Pole behaviour in identification

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Abstract. We show for a fractional filter with two cells that the rational models obtained by output error minimization have poles that cluster around the hyperbolic geodesic between the singular frequencies.

Introduction

Identification schemes of the maximum-likelihood type for stationary stochastic processes rest on the paradigm that the best estimate within a model class is the one that makes the observation most probable, the complexity of the model being assessed through a necessary trade-off between the bias and the variance of the error. In identification of linear discrete systems, where the standard model class is the set of stable transfer functions fed with observed inputs and white noise and where complexity is measured by the McMillan degree, several methods of equivalent asymptotic efficiency have been designed this way such as the maximization of the Gaussian log-likelihood of the sample or the minimization of the prediction error thereof, see *e.g.* [7, 9].

Of course the stochastic paradigm in itself does not help much in actually solving the underlying optimization problem which is the core of the identification procedure, nor in assessing properties of the identified model. Therefore the practice of identification is largely conditionned by the tractability of the numerics, the desire to match certain features for the model, and the amount of prior knowledge one has on the process.

One theoretical way to gain information on the outcome of the identification procedure under suitable ergodicity assumptions is to replace the optimization problem by its asymptotic form when the number of samples grows large, thereby going over to a rational approximation problem in the frequency domain, and to use tools from complex analysis and approximation theory. The difficulty here is twofold: first, one has to ensure that the convergence of the optimization problem to its asymptotic form is strong enough for the solutions themselves to converge; and second, most results on best rational approximants are of asymptotic nature and can hardly be used if the degree of the approximant is kept fixed from statistical considerations.

The above program has been carried over under the greatly simplifying assumption that the process to be identified lies in the model class. With this hypothesis the second difficulty listed above disappears, since the process itself is the obvious candidate to its own best approximation. Even then, the first difficulty is not so easily overcome as it requires what is called a consistency theorem, like the one given in [7] for the case where there is no input.

The present paper offers a modest excursion off the realm where "the true system belongs to the model class", in the special case where a single-input single-output fractional filter with two singular frequencies is fed with white noise and made subject to the ouput error method. We choose a fractional filter because it generates a long memory process and therefore has attracted attention from the statistical community [11] while at the same time the singularities of the transfer function are branch points that were much considered in approximation theory [1, 10]. The output error method with white noise input has been selected because of the simple form of the asymptotic optimization problem in this case, namely best rational approximation in the Hardy space $\overline{H}^2$ of the complement of the unit disk. We shall derive a geometric constraint on the poles of the identified transfer function for large sample size, which is of interest because the location of these poles is the nonlinear and most difficult aspect of the optimization problem to be solved, and their location is a fundamental aspect of the model's dynamics (the poles, by the way, are called the zeros in prediction theory). This constraint shows that the poles essentially gather in the unit disk to the hyperbolic geodesic between the singular frequencies, which is a circular arc orthogonal to the unit circle.

System, model class, and identification scheme

We consider a fractional filter with transfer function

$$f(z) = \left(c - 1/z\right)^{\delta}\left(\overline{c} - 1/z\right)^{\delta}, \tag{1}$$

where $0 < \operatorname{Im} c \leq |c| = 1$ and $\delta > 0$, but not integer. It is understood for the function $w \mapsto w^{\delta}$ that we choose its principal branch such that $w^{\delta} > 0$ for $w > 0$. The function $f(z)$ is analytic for $|z| > 1$ and uniformly continuous for $|z| \geq 1$. Moreover it has an absolutely

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convergent power series expansion

$$f(z)=\sum_{k=0}^{\infty} f_k z^{-k}, \qquad |z|\geq 1,$$

and the boundedness of f on the unit circle $\mathbb{T}$ enables it to act by convolution on any complex-valued stationary process u to yield a new process

$$f*u(t)=\sum_{k=0}^{\infty} f_k u(t-k). \tag{2}$$

For $n\in\mathbb{N}$ we let $\mathbb{R}_n[z]$ denote the space of all polynomials with real coefficients and of degree at most n. By definition, a transfer function of degree n is a rational function $h(z)=p_n(z)/q_n(z)$ where $p_n, q_n\in\mathbb{R}_n[z]$ are coprime, and q_n is monic of degree n with all its roots in the open unit disk $\mathbb{U}$. A transfer function also has an absolutely convergent power series expansion

$$h(z)=\sum_{k=0}^{\infty} h_k z^{-k}, \qquad |z|\geq 1,$$

and gives rise by convolution on u to the process

$$h*u(t)=\sum_{k=0}^{\infty} h_k u(t-k).$$

The set of all transfer functions of degree *at most* n is denoted by $\mathcal{S}_n$.

In order to model a functional correspondence between two discrete-time real-valued stochastic processes $y(t)$ and $u(t)$ observed, say for $t\geq 0$, the output error method consists in fitting to y a process of the form $h*u+e$, where e is some white noise independent from u and h is a transfer function of degree at most n, so as to minimize the l^2-norm of the prediction error achieved by the model over the sample. Thus if the sample has length $N+1$, the underlying optimization problem is to find

$$\inf_{h\in\mathcal{S}_n}\frac{1}{N+1}\sum_{t=0}^{N}\left|y_t-(h*u)_t\right|^2 \tag{3}$$

where the subscript t indicates the value of the sample at time t. The value of n has to be determined from the trade-off between the bias and the variance, also possibly from physical and practical constraints one puts on the model. Of course, it is impossible to compute the criterion exactly in the right-hand side of (3) because the values of u_t are unknown for $t<0$, but setting these to zero (which is the mean of $u(t)$) does not affect the asymptotic behaviour of the method when N goes large as explained in the next section.

The output error method coincides with likelihood maximization on the considered model class, although the latter is not the most general in linear identification because one would typically consider systems of the form $h*u+g*e$ where g is some invertible transfer function, and the degree of the $\mathbb{C}^2$-valued rational function $(h(z),g(z))$ is at most n. In this paper, we shall apply the output error method to the process (2) *in the special case where u is white noise.*

Asymptotic form of the optimization problem

The Hardy space $\overline{H}^2$ of the complement of the unit disk is the subspace of $L^2(\mathbb{T})$ consisting of the functions whose Fourier coefficients of positive index do vanish. Hence $g\in\overline{H}^2$ if, and only if

$$g(z)=\sum_{k=0}^{\infty} g_k z^{-k} \text{ a. e. on } \mathbb{T} \text{ and } \|g\|^2=\sum_{k=0}^{\infty}|g_k|^2<\infty.$$

Then the power series $\sum_{k=0}^{\infty} g_k z^{-k}$ defines an analytic function for $|z|>1$ whose nontangential limit on $\mathbb{T}$ is equal to g almost everywhere by a classical theorem of Fatou [6]. This establishes a one-to-one correspondence between $\overline{H}^2$ and the space of analytic functions whose Taylor coefficients at infinity are square summable, and accounts for the fact that the Hardy space $\overline{H}^2$ is regarded both as a subspace of $L^2(\mathbb{T})$ and as a Hilbert space of analytic functions in the complement of the closed unit disk. We shall actually restrict ourselves to functions with real Fourier coefficients, that form a *real* subspace of $\overline{H}^2$ called the *conjugate-symmetric Hardy space* and denoted by $\overline{H}^2_{\mathbb{R}}$. The rational functions in the conjugate-symmetric Hardy space are precisely the transfer functions defined previously and the nonrational function f in (1) belongs to $\overline{H}^2_{\mathbb{R}}$, too. Any function $g\in\overline{H}^2_{\mathbb{R}}$ has at least one best rational approximant of degree at most n, in other words a best approximant out of $\mathcal{S}_n$, and any of them belongs to $\mathcal{S}_n\setminus\mathcal{S}_{n-1}$ unless $g\in\mathcal{S}_{n-1}$, see [8, 2]. Best rational approximants to the function f in (1) will be a major concern to us because of the following result:

Theorem 1 *Assume that $y=f*u$ where f is given by* (1) *and u is white noise. Then, almost surely, the infimum* (3) *is attained in at least one element of $\mathcal{S}_n$ when N is sufficiently large. Moreover, the set of all minimizers as N ranges over $\mathbb{N}$ is compact in $\overline{H}^2_{\mathbb{R}}$, and any limit point is a best approximant to f from $\mathcal{S}_n$.*

Theorem 1 is actually a special case of a more general result on the consistency of prediction error minimization for linear models. Specifically, one can prove that minimizers of the prediction error converge to a best estimate of the observed process out of the model class when the number of samples goes large, provided that u, y are ergodic, that the spectral density of u is bounded away from 0 on $\mathbb{T}$, and that the trigonometric series

$$W(z) = \sum_{k=-\infty}^{\infty} \mathbf{E}\{y(k)y(0)\}e^{-ik\theta},$$

$$F(z) = \sum_{k=-\infty}^{\infty} \mathbf{E}\{y(k)u(0)\}e^{-ik\theta},$$

$$F^{-}(z) = \sum_{k=0}^{\infty} \mathbf{E}\{y(k)u(0)\}e^{-ik\theta}$$

lie in $L^1(\mathbb{T})$, $L^2(\mathbb{T})$, and $\overline{H}^2_{\mathbb{R}} \backslash \mathcal{S}_{n-1}$ respectively [3]; here the symbol $\mathbf{E}$ stands for the mathematical expectation. In the present case $W = |f|^2$ while $F = F^- = f$ and the spectral density of u is 1, so Theorem 1 follows from the quoted result. Because of the compactness assertion in Theorem 1, the minimizers of (3) have uniform geometric decay of their Fourier coefficients as soon as N is large enough, and this accounts for our previous contention that setting $u_t = 0$ for $t < 0$ does not affect the asymptotics of the minimization.

Rational approximants to the fractional transfer function

If the function g belongs to $\overline{H}^2_{\mathbb{R}} \setminus \mathcal{S}_{n-1}$ then the monic denominator q_n of any best rational approximant p_n/q_n to g from $\mathcal{S}_n$ has degree n and moreover it satisfies the orthogonality relation

$$\langle\, g\,, z^k \widetilde{q_n}/q_n^2 \,\rangle = 0, \qquad 0 \le k \le n-1, \qquad (4)$$

in which $\langle\,\cdot\,,\cdot\,\rangle$ denotes the inner product in $L^2(\mathbb{T})$ and $\widetilde{q_n}(z) = z^n \overline{q_n(1/\overline{z})}$ the reciprocal polynomial of q_n. Relation (4) can be obtained upon equating to zero the derivatives of $\|g - p_n/q_n\|^2 = \langle\, g - p_n/q_n\,, g - p_n/q_n \,\rangle$ with respect to the coefficients of p_n and q_n. Indeed, since p_n and q_n are coprime, p_n/q_n must be the orthogonal projection of g onto the space $\mathbb{R}_{2n}[z]/q_n^2$ of all rational functions of degree an most $2n$ with denominator q_n^2 in $\overline{H}^2_{\mathbb{R}}$, and consequently g must be orthogonal to the orthogonal complement of $\mathbb{R}_n[z]/q_n$ in $\mathbb{R}_{2n}[z]/q_n^2$ which is precisely $\mathbb{R}_{n-1}[z]\widetilde{q_n}/q_n^2$. The orthogonality relation (4) is proved in [4]; but it is also easily checked to be equivalent to the fact that p_n/q_n interpolates g of order 2 at the reflections of the zeros of q_n across $\mathbb{T}$, which is the classical necessary condition for optimality, see [8]. When we use the integral representation for the inner product, then we find that relation (4) is equivalent to

$$\frac{1}{2\pi i}\int_{\mathbb{T}} g(\xi)\frac{\xi^k q_n(\xi)}{\widetilde{q_n}^{\,2}(\xi)}\,d\xi = 0, \qquad 0 \le k \le n-1. \qquad (5)$$

Herein the switch of the tilde is not a mistake, but a consequence of conjugation in the inner product; and as usual, the unit circle is oriented counter-clockwise.

We let Γ denote the hyperbolic geodesic from $\overline{c}$ to c, that is the arc of circle in the closed unit disk $\overline{\mathbb{U}}$ that both in the initial point $\overline{c}$ and in the endpoint c is orthogonal to the unit circle. When we cut the extended complex plane $\widehat{\mathbb{C}}$ along Γ, then the function f in (1) admits a single-valued analytic continuation to the cut-plane $\widehat{\mathbb{C}} \setminus \Gamma$; and for all $\zeta \in \Gamma$ the limits $f_-(\zeta)$ and $f_+(\zeta)$ from the left respectively right side of Γ exist, but they differ by the constant factor $e^{-2\pi i\delta} \neq 1$, that is they satisfy $f_+(\zeta) = e^{-2\pi i\delta} f_-(\zeta)$. Moreover these two limit functions are continuous on Γ and as a limit case of Cauchy's formula we obtain in $\widehat{\mathbb{C}} \setminus \Gamma$ for f the integral representation

$$f(z) = 1 + \frac{K}{2\pi i}\int_{\Gamma}\frac{f_-(\zeta)}{\zeta - z}\,d\zeta, \qquad z \in \widehat{\mathbb{C}} \setminus \Gamma, \qquad (6)$$

where $K = 1 - e^{-2\pi i\delta} \neq 0$ and Γ is oriented from $\overline{c}$ to c. Now, in (4) and (5) we choose $g = f$. Since all zeros of q_n have modulus less than 1 and consequently all zeros of $\widetilde{q_n}$ have modulus greater than 1, there is $R > 1$ such that $q_n/\widetilde{q_n}^{\,2}$ is analytic for $|z| < R$. Using the fact that f is analytic for $|z| > 1$ and uniformly continuous for $|z| \ge 1$, and in addition Cauchy's formula, Fubini's theorem and (6), we then obtain for all $r \in [1, R)$ and all $k \in \mathbb{N}$ the identity

$$\frac{1}{2\pi i}\int_{r\mathbb{T}} f(\xi)\frac{\xi^k q_n(\xi)}{\widetilde{q_n}^{\,2}(\xi)}\,d\xi = \frac{-K}{2\pi i}\int_{\Gamma}\frac{\zeta^k q_n(\zeta)}{\widetilde{q_n}^{\,2}(\zeta)} f_-(\zeta)\,d\zeta, \qquad (7)$$

with $r\mathbb{T}$ oriented counter-clockwise and Γ from $\overline{c}$ to c. Thus, as $K \neq 0$, we get with $r = 1$ from (7) that for $g = f$ the orthogonality relation (5) is equivalent to

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\zeta^k q_n(\zeta)}{\widetilde{q_n}^{\,2}(\zeta)} f_-(\zeta)\,d\zeta = 0, \qquad 0 \le k \le n-1. \qquad (8)$$

We set $a := (c - i)/(1 - ic) = \operatorname{Re} c/(1 + \operatorname{Im} c) \in (-1, 1)$ and by means of a we define the unit disk automorphism

$$T(z) := \frac{iz + a}{1 + aiz}, \quad \text{with} \quad T'(z) = i\frac{1 - a^2}{(1 + aiz)^2}, \qquad (9)$$

which maps the interval $[-1, 1]$ onto the geodesic Γ so that $T(-1) = \overline{c}$ and $T(1) = c$. Furthermore we define $d_n := \widetilde{q_n}(a) \in \mathbb{R} \setminus \{0\}$ and the polynomial

$$Q_n(z) := (1/d_n)(az - i)^n q_n\big(T(z)\big) \qquad (10)$$

which is also monic of degree n but in general with non-real coefficients, and whose zeros are the images under T^{-1} of the zeros of q_n. As $T(1/\overline{z}) = 1/\,\overline{T(z)}$ we get

$$\widetilde{Q_n}(z) = (1/d_n)(1 + aiz)^n \widetilde{q_n}\big(T(z)\big)$$

and, since $az - i = (1 + aiz)/i$, we obtain

$$\frac{T(z)^k q_n\big(T(z)\big)}{\widetilde{q_n}^{\,2}\big(T(z)\big)} = \frac{i^n}{d_n}(1 + aiz)^n T(z)^k \frac{Q_n(z)}{\widetilde{Q_n}^{\,2}(z)}. \qquad (11)$$

As the mapping $p(z) \mapsto (1+\bar{a}iz)^{n-1}p((T(z))$ is a linear bijection of the complex polynomial space $\mathbb{C}_{n-1}[z]$ onto itself, it follows from (11) and (9) that the orthogonality relation (8) is equivalent to

$$\int_{-1}^{1} \frac{s^k Q_n(s)}{\widetilde{Q_n}^2(s)} \, \frac{\big(c - 1/T(s)\big)^{\delta}\big(\overline{c} - 1/T(s)\big)^{\delta}}{1 + ais} \, ds = 0, \qquad (12)$$
$$0 \le k \le n-1.$$

A closer look at the function

$$s \mapsto w(s) := \frac{\big(c - 1/T(s)\big)^{\delta}\big(\overline{c} - 1/T(s)\big)^{\delta}}{1 + ais}$$

yields that on $[-1,1]$ the total variation of its argument is equal to

$$\begin{aligned} V &:= 2\,\delta\Big(\pi - \big|\mathrm{Arg}(c/i)\big|\Big) + 2\,\mathrm{Arg}(1 + i\,|a|) \\ &= 2\,\delta\Big(\pi - \mathrm{Arcsin}\,|\mathrm{Re}\,c|\Big) + 2\,\mathrm{Arctan}\,\frac{|\mathrm{Re}\,c|}{1 + \mathrm{Im}\,c} \\ &\le 2\,\delta\,\pi + 2(1-\delta)\,\mathrm{Arcsin}\,|\mathrm{Re}\,c| \le \big(\delta + \max(1,\delta)\big)\pi. \end{aligned}$$

Here $\mathrm{Arg}(z) \in (-\pi, \pi]$ denotes the principal branch of the argument; and correspondingly Arcsin and Arctan. The function $w(s)$ is continuous on $[-1,1]$ and vanishes only for $s = \pm 1$; therefore its moments $\int_{-1}^{1} s^k w(s) ds$ exist for all $k \in \mathbb{N}$ and, when continuously extended to $s = \pm 1$, the function $w(s)/\big|w(s)\big|$ also has a continuous argument of total variation V on $[-1,1]$. As was shown in [5], it then follows from the orthogonality relation (12) that the zeros $b_1, \ldots, b_n \in \mathbb{U}$ of Q_n satisfy the inequality

$$\sum_{k=1}^{n} \big(\pi - \Theta(b_k)\big) \le V \le \big(\delta + \max(1,\delta)\big)\pi,$$

where for $z \in \mathbb{U}$ the number

$$\Theta(z) := \big|\,\mathrm{Arg}(z-1) - \mathrm{Arg}(z+1)\,\big| \in \big(\pi/2, \pi\big]$$

is the size of the angle under which the interval $[-1,1]$ is seen from the point z. Since the points $T(b_k)$ are the zeros of q_n, we finally obtain the following theorem:

Theorem 2 *Let $n \in \mathbb{N}$, $0 < \mathrm{Im}\,c \le |c| = 1$, and $\delta > 0$ but not integer. Then every best rational approximant from $\mathcal{S}_n$ to the function f in* (1) *has degree n, and its poles $a_1, \ldots, a_n \in \mathbb{U}$ satisfy the inequality*

$$\sum_{k=1}^{n} \Big(\pi - \Theta\big(T^{-1}(a_k)\big)\Big) \le V \le \big(\delta + \max(1,\delta)\big)\pi.$$

In particular, for $\beta \in \big(0, \pi/2\big)$ all except for at most $[V/\beta]$ poles a_k lie in the neighbourhood

$$\big\{z \in \mathbb{U} : \pi - \Theta\big(T^{-1}(z)\big) < \beta\big\}$$

of the hyperbolic geodesic from $\overline{c}$ to c in the unit disk. (For $c \ne i$ this neighbourhood is sickle-shaped, but when $c = i$ then it is lenticular.)

Pole behaviour in identification

We are now ready to draw a conclusion on the outcome of our identification procedure for large sample size. In effect, the contents of Theorems 1 and 2 team up to yield the following result:

Theorem 3 *Assume that $y = f * u$ where f is given by* (1) *and u is white noise. Then, almost surely, a minimizer of* (3) *is a member of $\mathcal{S}_n \backslash \mathcal{S}_{n-1}$ whose poles $a_1, \ldots, a_n$ satisfy the conclusion of Theorem 2 when N is large enough.*

Theorem 3 entails that the poles tend to cluster on the geodesic Γ when n increases. To our knowledge, this is the first result that gives information on the model when the true system does not belong to the model class.

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