{m_{\rm c}}$,

$$\begin{aligned} u_{\rm c}^{\rm T} R_{2{\rm c}}(x) u_{\rm c} &\le L_{1{\rm c}}(x) + u_{\rm c}^{\rm T} R_{2{\rm c}}(x) u_{\rm c} \\ &= \phi_{\rm c}^{\rm T}(x) R_{2{\rm c}}(x) \phi_{\rm c}(x) - V'(x) f_{\rm c}(x) \\ &\quad + u_{\rm c}^{\rm T} R_{2{\rm c}}(x) u_{\rm c} \\ &= y_{\rm c}^{\rm T} R_{2{\rm c}}(x) y_{\rm c}(x) + 2 y_{\rm c}^{\rm T} R_{2{\rm c}}(x) u_{\rm c} \\ &\quad - V'(x)[f_{\rm c}(x) + G_{\rm c}(x) u_{\rm c}] + u_{\rm c}^{\rm T} R_{2{\rm c}}(x) u_{\rm c} \\ &= [u_{\rm c} + y_{\rm c}]^{\rm T} R_{2{\rm c}}(x)[u_{\rm c} + y_{\rm c}] \\ &\quad - V'(x)[f_{\rm c}(x) + G_{\rm c}(x) u_{\rm c}], \end{aligned}$$

which implies that, for all $u_{\rm c}(\cdot) \in \mathbb{R}^{m_{\rm c}}$ and $t \ge 0$, $t \ne t_k$, $k \in \mathcal{N}$,

$$\begin{aligned} & u_{\rm c}^{\rm T}(t) R_{2{\rm c}}(x(t)) u_{\rm c}(t) \\ &\quad \le [u_{\rm c}(t) + y_{\rm c}(t)]^{\rm T} R_{2{\rm c}}(x(t))[u_{\rm c}(t) + y_{\rm c}(t)] \\ &\quad - \dot{V}(x(t)). \end{aligned}$$

Now, integrating over $[t, \hat{t}]$ yields (33).

Next, it follows from (30) that for all $x \in \mathbb{R}^n$ and $u_{\rm d} \in \mathbb{R}^{m_{\rm d}}$,

$$\begin{aligned} u_{\rm d}^{\rm T} R_{2{\rm d}}(x) u_{\rm d} &\le L_{1{\rm d}}(x) + u_{\rm d}^{\rm T} R_{2{\rm d}}(x) u_{\rm d} \\ &= \phi_{\rm d}^{\rm T}(x)(R_{2{\rm d}}(x) + P_2(x)) \phi_{\rm d}(x) \\ &\quad - V(x + f_{\rm d}(x)) + V(x) + u_{\rm d}^{\rm T} R_{2{\rm d}}(x) u_{\rm d} \\ &= y_{\rm d}^{\rm T}(R_{2{\rm d}}(x) + P_2(x)) y_{\rm d}(x) \\ &\quad + 2 y_{\rm d}^{\rm T}(R_{2{\rm d}}(x) + P_2(x)) u_{\rm d} \\ &\quad - V(x + f_{\rm d}(x) + G_{\rm d}(x) u_{\rm d}) \\ &\quad + V(x) + u_{\rm d}^{\rm T} R_{2{\rm d}}(x) u_{\rm d} + u_{\rm d}^{\rm T} P_2(x) u_{\rm d} \\ &= [u_{\rm d} + y_{\rm d}]^{\rm T}(R_{2{\rm d}}(x) + P_2(x))[u_{\rm d} + y_{\rm d}] \\ &\quad - V(x + f_{\rm d}(x) + G_{\rm d}(x) u_{\rm d}) + V(x), \end{aligned}$$

which implies (34), for all $u_{\rm d}(t_k) \in \mathbb{R}^{m_{\rm d}}$, $k \in \mathcal{N}$. ☐

Note that with $R_{2{\rm c}}(x) \equiv I_{m_{\rm c}}$ and $R_{2{\rm d}}(x) \equiv I_{m_{\rm d}}$ conditions (33) and (34) are precisely the hybrid counterpart of the return difference condition for continuous-time and discrete-time systems [8, 11, 12]. However, in the continuous-time case an optimal feedback control law $\phi(x)$ satisfying the return difference condition is equivalent to the fact that a continuous-time nonlinear affine system with input u and output $y = -\phi(x)$ is dissipative with respect to the quadratic supply rate $[u+y]^{\rm T}[u+y] - u^{\rm T}u$. Hence, using the nonlinear Kalman-Yakubovich-Popov lemma [13] one can show that a feedback control law $\phi(x)$ satisfies the return difference inequality if and only if $\phi(x)$ is optimal with respect to a performance criterion involving a nonnegative-definite weighting function on the state. Alternatively, in the hybrid case, (33) and (34) are not equivalent to the dissipativity of (16)–(19) due to the presence of $P_{2u}(x)$ in (34). However, it follows from Lemma 3.1 and Theorem 2.1 that (33) and (34) do imply that if $(\phi_{\rm c}(x), \phi_{\rm d}(x))$ is a stabilizing optimal hybrid control law then $\mathcal{G}$ is dissipative with respect to a quadratic supply rate.

Next, we present our main result which provides hybrid disk margins for the optimal hybrid regulator given by Theorem 3.1. For the following result define

$$\overline{\gamma}_{\rm c} \triangleq \sup_{x \in \mathbb{R}^n} \sigma_{\max}(R_{2{\rm c}}(x)), \ \underline{\gamma}_{\rm c} \triangleq \inf_{x \in \mathbb{R}^n} \sigma_{\min}(R_{2{\rm c}}(x)), \quad (35)$$

$$\overline{\gamma}_{\rm d} \triangleq \sup_{x \in \mathbb{R}^n} \sigma_{\max}(R_{2{\rm d}}(x) + P_2(x)), \ \underline{\gamma}_{\rm d} \triangleq \inf_{x \in \mathbb{R}^n} \sigma_{\min}(R_{2{\rm d}}(x)). \quad (36)$$

Furthermore, assume that $\overline{\gamma}_{\rm c} = \overline{\gamma}_{\rm d}$.

Theorem 3.2. Consider the impulsive nonlinear dynamical system $\mathcal{G}$ given by (16)–(19) where $(\phi_{\rm c}(x), \phi_{\rm d}(x))$ is an optimal stabilizing control law given by (27), (28) and where $V(x)$, $P_{12}(x)$, and $P_2(x)$ are such that $V(0) = 0$, $V(x) > 0$, $x \in \mathbb{R}^n$, $x \ne 0$, and satisfy (21)–(24). Then the impulsive nonlinear system $\mathcal{G}$ given by (16)–(19) is dissipative with respect to the hybrid supply rate $(r_{\rm c}(u_{\rm c}, y_{\rm c}), r_{\rm d}(u_{\rm d}, y_{\rm d})) = (u_{\rm c}^{\rm T} y_{\rm c} + \frac{(1-\theta_{\rm c}^2)}{2} u_{\rm c}^{\rm T} u_{\rm c} + \frac{1}{2} y_{\rm c}^{\rm T} y_{\rm c}, u_{\rm d}^{\rm T} y_{\rm d} + \frac{(1-\theta_{\rm d}^2)}{2} u_{\rm d}^{\rm T} u_{\rm d} + \frac{1}{2} y_{\rm d}^{\rm T} y_{\rm d})$ and has a hybrid disk margin $(\frac{1}{1+\theta_{\rm c}}, \frac{1}{1-\theta_{\rm c}}), (\frac{1}{1+\theta_{\rm d}}, (\frac{1}{1-\theta_{\rm d}})$, where $\theta_{\rm c} \triangleq \sqrt{\underline{\gamma}_{\rm c}/\overline{\gamma}_{\rm c}}$ and $\theta_{\rm d} \triangleq \sqrt{\underline{\gamma}_{\rm d}/\overline{\gamma}_{\rm d}}$.

Proof. Note that for all $u_{\rm c}(\cdot) \in \mathbb{R}^{m_{\rm c}}$ and $u_{\rm d}(\cdot) \in \mathbb{R}^{m_{\rm d}}$, the solution $x(t)$, $t \ge 0$, to (16), (17) satisfies (33) and (34) which implies that

$$\begin{aligned} & V(x(\hat{t})) - V(x(t)) \\ &\qquad \le \int_t^{\hat{t}} \{\overline{\gamma}_{\rm c}[u_{\rm c}(t) + y_{\rm c}(t)]^{\rm T}[u_{\rm c}(t) + y_{\rm c}(t)] \\ &\qquad - \underline{\gamma}_{\rm c} u_{\rm c}^{\rm T}(t) u_{\rm c}(t)\} {\rm d}t, \ t_k \le t < \hat{t} < t_{k+1}, \end{aligned}$$

and

$$\begin{aligned} & V(x(t_k^+)) - V(x(t_k)) \\ &\qquad \le \{\overline{\gamma}_{\rm d}[u_{\rm d}(t_k) + y_{\rm d}(t_k)]^{\rm T}[u_{\rm d}(t_k) + y_{\rm d}(t_k)] \\ &\qquad - \underline{\gamma}_{\rm d} u_{\rm d}^{\rm T}(x(t_k)) u_{\rm d}(t_k)\}, \qquad k \in \mathcal{N}. \end{aligned}$$

Hence, with the storage function $V_{\rm s}(x) = \frac{1}{2\overline{\gamma}_{\rm c}} V(x)$, it follows from Theorem 2.1 that $\mathcal{G}$ is dissipative with respect to hybrid supply rate $(r_{\rm c}(u_{\rm c}, y_{\rm c}), r_{\rm d}(u_{\rm d}, y_{\rm d})) = (u_{\rm c}^{\rm T} y_{\rm c} + \frac{(1-\theta_{\rm c}^2)}{2} u_{\rm c}^{\rm T} u_{\rm c} + \frac{1}{2} y_{\rm c}^{\rm T} y_{\rm c}, u_{\rm d}^{\rm T} y_{\rm d} + \frac{(1-\theta_{\rm d}^2)}{2} u_{\rm d}^{\rm T} u_{\rm d} + \frac{1}{2} y_{\rm d}^{\rm T} y_{\rm d})$. Now, the result follows immediately from Lemma 3.1 and Definition 2.5 with $\alpha_{\rm c} = \frac{1}{1+\theta_{\rm c}}$, $\beta_{\rm c} = \frac{1}{1-\theta_{\rm c}}$, $\alpha_{\rm d} = \frac{1}{1+\theta_{\rm d}}$, and $\beta_{\rm d} = \frac{1}{1-\theta_{\rm d}}$. ☐

Remark 3.1. Note that in the case where $R_{2{\rm c}}(x) \equiv I_{m_{\rm c}}$ it follows that $\theta_{\rm c} = 1$. Hence, the continuous-time dynamics of $\mathcal{G}$ has a disk margin of $(\frac{1}{2}, \infty)$. This of course does not imply that the hybrid optimal nonlinear regulator has hybrid disk margin of $\big((\frac{1}{2}, \infty), (\frac{1}{2}, \infty)\big)$.

4. Specialization to Linear Impulsive Dynamical Systems

In this section we specialize our main results to the case of linear impulsive systems. First, however, we recall the specialization of Theorem 3.1 to linear impulsive systems. For the following result let $A_{\rm c} \in \mathbb{R}^{n \times n}$, $B_{\rm c} \in \mathbb{R}^{n \times m_{\rm c}}$, $A_{\rm c} \in \mathbb{R}^{n \times n}$, $B_{\rm d} \in \mathbb{R}^{n \times m_{\rm d}}$, $R_{1{\rm c}} \in \mathbb{R}^{n \times n}$, $R_{2{\rm c}} \in \mathbb{R}^{m_{\rm c} \times m_{\rm c}}$, $R_{1{\rm d}} \in \mathbb{R}^{n \times n}$, and $R_{2{\rm d}} \in \mathbb{R}^{m_{\rm d} \times m_{\rm d}}$ be given, where $R_{1{\rm c}}$, $R_{2{\rm c}}$, $R_{1{\rm d}}$, and $R_{2{\rm d}}$ are positive definite.

Corollary 4.1 [7]. Consider the linear controlled impulsive system

$$\dot{x}(t) = A_{\rm c}x(t) + B_{\rm c}u_{\rm c}(t),\ x(0) = x_0,\ x(t) \notin \mathcal{Z}, \quad (37)$$

$$\Delta x(t) = (A_{\rm c} - I_n)x(t) + B_{\rm d}u_{\rm d}(t),\ x(t) \in \mathcal{Z}, \quad (38)$$

with quadratic hybrid performance functional

$$\begin{aligned} J(x_0, u_{\rm c}(\cdot), u_{\rm d}(\cdot)) &= \int_0^\infty [x^{\rm T}(t)R_{1\rm c}x(t) + u_{\rm c}^{\rm T}(t)R_{2\rm c}u_{\rm c}(t)]{\rm d}t \\ &+ \sum_{k\in\mathcal{N}_{[0,\infty)}} [x^{\rm T}(t_k)R_{1\rm d}x(t_k) + u_{\rm d}^{\rm T}(t_k)R_{2\rm d}u_{\rm d}(t_k)], \quad (39) \end{aligned}$$

where $(u_{\rm c}(\cdot), u_{\rm d}(\cdot))$ is an admissible hybrid control. Furthermore, assume there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$0 = x^{\rm T}(A_{\rm c}^{\rm T}P + PA_{\rm c} + R_{1\rm c} - PB_{\rm c}R_{2\rm c}^{-1}B_{\rm c}^{\rm T}P)x,\ x \notin \mathcal{Z}, \quad (40)$$

$$\begin{aligned} 0 = x^{\rm T}(&A_{\rm c}^{\rm T}PA_{\rm c} - P + R_{1\rm d} \\ &-A_{\rm c}^{\rm T}PB_{\rm d}(R_{2\rm d} + B_{\rm d}^{\rm T}PB_{\rm d})^{-1}B_{\rm d}^{\rm T}PA_{\rm c})x,\ x \in \mathcal{Z}. \quad (41) \end{aligned}$$

Then, the zero solution $x(t) \equiv 0$ to (37), (38) is globally asymptotically stable with the hybrid feedback controller

$$u_{\rm c} = \phi_{\rm c}(x) = -R_{2\rm c}^{-1}B_{\rm c}^{\rm T}Px, \qquad x \notin \mathcal{Z}, \quad (42)$$

$$u_{\rm d} = \phi_{\rm d}(x) = -(R_{2\rm d} + B_{\rm d}^{\rm T}PB_{\rm d})^{-1}B_{\rm d}^{\rm T}PA_{\rm c}x,\ x \in \mathcal{Z}, \quad (43)$$

and

$$J(x_0, \phi_{\rm c}(\cdot), \phi_{\rm d}(\cdot)) = x_0^{\rm T}Px_0, \qquad x_0 \in \mathbb{R}^n. \quad (44)$$

Furthermore,

$$J(x_0, \phi_{\rm c}(\cdot), \phi_{\rm d}(\cdot)) = \min_{(u_{\rm c}(\cdot),u_{\rm d}(\cdot))\in\mathcal{C}(x_0)} J(x_0, u_{\rm c}(\cdot), u_{\rm d}(\cdot)), \quad (45)$$

where $\mathcal{C}(x_0)$ is the set of asymptotically stabilizing controllers for (37), (38) and $x_0 \in \mathbb{R}^n$.

Next, we present our main result of this section which provides hybrid disk margins for the optimal hybrid regulator given by Corollary 4.1. For the following result define

$$\overline{\gamma}_{\rm c} \triangleq \sigma_{\max}(R_{2\rm c}),\ \underline{\gamma}_{\rm c} \triangleq \sigma_{\min}(R_{2\rm c}), \quad (46)$$

$$\overline{\gamma}_{\rm d} \triangleq \sigma_{\max}(R_{2\rm d} + B_{\rm d}^{\rm T}PB_{\rm d}),\ \underline{\gamma}_{\rm d} \triangleq \sigma_{\min}(R_{2\rm d}). \quad (47)$$

Furthermore, assume that $\overline{\gamma}_{\rm c} = \overline{\gamma}_{\rm d}$.

Corollary 4.2. Consider the linear impulsive system given by (37), (38), (18), and (19) where ($\phi_{\rm c}(x)$, $\phi_{\rm d}(x)$) is an optimal stabilizing control law given by (42), (43). Then the linear impulsive system given by (37), (38), (18), and (19) is dissipative with respect to the hybrid supply rate $(r_{\rm c}(u_{\rm c}, y_{\rm c}), r_{\rm d}(u_{\rm d}, y_{\rm d})) = (u_{\rm c}^{\rm T}y_{\rm c} + \frac{(1-\theta_{\rm c}^2)}{2}u_{\rm c}^{\rm T}u_{\rm c} + \frac{1}{2}y_{\rm c}^{\rm T}y_{\rm c}, u_{\rm d}^{\rm T}y_{\rm d} + \frac{(1-\theta_{\rm d}^2)}{2}u_{\rm d}^{\rm T}u_{\rm d} + \frac{1}{2}y_{\rm d}^{\rm T}y_{\rm d})$ and has a hybrid disk margin $(\frac{1}{1+\theta_{\rm c}}, \frac{1}{1-\theta_{\rm c}}), (\frac{1}{1+\theta_{\rm d}}, (\frac{1}{1-\theta_{\rm d}})$, where $\theta_{\rm c} \triangleq \sqrt{\underline{\gamma}_{\rm c}/\overline{\gamma}_{\rm c}}$ and $\theta_{\rm d} \triangleq \sqrt{\underline{\gamma}_{\rm d}/\overline{\gamma}_{\rm d}}$.

Proof. The result is a direct consequence of Theorem 3.2 with $f_{\rm c}(x) = A_{\rm c}x$, $f_{\rm d}(x) = A_{\rm c}x$, $G_{\rm c}(x) = B_{\rm c}$, $G_{\rm d}(x) = B_{\rm d}$, $V(x) = x^{\rm T}Px$, $L_{1\rm c} = x^{\rm T}R_{1\rm c}x$, $R_{2\rm c}(x) \equiv R_{2\rm c}$, $L_{1\rm d} = x^{\rm T}R_{1\rm d}x$, $R_{2\rm d}(x) \equiv R_{2\rm d}$, $P_1(x) = x^{\rm T}A_{\rm c}^{\rm T}PB_{\rm d}$, and $P_2(x) = B_{\rm d}^{\rm T}PB_{\rm d}$. □

5. Conclusion

Sufficient conditions for gain, sector, and disk margin guarantees for nonlinear hybrid systems controlled by nonlinear optimal and inverse optimal hybrid regulators that minimize a nonlinear-nonquadratic hybrid performance criterion were derived. Using these results, connections between dissipativity and optimality of nonlinear hybrid systems were established.

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