On the parametrization of J-spectral factors

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Abstract

This paper considers a parametrization of a class of J spectral factors of a J spectral density. The notion of a J-stable spectral factor is introduced and it is shown how four extremal J-spectral factors can be derived from each other. Some applications of J-spectral factorizations in representation theory and $\mathcal{H}_\infty$ control theory are discussed.

1 Introduction

The purpose of this paper is to develop new algorithms for the calculation of J-spectral factorizations of rational matrices. J-spectral factorizations and J-spectral factors play a dominant role in the theory of $\mathcal{H}_\infty$ control and quite some work has been reported on this topic. See, for example [4, 5, 10, 11, 14] and references therein.

Let p and q be nonnegative integers and let $J = J_{pq} := \mathrm{diag}(I_p, -I_q)$ be a signature matrix where I_p denotes the pth order identity matrix. A rational function Φ is said to be a *J-spectral density* if $\Phi(s) = \Phi^*(s)$ for all s on the imaginary axis $\mathbb{C}^0$ and if its signature matrix is constant and equal to J for all complex numbers $s \in \mathbb{C}^0 \cup \infty$. Here, as elsewhere, $\Phi^*(s) = \overline{\Phi(-\bar{s})}^\top$ is the conjugate of Φ. We say that a square matrix valued rational function W is a *J-spectral factor* of a J-spectral density Φ if

$$\Phi = W^* J W.$$

The aim of this investigation is to study the J-spectral factors W of a J-spectral density function Φ. Depending on the application, one usually wishes the spectral factor W to satisfy additional properties, such as stability, stability of its inverse, minimum phase, maximum phase, etc. This study will focus around the introduction of four *extremal J-spectral factors* which are defined in terms of specific stability requirements. Here, the stability requirements will be formulated by means of *J-stable* rational functions. In the next section, we will first introduce this concept and discuss its system theoretic relevance. We will then show that the extremal J-spectral factors define a partial ordering on the set of J-spectral factors of a given J-spectral density. Applications will be discussed briefly in Section 4. This work is closely related to and motivated by the results in [6, 7, 9]. The present work aims to generalize the methods employed in [9] to J-spectral densities.

2 J-inner and J-outer functions

It is well known that every proper rational matrix valued function W can be written as $W(s) = C(Is - A)^{-1}B + D$ for suitable matrices A, B, C, D. Throughout this paper we will denote

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \tag{1}$$

to identify W with such a (state space) representation. Let J be a signature matrix. A rational matrix valued function W is said to be *row J-stable* if it admits a representation (1) such that the equation

$$AP + PA^* + BJB^* = 0 \tag{2}$$

has a positive definite solution P. Similarly, W is said to be *column J-stable* if it admits a representation (1) such that

$$A^*P + PA + C^*JC = 0 \tag{3}$$

has a positive definite solution P. It is easily seen that these notions do not depend on the specific representation (1) of W. That is, if (2) or (3) admit a positive definite solution P for *some* representation (1) of W, then it admits such a solution for *all* representations (1) of W, as long as the dimension of A coincides with the McMillan degree of W. Note that if $J = I$, then W is row J-stable if and only if all eigenvalues of A are in the left-half complex plane for some (and hence all minimal) representations (1) of W. Equivalently, $W \in \mathcal{H}_\infty$ and W is said to be *stable* in that case. Likewise, for $J = -I$, W is row J-stable if and only if W is *anti-stable*. Similar notions are defined for anti-stable matrix valued functions. Specifically, a rational matrix valued function W is said to be *row J anti-stable* if its conjugate W^* is W column J-stable. W is said to be J-stable if it is row J-stable.

For the state space system

$$\begin{cases} \dot{x} &= Ax + Bv \\ w &= Cx + Dv \end{cases} \tag{4}$$

these notions have the following system theoretic interpretations.

Proposition 2.1 *Let $W \in \mathcal{H}_\infty$ be represented by (1).*

1. *If the input $v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ in (4) is partitioned accordingly with the signature matrix J then the following statements are equivalent.*
   - *(a) W is row J-stable.*
   - *(b) there exists $Q = Q^\top$ such that*

     $$\begin{aligned} V(x_0) &:= \inf_{v_1} \sup_{v_2} \big\{ \|v_1\|_2^2 - \|v_2\|_2^2 \mid v_1 \in \mathcal{L}_2^-, \\ &\quad v_2 \in \mathcal{L}_2^-, x \in \mathcal{L}_2 \text{ and } x(0) = x_0 \big\} \\ &= x_0^\top Q x_0. \end{aligned}$$

     is positive for all nonzero states x_0.
2. *If the output $w = \begin{pmatrix} w_1 & w_2 \end{pmatrix}$ in (4) is partitioned accordingly with the signature matrix J, then the following statements are equivalent.*
   - *(a) W is column J-stable.*
   - *(b) there exists $Q = Q^\top$ such that*

     $$V(x_0) := \|w_1\|_2^2 - \|w_2\|_2^2 = x_0^\top Q x_0$$

     is positive for all nonzero states x_0. Here, $w \in \mathcal{L}_2^+$ satisfies (4) subject to the initial condition $x(0) = x_0$ and $u = 0$.

Proof: (Sketch). (1) To prove the first claim, suppose that W is row J-stable. Define $Q := P^{-1}$ where $P = P^\top$ is the positive definite solution of (2). A completion of the squares argument then shows that for all $\mathcal{L}_2$ trajectories v, x, w satisfying (4) there holds

$$\langle v, Jv \rangle = x_0^\top Q x_0 + \langle \bar{v}, J\bar{v} \rangle$$

where $\bar{v} = v - JB^*Qx$ and $\langle a, b \rangle := \int_{-\infty}^0 b^\top(t) a(t) dt$ is the standard inner product on $\mathcal{L}_2^-$. With the given partitioning of v, it then follows that $V(x_0) = x_0^\top Q x_0$ where the min-max equilibrium (if it exists) is obtained for $\bar{v} = 0$, i.e., with $v_1 = B_1^* Q x$ and $v_2 = -B_2^* Q x$ and with B_1 and B_2 the first and last rows of B corresponding to the partitioning of J. Since $P > 0$, also $Q > 0$, and hence $V(x_0)$ is positive for all nonzero x_0. Conversely, if $V(x_0) = x_0^\top Q x_0$ defines a positive equilibrium for any nonzero x_0, then from standard LQ theory one infers that $P = Q^{-1}$ is a positive definite solution of (2). Hence W is row J-stable.

(2) The second claim follows by observing that $w \in \mathcal{L}_2^+$ is a solution of (4) subject to $v = 0$ and $x(0) = x_0$ if and only if $w(t) = C\exp(At)x_0$. The result then follows from

$$\|w_1\|_2^2 - \|w_2\|_2^2 = x_0^\top Q x_0$$

where $Q = \int_0^\infty \exp(A^* t) C^* J C \exp(At)\, dt$ is positive definite if and only if $P = Q$ is a positive definite solution of (3). ∎

We proceed with definitions of J-inner and J-outer functions.

Definition 2.2 A square rational matrix W is called *J-outer* if it is row J-stable, invertible in $\mathcal{L}_\infty$ and its inverse W^{-1} is column J-stable.

Note that if $J = I$, a square matrix W is J-outer if and only if both W and W^{-1} belong to $\mathcal{H}_\infty$. In that case, W is a unitary element in $\mathcal{H}_\infty$ and is usually called an *outer* or *minimum phase* function. Likewise, for $J = -I$, a square matrix W is J-outer if and only if it is anti-stable and maximum phase. As mentioned before, these notions are well defined in the sense that they do not depend on the state space realization (1) of W. Let $\mathbb{C}^+$ denote the right-half complex plane, i.e. $\mathbb{C}^+ := \{s \in \mathbb{C} \mid \Re(s) > 0\}$. J-unitary and J-inner functions are defined as follows.

Definition 2.3 A square matrix W is *J-unitary* if $W^*(s) J W(s) = J$ for all $s \in \mathbb{C}^0$. It is *J-inner* if it is J-unitary and, in addition, $[W(s)]^* J W(s) \le J$ for all but finitely many points $s \in \mathbb{C}^+$.

These notions generalize to non-square matrices as follows.

Definition 2.4 Let W have dimension $p \times q$ and let $J = J_{p_+ p_-}$ and $\hat{J} = J_{q_+ q_-}$ with $p = p_+ + p_-$ and $q = q_+ + q_-$ two signature matrices. We call W

1. *wide $(\hat{J}, J)$-unitary* if $p \le q$ and $W(s)\hat{J}W(s)^* = J$ for all $s \in \mathbb{C}^0$.
2. *wide $(\hat{J}, J)$-inner* if it is wide $(\hat{J}, J)$-unitary and $W(s)\hat{J}[W(s)]^* \le J$ for all but finitely many $s \in \mathbb{C}^+$.
3. *tall $(J, \hat{J})$-unitary* if $p \ge q$ and $[W(s)]^* J W(s) = \hat{J}$ for all $s \in \mathbb{C}^0$.
4. *tall $(J, \hat{J})$-inner* if it is tall $(J, \hat{J})$-unitary and $[W(s)]^* J W(s) \le \hat{J}$ for almost all $s \in \mathbb{C}^+$.

It is well known [3,5,10] that if $W \in \mathcal{H}_\infty$, then W is tall $(J,\hat{J})$-inner if and only if the input and output variable of any representation (4) of W satisfy

$$\int_{-\infty}^{T} \langle w(t), Jw(t)\rangle \, dt \le \int_{-\infty}^{T} \langle v(t), \hat{J}v(t)\rangle \, dt$$

for all $T \in \mathbb{R}$ and all square integrable trajectories $v, w \in \mathcal{L}_2$ that satisfy (4), with equality holding for $T = \infty$.

The following result provides an algebraic characterization of these notions.

Proposition 2.5 *Let W be a $p \times q$ proper rational matrix function and let (1) be a minimal realization of W. Then*

1. *the following statements are equivalent.*
   (a) *W is tall $(J,\hat{J})$-unitary.*
   (b) *there exists an invertible matrix P such that*
   $$\begin{aligned} &A^*P + PA + C^*JC = 0 \\ &D^*JD = \hat{J} \\ &PB + C^*JD = 0 \end{aligned}$$
   Moreover, W is tall $(J,\hat{J})$-inner if and only if, in addition, $P > 0$.
2. *the following statements are equivalent.*
   (a) *W is wide $(\hat{J},J)$-unitary.*
   (b) *there exists an invertible matrix P such that*
   $$\begin{aligned} &AP + PA^* + B\hat{J}B^* = 0 \\ &D\hat{J}D^* = J \\ &CP + D\hat{J}B^* = 0 \end{aligned}$$
   Moreover, W is wide $(\hat{J},J)$-inner if and only if, in addition, $P > 0$.

A characterization of J-unitary and J-inner square matrices is obtained as a simple corollary of Proposition 2.5 by taking $J = \hat{J}$. In particular, it follows that W is a J-inner function if and only if it has a representation

$$W = \left(\begin{array}{c|c} A & -P^{-1}C^*JC \\ \hline C & D \end{array}\right)$$

where $DJD^* = J$ and $P > 0$ is a solution of (3). Note that wide $(\hat{J},J)$-inner functions are necessarily row $\hat{J}$-stable and tall $(J,\hat{J})$-inner functions are necessarily column J-stable.

3 Extremal J-spectral factors

Let J be a signature matrices. Given a J-spectral density Φ, we will be interested in finding J-spectral factors

1. $W = W_-$ such that W_- is J-stable and W_-^{-1} is J-stable.
2. $W = W_+$ such that W_+ is J-stable and W_+^{-1} is J anti-stable.
3. $W = \overline{W}_-$ such that $\overline{W}_-$ is J anti-stable and $\overline{W}_-^{-1}$ is J-stable.
4. $W = \overline{W}_+$ such that $\overline{W}_+$ is J anti-stable and $\overline{W}_+^{-1}$ is J anti-stable.

We develop algorithms for the calculation of W_-, W_+, $\overline{W}_-$ and $\overline{W}_+$ from the J-spectral density Φ, i.e., we construct the mapping

$$\Phi \longrightarrow (W_-, W_+, \overline{W}_-, \overline{W}_+) \tag{5}$$

and show that the diagram below is commutative.

$$\begin{array}{ccc} W_- & \xrightarrow{Q_+} & W_+ \\ \uparrow{\scriptstyle K_-} & & \uparrow{\scriptstyle K_+} \\ \overline{W}_- & \xrightarrow{Q_-} & \overline{W}_+ \end{array}$$

The mappings in the above diagram are constructed as follows.

Theorem 3.1 *Let W_- be a J-outer spectral factor of a J spectral density Φ. Then there exist J-inner functions K_+, K_-, Q_+ and Q_- such that*

$$\begin{aligned} \overline{W}_- &= K_-^* W_- \\ \overline{W}_+ &= Q_- \overline{W}_- \\ W_+ &= Q_+ W_-. \end{aligned}$$

Moreover, we have that

$$Q_+K_- = K_+Q_-$$

i.e., the diagram above is commutative.

4 Applications

4.1 System equivalence

Especially in representation theory of dynamical systems, J spectral densities are usually of the form $\Phi = KJK^*$ or $\Phi = K^*JK$ where K is a rational function which defines a dynamical system. We will briefly comment on the relevance of the mapping (5) in this context.

Following the behavioral approach, dynamical systems will be viewed as sets of trajectories evolving over time. Let K be a proper rational matrix in $\mathcal{L}_\infty$ and suppose that

$$K = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

defines a minimal realization of K. Any such K defines the *driving variable system*

$$\mathcal{B}_{\rm dv}(K) := \{w \in \mathcal{L}_2 \mid \hat{w} \in K\mathcal{L}_2\} \tag{6}$$

where $\hat{w}$ denotes the Laplace transform of $w \in \mathcal{L}_2$. That is, the system $\mathcal{B}_{\rm dv}(K)$ consists of all square integrable trajectories w whose Laplace transform is in the $\mathcal{L}_2$-image of K. Equivalently,

$$\mathcal{B}_{\rm dv}(K) = \{w \in \mathcal{L}_2 \mid \exists v, x \in \mathcal{L}_2 \text{ such that (4) holds}\}.$$

Obviously, these systems define linear, shift invariant subspaces of the Hilbert $\mathcal{L}_2$. See [13] for a detailed study of this class of systems. We note that the $\mathcal{L}_\infty$ assumption of K can be made without loss of generality to define systems of the form (6). We say that two $\mathcal{L}_\infty$ functions K_1 and K_2 are *equivalent* if they define the same system, i.e, K_1 is equivalent to K_2 if $\mathcal{B}_{\rm dv}(K_1) = \mathcal{B}_{\rm dv}(K_2)$.

Proposition 4.1 *Let J be a signature matrix. With the above notion of equivalence, every rational function $K \in \mathcal{L}_\infty$ is equivalent to a tall $(J, \hat{J})$-inner matrix $K_0 \in \mathcal{L}_\infty$. Moreover, any such K_0 can be written as*

$$K_0 = KW^{-1}$$

where W is a $\hat{J}$ spectral factor of the $\hat{J}$-spectral density $\Phi = K^* JK$.

Proof: (Sketch). Let K have a minimal representation (4). It has been shown in [12] that K_0 is equivalent to K if and only if

$$K_0 = \left(\begin{array}{c|c} S^{-1}(A + BF)S & S^{-1}BR \\ \hline (C + DF)S & DR \end{array}\right) \tag{7}$$

for some F and nonsingular matrices S and R. Let P solve

$$\begin{aligned} A^*P + PA - (PB + C^*JD)(D^*JD)^{-1}(B^*P + D^*JC) \\ + C^*JC = 0 \end{aligned}$$

and put $F = -(D^*JD)^{-1}(B^*P + D^*JC)$, $S = I$ and a nonsingular R such that $D^*JD = R^{-*}\hat{J}R^{-1}$, where $\hat{J}$ is the signature of D^*JD. This defines K_0 through (7) and a completion of the squares argument yields that

$$\langle w, Jw\rangle = \langle \bar{v}, \hat{J}\bar{v}\rangle$$

where $\bar{v} = R^{-1}(v - Fx)$ is any $\mathcal{L}_2$ input (or *driving variable* of K_0 and $w \in \mathcal{L}_2$ its corresponding output. Infer from Proposition (2.5) that K_0 is tall $(J, \hat{J})$-inner. Moreover, the mapping $W : v \mapsto \bar{v}$ is bijective on $\mathcal{L}_2$, and satisfies $K_0 W = K$. Since

$$\Phi := K^*JK = W^*K_0^*JK_0W = W^*\hat{J}W$$

it follows that W is a $\hat{J}$-spectral factor of K^*JK, as claimed. ■

A similar result can be obtained for wide $(\hat{J}, J)$ inner functions if one associates sets of the form $\{w \in \mathcal{L}_2 \mid K\hat{w} = 0\}$ with a rational matrix K.

4.2 J-dissipativity

Let $K \in \mathcal{L}_\infty$ and consider the system $\mathcal{B} := \mathcal{B}_{dv}(K)$. Then $\mathcal{B}$ is said to be J-dissipative if

$$\int_{-\infty}^{t} \langle w(\tau), Jw(\tau)\rangle \, d\tau \;\geq\; 0$$

for all $t \in \mathbb{R}$ and $w \in \mathcal{B}$. Clearly, this notion implies that for a suitable partitioning $w = \begin{pmatrix} w_1 & w_2 \end{pmatrix}$ of w, we have that

$$\|w_1\|^2 - \|w_2\|^2 \geq 0$$

for all $w \in \mathcal{B}$. The following result characterizes this property.

Theorem 4.2 *Equivalent statements are:*

1. *$\mathcal{B}$ is J-dissipative.*
2. *There exist $P = P^\top \leq 0$ such that*
$$\begin{pmatrix} A^*P + PA + C^*JC & PB + C^*JD \\ B^*P + D^*JC & D^*JD \end{pmatrix} \geq 0.$$
3. *The spectral density $\Phi(s) := K^*(s)JK(s) \geq 0$ for all $s \in \mathbb{C}^0$.*

If $\mathcal{B}$ is not J-dissipative, then we may consider the control problem to synthesize a controller $\mathcal{C}$ for $\mathcal{B}$, such that the controlled system becomes J-dissipative. In the simplest version of this problem, the controller is assumed to have access to the driving variables v of (4) and is a linear time-invariant system of the form $\mathcal{C} = \mathcal{B}_{\rm dv}(C)$ where C is a proper rational matrix in $\mathcal{L}_\infty$. The controlled system is then defined by $\mathcal{B}_{\rm dv}(KC)$. Now consider the control problem to synthesize C, given K, such that

1. $\mathcal{B}_{\rm dv}(KC)$ is J-dissipative and
2. $\mathcal{B}_{\rm dv}(KC)$ is maximal free in the sense that there exists an injective matrix S of rank $r \geq 0$ such that
$$S\mathcal{L}_2 \subseteq \mathcal{B}_{\rm dv}(KC) \tag{8}$$
while no S of rank $r' > r$ exists such that (8) holds.

The rank r in condition 2 has the interpretation of the input dimension of the controlled system in any input-output representation of $\mathcal{B}_{\mathrm{dv}}(KC)$. This condition therefore means that the controller should not impose undue restrictions on the free variables of the system so as to achieve J-dissipativity of the controlled system. In view of the previous results, the solution of this problem is straightforward. For given $K \in \mathcal{L}_\infty$, it easily follows from Proposition 4.1 that all controllers C for which the controlled system $\mathcal{B}_{\mathrm{dv}}(KC)$ meets the above requirements are parametrized by

$$C = W^{-1} \begin{pmatrix} I \\ U \end{pmatrix}$$

where W is the $\hat{J} = \mathrm{diag}(I_{q_+}, -I_{q_-})$ spectral factor of Proposition 4.1 I is the identity matrix of dimension q_+ and $U \in \mathcal{L}_\infty$ is such that $\|U\|_\infty \leq 1$. The dimension r of condition 2 satisfies $r = q_+$. We refer to [10, 14] for more details on the $\mathcal{H}_\infty$ control problem in this behavioral setting.

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