Stability and Stabilization of Implicit Systems

Izumi Masubuchi
Department of Computer and Systems, Kobe University
1-1 Rokkodai, Nada, Kobe 657-8501, Japan
masubuti@cs.kobe-u.ac.jp

Abstract

This paper is concerned with stability and stabilization of implicit systems. Stability criteria are provided in terms of the Kronecker form, Lyapunov equation and inequality and conditions on extended rank and invertibility of the system pencil. Then stabilization of implicit systems via interconnection is considered based on the notion of initial-freedom preservation as a formulation of regularity of interconnection. To analyze stabilization with initial-freedom preservation, we define complete-stabilizability and zero-detectability. It is shown that stabilizability via interconnection with initial-freedom preservation is equivalent to complete-stabilizability and zero-detectability. Thus standard results on output feedback stabilization of state-space and descriptor systems are generalized to stabilization of implicit systems via interconnection.

1 Introduction

Implicit systems are mathematical models defined with variables of interest and dynamic and static relations among them. While analysis of dynamical systems based on implicit system models has been mainly based on descriptor systems with the regular system pencil [1,6,7], in the last decade lots of new insight and refinement of theory have been provided for differential systems as an application of the Behavioral approach [4,9]. In the descriptor systems context, differential-algebraic equations have been considered mainly in the distribution sense with *initial condition* described with Dirac's delta, such as

$$E\dot{x} = Ax + E\delta x_0, \tag{1}$$

where x_0 is intended to represent the initial condition at $t = 0$. On the other hand, The differential systems of the Behavioral approach considers e.g. $E\dot{x} = Ax$ as first-order representations. These are distinct systems providing different mathematical models. Also in Behavioral approaches, Geerts and Schumacher [10,11] proposed impulsive-smooth Behavior defined by the first-order differential-algebraic equations of (1).

There have been several results on analysis of the impulsive-smooth Behavior or implicit systems proposed based on the equation (1). Geerts [10,11], Geerts et al. [10,11]. have shown structure analysis of the impulsive-smooth Behavior. Masubuchi [15], Masubuchi and Shimemura [16] have proposed Lyapunov inequality and equation for implicit systems (1) with no assumptions on the regularity of the system pencil. Recently Takaba [13] has shown a Lyapunov equation for implicit systems with a detectability condition. On the other hand, many of synthesis problems for implicit systems, which have potential applications for such as combined design of control and structure in mechanical systems, are still open. One of the most important synthesis problems is stabilization via interconnection [3,26]. Since interconnection has been proposed in the Behavioral context as a basic notion including input-output or feedback connections of systems, several approaches have been shown for stabilization [3], pole assignment [26], LQ-optimal control [24,25] and H_∞-optimal control [20,28,29] via interconnection. There can be different types of interconnections according to restriction on dynamics of interconnected systems, such pointed out in [26] as singular/regular interconnections and singular/regular feedback interconnections. These are defined in terms of kernel representations of the Behavior of differential systems [26]. One important issue in stabilization via interconnection is to characterize each of them in terms of data of the system or Behavior to be stabilized. Problems of interconnection with certain specified regularity have never been considered before particularly for implicit systems or impulsive-smooth Behavior.

This paper is concerned with stability and stabilizability of implicit systems. In particular, we consider 'stabilization with initial-freedom preservation,' which is one formulation of regularity of interconnections and gives an answer to a problem of interconnection with regularity. For this purpose, first we show an explicit representation of the all solutions of implicit systems via the Kronecker form, following which we summarize several stability criteria. Next, we define stabilizability of implicit systems via interconnection with initial freedom preservation, where 'initial-freedom' means the freedom of x_0 in (1) that the implicit system possesses when it is not interconnected to other systems. The initial-freedom corresponds to the freedom of dynamics of implicit systems and preservation of initial-freedom avoids stabilization achieved by limiting the inherent freedom of dynamics of the implicit system through interconnection.

To analyze the stabilizability with initial-freedom preservation, we define complete-stabilizability and zero-detectability, which correspond to stabilizability plus impulse-controllability and detectability plus impulse-observability of descriptor systems, respectively. We give a necessary and sufficient condition for each to hold in terms of the Kronecker form of the system pencil. We prove that a implicit system to be stabilizable via interconnection with initial-freedom preservation if and only if it is completely-stabilizable and zero-detectable. Thus the equivalence of stabilizability plus detectability and (well-posed) output-feedback in state space systems is generalized to implicit systems due to the notion of initial-freedom preservation.

The organization of the paper is as follows: In Section 2, we define the implicit system and show an explicit representation of the solutions. Section 3 is devoted to stability analysis, where previous results for stability are summarized as well as new stability criteria are shown. Stability criteria under a certain minimality condition are provided. In Section 4, we define stabilizability with initial-freedom preservation as well as complete-stabilizability and zero-detectability. Then we show the main results on stabilizability with initial-freedom preservation.

Notation

$\mathcal{D}^n$: The set of n-component vectors of distributions.
$\mathcal{C}^n_\infty$: The set of n-component vectors of infinitely differentiable functions defined on $(-\infty, +\infty)$.
$\overline{\mathbf{C}}_+$: The closed right half complex plane.

2 Preliminaries

2.1 Description of Implicit Systems

Throughout this paper we consider implicit systems represented as follows:

$$E\dot{x} = Ax + Ex_0\delta, \quad x \in \mathcal{D}^n_+, \ \ x_0 \in \mathbf{R}^n, \tag{2}$$

where $E, A \in \mathbf{R}^{m\times n}$, and δ is the Dirac's delta distribution. We denote by $\mathcal{D}^n_+$ the following set of n-component vectors of distributions with support $t \geq 0$:

$$\mathcal{D}^n_+ = \{x = x_c\mathbf{1} + \sum_{i=0}^{N_x} \xi_i\delta^{(i)} : x_c \in \mathcal{C}^n_\infty, \xi_i \in \mathbf{R}^n\},$$

where $\mathbf{1}$ is the unit step function; $\mathbf{1}(t) = 1, t \geq 0$ and $\mathbf{1}(t) = 0, t < 0$ and $\delta^{(i)}$ represents the i-th derivative of δ. The number of the terms in the summation N_x is finite and depends on x. This is a class of distributions adopted in [10,11,22,23], particularly in [10,11] it is used to define the impulsive-smooth Behavior. The system (2) is a generalization of descriptor systems including impulsive phenomena [7] or switching [10] and intended to describe trajectories on $t \geq 0$ according to 'initial condition' x_0 at $t = 0$.

In the following we call the equation (2) *system* $\Sigma(E, A)$. The set of $x \in \mathcal{D}^n_+$ satisfying (2) is defined as:

$$\mathcal{X}(E, A) = \{x \in \mathcal{D}^n_+ : \exists x_0 \in \mathbf{R}^n, E\dot{x} = Ax + Ex_0\delta\},$$

which exactly coincides with the impulsive-smooth Behavior [10,11]. It has been pointed out in [15] that the 'initial value' x_0 can not always take arbitrary value of $\mathbf{R}^n$, which will be reviewed below. Accordingly we define the following set:

$$\mathbf{X}_0(E, A) = \{x_0 \in \mathbf{R}^n : \exists x \in \mathcal{D}^n_+, E\dot{x} = Ax + Ex_0\delta\}.$$

We call $\mathbf{X}_0$ the *initial-freedom* of system $\Sigma(E, A)$. The sets $\mathcal{X}(E, A)$ and $\mathbf{X}_0(E, A)$ are linear subspaces of $\mathcal{D}^n_+$ and $\mathbf{R}^n$, respectively.

2.2 Explicit representation of the solutions of (2) via the Kronecker form

In this subsection we show an explicit representation of the solutions $x \in \mathcal{X}(E, A)$. The Kronecker form of the pencil $sE - A$ gives a blockwise representation of the solutions and later plays more important role to analyze stability and stabilization of implicit systems.

It is widely known that for a pencil $sE - A$ there exist nonsingular matrices L, R that lead $sE - A$ to the Kronecker form [5]:

$$L^{\mathrm{T}}(sE - A)R = \mathrm{diag}\{L(s), M(s), N(s), E(s), 0\} \tag{3}$$

with the following structure of each block:

$$\begin{aligned} L(s) &= \begin{bmatrix} sI - A_L & B_L \end{bmatrix}, & M(s) &= \begin{bmatrix} sI - A_M \\ C_M \end{bmatrix}, \\ N(s) &= I - s\Lambda, & E(s) &= sI - A_E, \end{aligned}$$

where (A_L, B_L) and (C_M, A_M) are controllable and observable pairs in the state space sense, respectively, and Λ is a nilpotent matrix. This block structure is unique up to $sE - A$. Though the Kronecker form has more specific structure in the blocks $L(s), M(s)$, etc., we only indicate their structure required in the paper.

Now we can show an explicit representation of the solutions to (2) in the Kronecker form. According to the above block structure, we represent

$$\begin{aligned} R^{-1}x &= \begin{bmatrix} x_L^{\mathrm{T}} & x_M^{\mathrm{T}} & x_N^{\mathrm{T}} & x_E^{\mathrm{T}} & x_Z^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\ R^{-1}x_0 &= \begin{bmatrix} x_{0L}^{\mathrm{T}} & x_{0M}^{\mathrm{T}} & x_{0N}^{\mathrm{T}} & x_{0E}^{\mathrm{T}} & x_{0Z}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}. \end{aligned}$$

and $x_L = \begin{bmatrix} x_{L1}^{\mathrm{T}} & x_{L2}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, $x_{0L} = \begin{bmatrix} x_{0L1}^{\mathrm{T}} & x_{0L2}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, the partitions of which agree with that of $L(s)$. Then the possible values of x_0 in (2) are such that x_{0L}, x_{0N}, x_{0E} and x_{0Z} are arbitrary, while $x_{0M} = 0$. Every solution x to the equation (2) is represented componentwise as follows:

$$\begin{aligned}
&x_{L1}(s) = (sI - A_L)^{-1}(x_{0L1} + B_L x_{L2}(s)),\\
&x_{L2} \in \mathcal{D}_+^\bullet : \text{ arbitrary},\\
&x_M = 0,\\
&x_N = (I - s\Lambda)^{-1}\Lambda x_{0N},\\
&x_E(s) = (sI - A_E)^{-1}x_{0E},\\
&x_Z : \text{ arbitrary}.
\end{aligned}$$

In the following sections we will frequently exploit the Kronecker form and the equivalence of pencils by pre- and post-multiplying of regular matrices, which we call coordinate transformation of implicit systems.

Definition 1 *Two polynomial matrices $P(s), \hat{P}(s)$ are said to be equivalent if there exist regular matrices L, R such that $L^{\mathrm{T}}P(s)R = \hat{P}(s)$. This is an equivalence relation and denoted as $P(s) \simeq \hat{P}(s)$.* ■

If $sE - A \simeq s\hat{E} - \hat{A}$, it holds that $\mathcal{X}(\hat{E}, \hat{A}) = R\mathcal{X}(E, A)$ and $\mathbf{X}_0(\hat{E}, \hat{A}) = R\mathbf{X}_0(E, A)$ for some regular matrix R.

3 Analysis of the internal stability

We consider *internal stability* of the linear system $\Sigma(E, A)$ defined as follows.

Definition 2 *System $\Sigma(E, A)$ is said to be internally stable (or admissible) if $\mathcal{X}(E, A) \subset \mathcal{A}^n$, where*

$$\mathcal{A}^n := \{x = x_c\mathbf{1} : x_c \in \mathcal{C}_\infty^n, \lim_{t\to\infty} x_c(t) = 0\}.$$

This definition of the internal stability of $\Sigma(E, A)$ corresponds to that of descriptor systems as being impulse free and having no unstable modes (e.g., [1]). Note that the internal stability is invariant under coordinate transformations.

If $\Sigma(E, A)$ is internally stable, it can not have columns of zeros and block $L(s)$, since they bring arbitrary components in x. Thus if $\Sigma(E, A)$ is internally stable, only one element $x \in \mathcal{X}$ corresponds to one element $x_0 \in \mathbf{X}_0$, i.e., internal stability implies autonomy [15]. Further, the following criterion is easily verified which characterizes the internal stability via the Kronecker form:

Lemma 1 *[15] System $\Sigma(E, A)$ is internally stable if and only if $sE - A$ has the following Kronecker form*

$$sE - A \simeq \begin{bmatrix} M(s) & & \\ & I & \\ & & sI - A_E \\ 0 & \cdots & 0 \end{bmatrix} \qquad (4)$$

with A_E Hurwitz.

Further, the following criteria are proved via Lemma 1 to characterize the internal stability.

Theorem 1 *The system $\Sigma(E, A)$ is internally stable if and only if any one of the following conditions holds.*

i) The following generalized Lyapunov equation

$$E^{\mathrm{T}}X = X^{\mathrm{T}}E, \quad A^{\mathrm{T}}X + X^{\mathrm{T}}A + Q = 0 \qquad (5)$$

has a solution $X \in \mathbf{R}^{m\times n}$ satisfying $X^{\mathrm{T}}E \geq 0$.

ii) The following generalized Lyapunov equation

$$E^{\mathrm{T}}X = X^{\mathrm{T}}E, \quad A^{\mathrm{T}}X + X^{\mathrm{T}}A < 0 \qquad (6)$$

has a solution $X \in \mathbf{R}^{m\times n}$ satisfying $X^{\mathrm{T}}E \geq 0$.

iii) The pencil $sE - A$ is column proper and of full column rank at each $s \in \overline{\mathbf{C}}_+$.

iv) There exists $F(s) \in RH_\infty^{n\times m}$ such that

$$F(s)(sE - A) = I.$$

Proof: Omitted (See [15–17]). ■

Next, let us consider a minimality of $\Sigma(E, A)$ in the following sense. It will be invoked in the next section.

Definition 3 *Representation (E, A), $E, A \in \mathbf{R}^{m\times n}$, is minimal if any representation $(\hat{E}, \hat{A})$ satisfying $\mathcal{X}(E, A) = \mathcal{X}(\hat{E}, \hat{A})$ has the size of $\hat{m} \times n$ with $\hat{m} \geq m$.*

The sets of initial-freedom $\mathbf{X}_0(E, A)$ and $\mathbf{X}_0(\hat{E}, \hat{A})$ can be different even though $\mathcal{X}(E, A) = \mathcal{X}(\hat{E}, \hat{A})$. Thus the equivalence '$\simeq$' in Definition 1 is stronger than the equivalence induced by $\mathcal{X}(E, A) = \mathcal{X}(\hat{E}, \hat{A})$.

In modeling of a certain system by using implicit equations, one can always use a minimal representation and it is not likely to pick a nonminimal representation. On the other hand, implicit systems such as derived through interconnection can be nonminimal.

The definition of the internal stability does not depend on the representation of the system and criteria shown above do not assume the minimality. However, in this subsection we consider minimality and show criteria about it because it is related to theory of descriptor systems and some duality-type results are shown later with the minimality.

Lemma 2 *Representation (E, A) is minimal if and only if $sE - A$ has the following Kronecker form*

$$sE - A \simeq \begin{bmatrix} L(s) & & & 0 \\ & N(s) & & \vdots \\ & & sI - A_E & 0 \end{bmatrix}. \qquad (7)$$

4 Stabilization of implicit systems with initial-freedom preservation

4.1 Interconnection and the sets of the solutions and initial-freedom

Let us consider stabilization via interconnection. For the system $\Sigma(E, A)$ of (2), let

$$w = Cx$$

be the variable of p components that is accessible for interconnection. An implicit system is denoted as $\Sigma(E, A, C)$ if interconnection through w is considered. The whole expression of $\Sigma(E, A, C)$ is

$$\begin{cases} E\dot{x} = Ax + Ex_0\delta \\ w = Cx. \end{cases} \tag{8}$$

Consider also the following compensating implicit system $\Sigma(E^c, A^c, C^c)$

$$\begin{cases} E^c\dot{x}^c = A^c x^c + E^c x_0^c \delta \\ w^c = C^c x^c \end{cases} \tag{9}$$

defined in the same way as the system (8), where $E^c, A^c \in \mathbf{R}^{m^c \times n^c}$ and $C^c \in \mathbf{R}^{p \times n^c}$.

The interconnection of the systems (8) and (9) via

$$w = w^c \tag{10}$$

is represented by

$$E^{ic}\dot{x}^{ic} = A^{ic}x^{ic} + E^{ic}x_0^{ic}\delta, \tag{11}$$

where

$$x^{ic} = \begin{bmatrix} x \\ x^c \end{bmatrix}, \quad E^{ic} = \begin{bmatrix} E & 0 \\ 0 & E^c \\ 0 & 0 \end{bmatrix}, \quad A^{ic} = \begin{bmatrix} A & 0 \\ 0 & A^c \\ C & -C^c \end{bmatrix}$$

with $E^{ic}, A^{ic} \in \mathbf{R}^{(m+m^c+p)\times(n+n^c)}$. Below we abbreviate as $\Sigma = \Sigma(E, A, C)$, $\Sigma^c = \Sigma(E^c, A^c, C^c)$ and $\Sigma^{ic} = \Sigma(E^{ic}, A^{ic})$. Similarly $\mathcal{X} = \mathcal{X}(E, A)$, $\mathcal{X}^c = \mathcal{X}(E^c, A^c)$, and $\mathcal{X}^{ic} = \mathcal{X}(E^{ic}, A^{ic})$ and so are $\mathbf{X}_0$, $\mathbf{X}_0^c$ and $\mathbf{X}_0^{ic}$. Since

$$\mathcal{X}^{ic} = \left\{ \begin{bmatrix} x \\ x^c \end{bmatrix} : x \in \mathcal{X}, x^c \in \mathcal{X}^c, Cx = C^c x^c \right\}, \tag{12}$$

obviously we have

$$\mathcal{X}^{ic} \subset \mathcal{X} \times \mathcal{X}^c, \quad \mathbf{X}_0^{ic} \subset \mathbf{X}_0 \times \mathbf{X}_0^c \tag{13}$$

and setting $\Pi = \begin{bmatrix} I_n & 0 \end{bmatrix}$, $\Pi^c = \begin{bmatrix} 0 & I_{n^c} \end{bmatrix}$ gives the following relations of inclusion:

$$\Pi\mathcal{X}^{ic} \subset \mathcal{X}, \qquad \Pi\mathbf{X}_0^{ic} \subset \mathbf{X}_0, \tag{14}$$

$$\Pi^c\mathcal{X}^{ic} \subset \mathcal{X}^c, \quad \Pi^c\mathbf{X}_0^{ic} \subset \mathbf{X}_0^c. \tag{15}$$

4.2 Formulation of stabilization problem with initial-freedom preservation

In the rest of this section we consider the following conditions for the interconnected system Σ^{ic}.

1. The interconnected system Σ^{ic} is internally stable;

$$\mathcal{X}^{ic} \subset \mathcal{A}^{n+n_c}. \tag{16}$$

2. The 'plant-subspace' of the initial-freedom of the interconnected system Σ^{ic} includes the initial-freedom of the plant alone;

$$\mathbf{X}_0 \subset \Pi\mathbf{X}_0^{ic} \tag{17}$$

From the relation (14) that always holds, the condition (17) is equivalent to $\mathbf{X}_0 = \Pi\mathbf{X}_0^{ic}$.

We say that a system Σ *is stabilizable with initial-freedom preservation* if there exists a compensating system Σ^c such that the interconnected system Σ^{ic} satisfies the above two conditions. In particular, the second condition is intended to avoid that stabilization is achieved by limiting the inherent freedom of dynamics of the plant. Thus it is one formulation of regularity of interconnection.

The purpose of this section is to derive a necessary and sufficient condition to the stabilization with initial-freedom preservation. In the following two subsections we define 'complete-stabilizability' and 'zero-detectability' conditions and prove each of them is necessary to the stabilization with initial-freedom preservation, as well as we show Kronecker-form characterizations of the complete-stabilizability and the zero-detectability. Then, by proving the sufficiency of them, we show that the complete-stabilizability and zero-detectability are a necessary and sufficient condition for system Σ to be stabilizable with initial-freedom preservation.

4.3 Complete-stabilizability

Definition 4 *System Σ is said to be completely-stabilizable if for any $x_0 \in \mathbf{X}_0$ there exists $x \in \mathcal{X} \cap \mathcal{A}^n$ such that*

$$Ex(0) = Ex_0. \tag{18}$$

This may be an 'impulse-smooth behavior' version of stabilizability defined in the behavioral context in e.g. [4]. Complete-stabilizability is invariant under coordinate transformations.

Lemma 3 *System Σ is stabilizable with initial-freedom preservation only if Σ is completely-stabilizable.*

Proof: Suppose that the interconnected system Σ^{ic} is internally stable, i.e., $\mathcal{X}^{ic} \subset \mathcal{A}^{n+n_c}$. Then $\Pi\mathcal{X}^{ic} \subset \mathcal{A}^n$, which implies with the relation in (14) that

$$\Pi\mathcal{X}^{ic} = \Pi\mathcal{X}^{ic} \cap \mathcal{A}^n \subset \mathcal{X} \cap \mathcal{A}^n \tag{19}$$

holds. On the other hand, since Σ^{ic} preserves the initial-freedom, it holds that

$$\forall x_0 \in \mathbf{X}_0, \quad \exists x_0^c \in \mathbf{X}_0^c, \quad x_0^{ic} := \begin{bmatrix} x_0 \\ x_0^c \end{bmatrix} \in \mathbf{X}_0^{ic}.$$

From the definition of $\mathbf{X}_0^{ic}$, there exists x^{ic} which satisfies (11) for $x_0^{ic} \in \mathbf{X}_0^{ic}$ above. Denoting it by $x^{ic} = \begin{bmatrix} x \\ x^c \end{bmatrix}$, we see $x \in \Pi\mathcal{X}^{ic}$, and from (19) we have $x \in \mathcal{X} \cap \mathcal{A}^n$. Further, $Ex(0) = Ex_0$ holds since $E^{ic}x^{ic}(0) = E^{ic}x_0^{ic}$ is always true if $\Sigma(E^{ic}, A^{ic})$ is internally stable. ∎

Lemma 4 *System $\Sigma = \Sigma(E, A)$ is completely-stabilizable if and only if in the Kronecker form (3) of the pencil $sE - A$ it holds that $N(s) = I$ and A_E is Hurwitz.*

Proof: *(Sufficiency)* To the component of $x_{0M}(= 0)$ and x_{0N} with $N(s) = I$, corresponding component in the solution x is $0 \in \mathcal{A}^\bullet$, where $\mathcal{A}^\bullet$ means $\mathcal{A}^k$ for some k. Next, since A_E is Hurwitz, $x_e = \mathrm{e}^{A_E t} x_{0E} \mathbf{1} \in \mathcal{A}^\bullet$. Lastly, consider the block $L(s)$. Recall that $L_i(s)$ is represented as $L_i(s) = \begin{bmatrix} sI - A_i & -b_i \end{bmatrix}$, where

$$A_i = \begin{bmatrix} 0 & 1 & & 0 \\ 0 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \quad b_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

with a controllable pair (A_i, b_i). Taking an arbitrary stabilizing feedback gain f_i for (A_i, b_i), we obtain a stable element

$$x_{Li} = \begin{bmatrix} I \\ f_i \end{bmatrix} (sI - A_i - b_i f_i)^{-1} \begin{bmatrix} I & 0 \end{bmatrix} x_{0Li} \in \mathcal{A}^\bullet$$

satisfying $L_i(s) x_i = \begin{bmatrix} I & 0 \end{bmatrix} x_{0Li}$ and $\begin{bmatrix} I & 0 \end{bmatrix} x_{Li}(0) = \begin{bmatrix} I & 0 \end{bmatrix} x_{0Li}$.

(Necessity) If $N(s) \neq I$, i.e., if there exist some $N_i(s)$'s of the size larger than 1, the solution to (2) contains x_{Ni} with impulses $\delta^{(j)}$ for some nonzero x_{0Ni}. Similarly if A_E is not Hurwitz x_E does not converge to zero for some nonzero x_{0E}. ∎

Under the minimality of (E, A), we have duality-type results as follows.

Lemma 5 1. *Representation (E, A) is minimal and system $\Sigma = \Sigma(E, A)$ is completely-stabilizable if and only if $sE - A$ has the following Kronecker form*

$$sE - A \simeq \begin{bmatrix} L(s) & & & 0 \\ & I & & \vdots \\ & & sI - A_E & 0 \end{bmatrix} \qquad (20)$$

with A_E Hurwitz.

2. *Representation (E, A) is minimal and system $\Sigma = \Sigma(E, A)$ is completely-stabilizable if and only if $\Sigma(E^{\mathrm{T}}, A^{\mathrm{T}})$ is internally stable.*

Proof: The statement 1. is obvious from Lemmas 2 and 4. The statement 2. immediately follows from the statement 1. and Lemma 1. ∎

From Lemma 5, criteria for complete-stabilizability of system $\Sigma(E, A)$ with the minimal representation are automatically derived from those for the internal stability of $\Sigma(E^{\mathrm{T}}, A^{\mathrm{T}})$ shown in the previous section.

4.4 Zero-detectability

Definition 5 *System $\Sigma = \Sigma(E, A, C)$ is zero-detectable if $x \in \mathcal{A}^n$ holds for all $x \in \mathcal{X}$ if $w (= Cx) = 0$.*

This is a necessary and sufficient condition for Σ to be stabilized without caring its initial-freedom:

Theorem 2 *System Σ is stabilized via interconnection if and only if Σ is zero-detectable.*

Proof: *(Necessity)* Suppose that $\Sigma(E^{ic}, A^{ic})$ is internally stable. Then from the statement iii) of Theorem 1 the pencil $sE^{ic} - A^{ic}$ has full column rank at $s \in \overline{\mathbf{C}}_+ \cup \{\infty\}$. This implies that also $\begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} A \\ 0 \\ C \end{bmatrix}$ has full column rank at $s \in \overline{\mathbf{C}}_+ \cup \{\infty\}$ and we immediately see so is $s \begin{bmatrix} E \\ 0 \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix}$. Thus from Theorem 1 the system $\Sigma\left(\begin{bmatrix} E \\ 0 \end{bmatrix}, \begin{bmatrix} A \\ C \end{bmatrix}\right)$ is internally stable, which proves that $\Sigma(E, A, C)$ is zero-detectable.

(Sufficiency) If $\Sigma(E, A, C)$ is zero-detectable, the system $\Sigma\left(\begin{bmatrix} E \\ 0 \end{bmatrix}, \begin{bmatrix} A \\ C \end{bmatrix}\right)$ is internally stable. This obviously shows that Σ is stabilized via interconnection with a trivial system $w = 0$. ∎

Corollary 1 *System Σ is stabilizable with initial-freedom preservation only if Σ is zero-detectable.*

As seen in the proof above, the zero-detectability of $\Sigma(E, A, C)$ is characterized through the results for internal stability of system $\Sigma\left(\begin{bmatrix} E \\ 0 \end{bmatrix}, \begin{bmatrix} A \\ C \end{bmatrix}\right)$ obtained in the previous section.

4.5 Main result for stabilization with initial-freedom preservation

In Subsections 4.3 and 4.4, we proved that the stabilizability and zero-detectability are necessary conditions for stabilization with initial-freedom preservation. In this section, we prove that they are also sufficient, by utilizing the Kronecker-form results in previous subsections. Further, in the same time we obtain one of the interconnections that achieves the stabilization with initial-freedom preservation via well-developed state-space methods from data of $sE - A$ in the Kronecker form.

Theorem 3 *System Σ is stabilizable with initial-freedom preservation if and only if Σ is completely-stabilizable and zero-detectable.*

Proof: The necessity is shown in Lemma 3 and Corollary 1. The sufficiency is proved through the Kronecker form of a completely-stabilizable pencil with applying the rank condition that holds for zero-detectable system. The detail of the proof of the sufficiency is omitted. ∎

5 Conclusions

In this paper, we presented several stability criteria of implicit systems and considered stabilization of implicit systems via interconnection. Employing notion of stabilization with initial-freedom preservation, we formulated stabilization without prohibiting the inherent freedom of dynamics of the given implicit system. It was shown that complete-stabilizability and zero-detectability conditions are necessary and sufficient to the stabilizability with initial-freedom preservation.

References

[1] F. L. Lewis: A survey of linear singular systems; *Circuits Syst. Signal Process*, Vol. 5, pp. 3-36 (1986)

[2] J. D. Aplevich: *Implicit Linear Systems*, *Lecture Notes in Control and Information Science 152*, Springer-Verlag (1991)

[3] M. Kuijper: Why do stabilizing controllers stabilize?; *Automatica*, Vol.31, pp. 621–625 (1995)

[4] J. W. Polderman and J.C. Willems: *Introduction to Mathematical Systems Theory*, Springer-Verlag (1991)

[5] F. R. Gantmacher (Translation: K.A.Hirsch): *The Theory of Matrices*, Chelsea Publishing (1959)

[6] D. G. Luenberger: Dynamic Equations in Descriptor Form; *IEEE Trans. Automat. Contr.*, Vol. 22, No. 3, pp. 312–321 (1977)

[7] D. Cobb: On the Solutions of Linear Differential Equations with Singular Coefficients, *J. Differential Equations*, Vol. 46, pp. 310–323 (1982)

[8] J. C. Willems: Least Squares Stationary Optimal Control and the Algebraic Riccati Equation, *IEEE Trans. Automat. Contr.*, Vol. 16, No. 6, pp. 621–634 (1971)

[9] J. C. Willems: Paradigms and Puzzles in the Theory of Dynamical Systems; *IEEE Trans. Automat. Contr.*, Vol. 39, No. 12, pp. 259–294 (1991)

[10] A. H. W. (T.) Geerts and J. M. Schumacher: Impulsive-Smooth Behavior in Multimode Systems Part I: State-space and Polynomial Representations, *Automatica*, Vol. 32, No. 5, pp. 747–758 (1996)

[11] A. H. W. (T.) Geerts and J. M. Schumacher: Impulsive-Smooth Behavior in Multimode Systems Part II: Minimality and Equivalence, *Automatica*, Vol. 32, No. 6, pp. 810–832 (1996)

[12] K. Takaba, N. Morihira and T. Katayama: A Generalized Lyapunov Theorem for Descriptor System, *Systems Control Lett.*, Vol. 24, pp. 49–51 (1995)

[13] K. Takaba: Linear Quadratic Optimal Control for Linear Implicit System, *Proc. 38th CDC*, pp. 4074–4079 (1999)

[14] I. Masubuchi, Y. Kamitane, A. Ohara and N. Suda: H_∞ control for descriptor systems: A matrix inequalities approach, *Automatica*, Vol. 33, No. 4, pp. 669–673 (1997)

[15] I. Masubuchi and E. Shimemura: An LMI condition for stability of Implicit Systems, 36th IEEE Conf. on Decision and Control (1997)

[16] I. Masubuchi and E. Shimemura: Stability analysis of implicit systems with possible impulses, Trans. SICE, Vol. 35, No. 1, pp. 66–70 (1999) in Japanese.

[17] I. Masubuchi: Stability analysis of systems based on implicit representations, Systems, Control and Information, Vol. 44, No. 4, pp. 177–183 (2000) in Japanese.

[18] F. Paganini and J. Doyle: Analysis of Implicitly Defined Systems Proc. 33rd IEEE Conf. on Decision and Control, pp. 3673–3678 (1994)

[19] R. D'Andrea and F. Paganini: Controller synthesis for implicitly defined uncertain systems, Proc. 33rd IEEE Conf. on Decision and Control, pp. 3679–3684 (1994)

[20] R. D'Andrea: $\mathcal{H}_\infty$ optimal interconnections; *Systems & Control Lett.*, Vol.32 pp. 313–322 (1997)

[21] R. van der Geest: Quadratic performance of generalized first-order systems, Proc. 35th IEEE Conf. on Decision and Control, pp. 4553–4554 (1996)

[22] T. Geerts: Solvability conditions, consistency and weak consistency for linear differential-algebraic equations and time-invariant singular systems: the general case, *Linear Algebra and Its Applications* Vol. 181, pp. 111–130 (1993)

[23] T. Geerts: Invariant Subspaces and Invertibility properties for singular systems: the general case, *Linear Algebra and Its Applications* Vol. 183, pp. 61–83 (1993)

[24] J.C. Willems: LQ-control: a behavioral approach; *Proc. 32nd CDC*, pp. 3664–3668 (1993)

[25] J.C. Willems and H.L. Trentleman: On Quadratic Differential Forms; *Proc. 33rd CDC*, pp. 3690–3694 (1994)

[26] J. C. Willems and H. L. Trentleman: On Interconnections, control and feedback, *IEEE Trans. Automat. Contr.*, Vol. 42, No. 3, pp. 326–339 (1997)

[27] J. C. Willems and H. L. Trentleman: On Quadratic Differential Forms, *SIAM J. Control Optim.*, Vol. 36, No. 5, pp. 1703–1749 (1998)

[28] H.L. Trentleman and J.C. Willems: H_∞ Control in a Behavioral Context: The Full Information Case; *IEEE Trans. Automat. Contr.*, Vol.44, No.3 pp. 521–536 (1999)

[29] S. Weiland, A. Stoorvogel and B. de Jager: A behavioral approach to the $\mathcal{H}_\infty$ optimal control problem; *Systems & Control Lett.*, Vol.32 pp. 323–324 (1997)