Subspace angles between linear stochastic models

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Abstract

In this paper we define a notion of principal angles between two linear autoregressive (AR) models by considering the principal angles between the ranges of their infinite observability matrices. We show how a recently defined metric for these models, which is based on their cepstra, is related to the subspace angles between them. The definition of subspace angles is also extended to the linear autoregressive-moving-average (ARMA) model class.

Keywords: principal angles, ARMA models, linear systems, stochastic realization, distance measure, time series

1 Introduction

The concept of principal angles between subspaces of linear vector spaces is due to Jordan [9] in the previous century. This notion was translated into the statistical notion of canonical correlations by Hotelling [8]. Applications include data analysis [5], random processes [4] [10] and stochastic realization [1] [3] [13] (and references herein). Numerically stable methods to compute the canonical structure via a singular value decomposition have been proposed by Björk and Golub [2], Golub and Zha [7] and can also be found in [6].

In this paper we define a notion of principal angles and their corresponding principal directions between two linear autoregressive-moving-average (ARMA) models and show their relation to the metric for ARMA models defined by Martin [12]. The models we consider are SISO–LTI (single input single output linear time invariant).

The paper is organized as follows. In Section 2, we briefly recall the definition of principal angles between and corresponding principal directions in two subspaces. In Section 3, we discuss a distance measure for AR models, which has recently been defined by Martin [12]. Our definition of the subspace angles between AR models and their relation to the distance measure of Martin is given in Section 4. In Section 5, the definition of distance and subspace angles is extended to the ARMA model class. Finally in Section 6 we give the conclusions and point out possible further developments of these notions.

2 Principal angles between subspaces

In this section we discuss the notion of principal angles between and principal directions in two subspaces. We start with the definition in Section 2.1, and in Section 2.2 we show how the angles and directions can be characterized from a generalized eigenvalue problem.

2.1 Definition

Let $A \in \mathbb{R}^{m\times p}$ and $B \in \mathbb{R}^{m\times q}$ be given real matrices with the same number of rows and assume for convenience that A and B have full column rank and that $p \geq q$. We denote the range (column space) of a matrix A by range(A).

Definition 2.1. The q principal angles $\theta_k \in \left[0, \frac{\pi}{2}\right]$, between range($A$) and range($B$) are recursively defined for $k = 1, 2, \ldots, q$ as

$$\begin{aligned}
\cos\theta_1 &= \max_{\substack{x\in\mathbb{R}^p\\ y\in\mathbb{R}^q}} \frac{|x^T A^T B y|}{\|Ax\|_2\|By\|_2} \\
&= \frac{|x_1^T A^T B y_1|}{\|Ax_1\|_2\|By_1\|_2} \ , \\
\cos\theta_k &= \max_{\substack{x\in\mathbb{R}^p\\ y\in\mathbb{R}^q}} \frac{|x^T A^T B y|}{\|Ax\|_2\|By\|_2} \\
&= \frac{|x_k^T A^T B y_k|}{\|Ax_k\|_2\|By_k\|_2} \ , \text{ for } k = 2, \ldots, q \\
&\text{subject to } x_i^T A^T A x = 0 \text{ and } y_i^T B^T B y = 0, \\
&\text{for } i = 1, 2, \ldots, k-1 \ .
\end{aligned} \tag{1}$$

Note that the principal angles satisfy $0 \le \theta_1 \le \ldots \le \theta_q \le \frac{\pi}{2}$. The vectors $Ax_1, \ldots, Ax_q \in \mathbb{R}^m$ and $By_1, \ldots, By_q \in \mathbb{R}^m$ are called the principal directions of the pair of spaces. Following the notation in [13], the set of q principal angles between the ranges of the matrices A and B is denoted as $[A \lessdot B]$.

2.2 The principal angles and directions as the solution of a generalized eigenvalue problem

It can be shown (see e.g. [7]) that the principal angles and the principal directions between range(A) and range(B) follow from the symmetric generalized eigenvalue problem:

$$\begin{pmatrix} 0 & A^T B \\ B^T A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} A^T A & 0 \\ 0 & B^T B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \ , \quad \text{subject to } x^T A^T A x = 1 \text{ and } y^T B^T B y = 1. \tag{2}$$

The link between the formula in (2) and (1) goes via the so-called variational characterization of the eigenvalue problem.

Assume that the $p + q$ (real) eigenvalues λ_i are sorted in descending order as

$$\lambda_1 \ge \ldots \ge \lambda_{p+q} \ ,$$

then one can show that

$$\lambda_1 = \cos\theta_1, \ldots, \lambda_q = \cos\theta_q \ge 0 \ , \tag{3a}$$

$$\lambda_{p+q} = -\cos\theta_1, \ldots, \lambda_{p+1} = -\cos\theta_q \ , \tag{3b}$$

$$\lambda_{q+1} = \lambda_{q+2} = \cdots = \lambda_p = 0 \ . \tag{3c}$$

The vectors Ax_j and By_j, for $j = 1, \ldots, q$ where x_j and y_j satisfy (2) with $\lambda = \lambda_j$, are the principal directions corresponding to the principal angle θ_j. Note that when considering the principal angles between equidimensional subspaces ($p = q$), Equation (3c) does not come into play.

3 A metric for the set of SISO–LTI AR models

In [12] Martin defines a new metric for the set of SISO–LTI ARMA models. It is based on the inner product of the cepstra of ARMA models. Further on in the paper we will show that this metric is related to the principal angles between specific subspaces derived from the ARMA models. For the sake of completeness, we repeat in this section some results which have been reported in [12]. However, in this section, we restrict our discussion to the AR model class. The extension to the ARMA models will be made in Section 5.

Let M_1 and M_2 be stable AR models with cepstrum coefficients $c_1(n)$ and $c_2(n)$, $n = 0, \pm1, \pm2, \ldots$

Definition 3.1. **[12]** The distance between M_1 and M_2 is defined as

$$d(M_1, M_2) = \sqrt{\sum_{n=0}^{\infty} n|c_1(n) - c_2(n)|^2} \ . \tag{4}$$

Martin subsequently shows that for stable AR models M_1 with order n_1 and poles α_i and M_2 with order n_2 and poles β_i the following equality holds

$$d(M_1, M_2)^2 = \ln \frac{\prod_{i}^{n_1}\prod_{j}^{n_2} |1 - \bar{\alpha}_i \beta_j|^2}{\prod_{i,j}^{n_1}(1 - \bar{\alpha}_i \alpha_j)\prod_{i,j}^{n_2}(1 - \bar{\beta}_i \beta_j)} \ . \tag{5}$$

This equality basically follows from an expression that relates the cepstrum coefficients of an AR model to its poles, which can be found in [11]. Observe that if M_1 and M_2 are two first order stable AR models, their distance equals

$$d(M_1, M_2)^2 = \ln \frac{(1 - \alpha\beta)^2}{(1 - \alpha^2)(1 - \beta^2)} = \ln \frac{1}{\cos^2\theta} \ ,$$

where θ is the angle between the vectors

$$\begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \\ \vdots \end{pmatrix} \in \mathbb{R}^\infty \text{ and } \begin{pmatrix} 1 \\ \beta \\ \beta^2 \\ \vdots \end{pmatrix} \in \mathbb{R}^\infty \ .$$

It will become apparent in Section 4.3 that for higher order models, the squared distance as defined by Martin [12] can be expressed as the logarithm of a product of $\frac{1}{\cos^2\theta_i}$ (see Theorem 1). The angles θ_i will be called the subspace angles between the AR models.

4 Subspace angles between AR models

In this section we start the discussion of our new concept of angles between models, by considering AR models. The definition of the subspace angles between two AR models is given in Section 4.1. In Section 4.2 we show how the angles can be computed from the poles of the two AR models. The relation between the subspace angles between two AR models and their distance as defined in [12] (see also Definition 3.1) is given in Section 4.3.

4.1 Definition

Let two stable and observable nth order AR models M_1 and M_2 be characterized in state space terms by their system matrix A_1 and A_2 and output matrix C_1 and C_2 respectively.
Their infinite observability matrix,

$$\begin{pmatrix} C_i \\ C_iA_i \\ C_iA_i^2 \\ \vdots \end{pmatrix} \in \mathbb{R}^{\infty\times n} \ ,$$

is denoted as $\mathcal{O}_\infty(M_i)$ for $i = 1, 2$.

Definition 4.1. We define the subspace angles between M_1 and M_2 as the principal angles between the ranges of their infinite observability matrices:

$$[M_1 \sphericalangle M_2] = [\mathcal{O}_\infty(M_1) \sphericalangle \mathcal{O}_\infty(M_2)] \ . \qquad (6)$$

If two AR models do not have the same order, and assume the difference in order is equal to Δn, then we add Δn poles in zero to the model with the smallest order.
Consequently,

1. the number of subspace angles between two AR models of order n_1 and n_2 respectively is always equal to $\max(n_1, n_2)$,
2. the eigenvalues of the generalized eigenvalue problem (2) appear in pairs of opposite sign, since the considered subspaces are both of dimension $\max(n_1, n_2)$ (see (3)).

4.2 Computation of the subspace angles between AR models

For simplicity we take $n_1 = n_2 = n$.
Let the AR model M_1 have poles $\alpha_1, \ldots, \alpha_n$ and infinite observability matrix $\mathcal{O}_\infty(M_1)$ and let the AR model M_2 have poles $\beta_1, \ldots, \beta_n$ and infinite observability matrix $\mathcal{O}_\infty(M_2)$. Assume the two models are stable and observable.

The cosines of the subspace angles between M_1 and M_2 are equal to the largest n eigenvalues of $\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \in \mathbb{C}^{2n\times 2n}$, where

$$\begin{aligned} X(i,j) &= \frac{\prod_k^n(1-\bar{\alpha}_k\alpha_i)\prod_{k\neq i}^n(\beta_j-\alpha_k)}{\prod_{k\neq i}^n(\alpha_i-\alpha_k)\prod_k^n(1-\bar{\alpha}_k\beta_j)} \ , \\ Y(i,j) &= \frac{\prod_k^n(1-\bar{\beta}_k\beta_i)\prod_{k\neq i}^n(\alpha_j-\beta_k)}{\prod_{k\neq i}^n(\beta_i-\beta_k)\prod_k^n(1-\bar{\beta}_k\alpha_j)} \ , \end{aligned} \qquad (7)$$

in which $A(i,j)$ denotes the element of the matrix A on the ith row and the jth column.

This can be seen as follows.
The subspace angles between M_1 and M_2 are equal to the principal angles between $\text{range}(\mathcal{O}_\infty(M_1))$ and $\text{range}(\mathcal{O}_\infty(M_2))$. These subspaces only depend on the poles of the models:

$$\begin{aligned} \text{range}(\mathcal{O}_\infty(M_1)) &= \text{range}(\Gamma_1) \ , \\ \text{range}(\mathcal{O}_\infty(M_2)) &= \text{range}(\Gamma_2) \ , \end{aligned} \qquad (8)$$

where[1]

$$\Gamma_1 = \begin{pmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_n \\ \alpha_1^2 & \cdots & \alpha_n^2 \\ \vdots & \cdots & \vdots \end{pmatrix} \in \mathbb{C}^{\infty\times n} \ , \qquad (9)$$

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[1] For simplicity we assume that all poles are distinct. Otherwise, the structure of Γ_1 in (9) would be slightly different. However, it still would only depend on the poles.

and analogously for Γ_2.
Define the hermitian matrix $Q \in \mathbb{C}^{2n \times 2n}$ as

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} \Gamma_1^H \\ \Gamma_2^H \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix} ,$$

where Γ_i^H is the complex conjugate transpose of the matrix Γ_i. Due to the stability of M_1 and M_2, Q is a matrix with finite elements. By applying

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ for } |x| < 1 ,$$

we obtain

$$\begin{aligned} Q_{11}(i,j) &= \frac{1}{1-\bar{\alpha}_i \alpha_j} , \\ Q_{12}(i,j) &= \frac{1}{1-\bar{\alpha}_i \beta_j} , \\ Q_{21}(i,j) &= \frac{1}{1-\bar{\beta}_i \alpha_j} , \\ Q_{22}(i,j) &= \frac{1}{1-\bar{\beta}_i \beta_j} , \end{aligned} \qquad (10)$$

where $\bar{c}$ denotes the complex conjugate of c.
The subspace angles between M_1 and M_2 can then be obtained from the generalized eigenvalues of $\begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix}$ and $\begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix}$ (see Section 2.2). Since M_1 and M_2 are observable, the matrices Q_{11} and Q_{22} are non-singular. Therefore, the generalized eigenvalues are equal to the eigenvalues of $\begin{pmatrix} 0 & Q_{11}^{-1}Q_{12} \\ Q_{22}^{-1}Q_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$, and for these submatrices one can find the closed-form formula given in (7).

4.3 Relation of Martin's metric and the subspace angles between two AR models

The subspace angles between two AR models are related to the distance between AR models as defined in [12] (see also Section 3) in the following way.

Theorem 1. *For the stable and observable AR models M_1 of order n_1 and M_2 of order n_2,*

$$d(M_1, M_2)^2 = \ln \left(\prod_i^n \frac{1}{\cos^2 \theta_i} \right) , \qquad (11)$$

where $n = \max(n_1, n_2)$ and $\theta_1, \theta_2, \ldots, \theta_n$ are the subspace angles between M_1 and M_2.

Proof: For simplicity we take $n_1 = n_2 = n$.
As shown in Section 4.2, the cosines of the subspace angles between M_1 and M_2 are equal to the largest n eigenvalues of $\begin{pmatrix} 0 & Q_{11}^{-1}Q_{12} \\ Q_{22}^{-1}Q_{21} & 0 \end{pmatrix}$.
Since the eigenvalues $\lambda_1, \ldots, \lambda_{2n}$ appear in pairs of opposite sign (see Consequence 2 in Section 4.1), we obtain

$$\begin{aligned} \prod_i^n \cos^2 \theta_i &= (-1)^n \prod_i^{2n} \lambda_i \\ &= (-1)^n \det \begin{pmatrix} 0 & Q_{11}^{-1}Q_{12} \\ Q_{22}^{-1}Q_{21} & 0 \end{pmatrix} \\ &= \frac{\det Q_{12} \det Q_{21}}{\det Q_{11} \det Q_{22}} . \end{aligned} \qquad (12)$$

The matrices Q_{ij} $(i,j = 1,2)$ have a structure reminiscent to that of a Cauchy matrix, which has the form:

$$C = \begin{pmatrix} \frac{1}{x_1 - y_1} & \cdots & \frac{1}{x_1 - y_n} \\ \vdots & \ddots & \vdots \\ \frac{1}{x_n - y_1} & \cdots & \frac{1}{x_n - y_n} \end{pmatrix} .$$

A formula for its determinant was found by Cauchy and can be proven by induction:

$$\det C = \frac{\prod_{i>j}^n (x_i - x_j) \prod_{i<j}^n (y_i - y_j)}{\prod_{i,j}^n (x_i - y_j)} . \qquad (13)$$

The matrices Q_{ij} $(i,j = 1,2)$ can be written as the product of a diagonal matrix D_i and a Cauchy matrix C_{ij}:

$$Q_{ij} = D_i C_{ij} . \qquad (14)$$

As an example we take Q_{12}:

$$\begin{aligned} Q_{12} &= \begin{pmatrix} \frac{1}{1-\bar{\alpha}_1\beta_1} & \cdots & \frac{1}{1-\bar{\alpha}_1\beta_n} \\ \vdots & \ddots & \vdots \\ \frac{1}{1-\bar{\alpha}_n\beta_1} & \cdots & \frac{1}{1-\bar{\alpha}_n\beta_n} \end{pmatrix} \\ &= \operatorname{diag}\left(\frac{1}{\bar{\alpha}_i}\right) \begin{pmatrix} \frac{1}{\frac{1}{\bar{\alpha}_1} - \beta_1} & \cdots & \frac{1}{\frac{1}{\bar{\alpha}_1} - \beta_n} \\ \vdots & \ddots & \vdots \\ \frac{1}{\frac{1}{\bar{\alpha}_n} - \beta_1} & \cdots & \frac{1}{\frac{1}{\bar{\alpha}_n} - \beta_n} \end{pmatrix} , \end{aligned}$$

where $\operatorname{diag}\left(\frac{1}{\bar{\alpha}_i}\right)$ is the diagonal matrix with elements $\frac{1}{\bar{\alpha}_1}, \ldots, \frac{1}{\bar{\alpha}_n}$. Substituting (14) in (12) gives:

$$\prod_i^n \cos^2 \theta_i = \frac{\det C_{12} \det C_{21}}{\det C_{11} \det C_{22}} ,$$

which becomes after applying Cauchy's formula (13)

$$\prod_i^n \cos^2\theta_i = \frac{\prod_{i,j}^n (1-\bar{\alpha}_i\alpha_j)(1-\bar{\beta}_i\beta_j)}{\prod_{i,j}^n |1-\bar{\alpha}_i\beta_j|^2} ,$$

so that

$$\begin{aligned} \ln\prod_i^n \frac{1}{\cos^2\theta_i} &= \ln\frac{\prod_{i,j}^n |1-\bar{\alpha}_i\beta_j|^2}{\prod_{i,j}^n (1-\bar{\alpha}_i\alpha_j)(1-\bar{\beta}_i\beta_j)} \\ &= d(M_1,M_2)^2 \ . \end{aligned}$$

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5 Distance and angles between ARMA models

As mentioned in Section 3, Martin defined a metric, not only for AR models, but more general for ARMA models [12]. On the basis of this definition, which is repeated in Section 5.1, and a property of the distance measure, we define in Section 5.2 the subspace angles between two ARMA models.

5.1 The distance between ARMA models

Definition 3.1 can readily be used for the distance between ARMA models: Let M_1 and M_2 be two stable and minimum phase (i.e. all poles and zeros lie inside the unit circle) ARMA models with cepstrum coefficients $c_1(n)$ and $c_2(n)$, $n = 0, \pm 1, \pm 2, \ldots$.

Definition 5.1. [12] The distance between M_1 and M_2 is defined as

$$d(M_1,M_2) = \sqrt{\sum_{n=0}^{\infty} n|c_1(n)-c_2(n)|^2} \ .$$

Since the cepstrum is the inverse Fourier transformation of the logarithm of the spectrum:

$$\ln P(z) = \ln H(z)\bar{H}(z^{-1}) = \sum_{n\in\mathbb{Z}} c_n z^{-n} \ ,$$

the following property holds [12]

$$d(H_1H_3, H_2H_3) = d(H_1,H_2) \ ,$$

where H_i is the transfer function of the ARMA model M_i for $i = 1,2,3$. This property implies that in order to compute the distance between ARMA models, it is sufficient to consider AR models: for $H_1(z) = \dfrac{b_1(z)}{a_1(z)}$ and $H_2(z) = \dfrac{b_2(z)}{a_2(z)}$, take $H_3(z) = \dfrac{1}{b_1(z)b_2(z)}$, so that

$$d\left(\frac{b_1(z)}{a_1(z)}, \frac{b_2(z)}{a_2(z)}\right) = d\left(\frac{1}{a_1(z)b_2(z)}, \frac{1}{a_2(z)b_1(z)}\right) . \tag{15}$$

Because M_1 and M_2 are stable and minimum phase, the two AR models are stable.

Consider now the subspace angles between the AR models with respective transfer functions $\frac{1}{a_1(z)b_2(z)}$ and $\frac{1}{a_2(z)b_1(z)}$. In accordance with Definition 4.1, they are equal to the principal angles between range$\begin{pmatrix} \mathcal{O}_\infty(M_1) & \mathcal{O}_\infty(M_2^{-1}) \end{pmatrix}$ and range$\begin{pmatrix} \mathcal{O}_\infty(M_2) & \mathcal{O}_\infty(M_1^{-1}) \end{pmatrix}$, where the transfer function of M_1 is $\frac{b_1(z)}{a_1(z)}$, the transfer function of M_2 is $\frac{b_2(z)}{a_2(z)}$, $\mathcal{O}_\infty(M_i)$ is the infinite observability matrix of the model M_i and $\mathcal{O}_\infty(M_i^{-1})$ is the infinite observability matrix of the inverse model M_i^{-1}, for $i = 1,2$.

We now propose the following definition of the subspace angles between ARMA models.

5.2 Subspace angles between ARMA models

Assume M_1 and M_2 are stable, minimum phase ARMA models.

Definition 5.2. We define the subspace angles between M_1 and M_2 as the principal angles between the ranges of $\begin{pmatrix} \mathcal{O}_\infty(M_1) & \mathcal{O}_\infty(M_2^{-1}) \end{pmatrix}$ and $\begin{pmatrix} \mathcal{O}_\infty(M_2) & \mathcal{O}_\infty(M_1^{-1}) \end{pmatrix}$:

$$\begin{aligned} &[M_1 \lessdot M_2] = \\ &\left[\left(\mathcal{O}_\infty(M_1)\middle|\mathcal{O}_\infty(M_2^{-1})\right) \lessdot \left(\mathcal{O}_\infty(M_2)\middle|\mathcal{O}_\infty(M_1^{-1})\right)\right] . \end{aligned}$$

Or,

$$\left[\frac{b_1(z)}{a_1(z)} \lessdot \frac{b_2(z)}{a_2(z)}\right] = \left[\frac{1}{a_1(z)b_2(z)} \lessdot \frac{1}{a_2(z)b_1(z)}\right] ,$$

which reflects the property for the distance between two ARMA models in Equation (15).

Analogously to (8) and (9), the range of the observability matrix of the inverse model M^{-1} is only dependent on the zeros of M.

From Definition 5.2 and Equation (15) it is clear that Theorem 1, which was given for AR models, is also valid for ARMA models.

6 Conclusions

In this paper we have proposed a definition for the subspace angles and directions between two ARMA models and we have shown a relation between these angles and a recently defined distance measure for ARMA models [12].
In the near future, these new notions of distance and angles between models will be applied to several engineering applications, such as signal classification, fault detection, the calculation of so-called stabilization diagrams in vibrational analysis etc ...
Many questions remain to be tackled. Future developments will comprise the extension to MIMO (multiple input multiple output) models and to deterministic systems. Furthermore, the apparent relation with the notion of mutual information will be explored.

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References

[1] H. Akaike. Stochastic theory of minimal realization. *IEEE Transactions on Automatic Control*, 19:667–674, 1974.

[2] Å. Björck and G. H. Golub. Numerical methods for computing angles between linear subspaces. *Mathematics of computation*, 27(123):579–594, 1973.

[3] Peter E. Caines. *Linear stochastic systems.* Wiley New York, 1988.

[4] I. M. Gelfand and A. M. Yaglom. Calculation of the amount of information about a random function contained in another such function. *American Mathematical Society Translations, series 2*, 12:199–246, 1959.

[5] R. Gittins. *Canonical Analysis; A review with applications in ecology*, volume 12 of *Biomathematics.* Springer-Verlag, Berlin, 1985.

[6] G. H. Golub and C. F. Van Loan. *Matrix computations.* The Johns Hopkins University Press, second edition, 1989.

[7] G. H. Golub and H. Zha. The canonical correlations of matrix pairs and their numerical computation. In Adam Bojanczyk and George Cybenko, editors, *Linear algebra for signal processing*, volume 69, pages 59–82. Springer-Verlag, New York, 1995.

[8] H. Hotelling. Relations between two sets of variates. *Biometrika*, 28:321–372, 1936.

[9] C. Jordan. Essai sur la géométrie à n dimensions. *Bulletin de la Société Mathématique*, 3:103–174, 1875.

[10] T. Kailath. A view of three decades of linear filtering theory. *IEEE Transactions on Information Theory*, 20(2):146–181, 1974.

[11] R. J. Martin. Autoregression and cepstrum-domain filtering. *Signal Processing*, 76:93–97, 1999.

[12] R. J. Martin. A metric for ARMA processes. *IEEE Transactions on Signal Processing*, 48(4):1164–1170, April 2000.

[13] P. Van Overschee and B. De Moor. *Subspace identification for linear systems: Theory – Implementation – Applications.* Kluwer Academic Publishers, 1996.