Some links between 2D Laplace inverse problems and approximation in Hardy classes: constructive identification and stability properties

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Abstract

We consider the inverse problem of identifying a Robin coefficient on some part of the boundary of a smooth 2D domain from overdetermined data available on the other part of the boundary, for Laplace equation in the domain. Using tools from complex analysis and analytic functions theory, we provide a constructive and convergent identification scheme, together with a stability result for this inverse problem.

1 Introduction

Several inverse problems may need to be solved that some lacking data on a part of the boundary be recovered from overdetermined data on another part of the boundary. In this paper, the issue we are interested in is the recovery of a Robin coefficient from measurements performed on some part of the boundary. Such an issue arises for example in corrosion detection by electrical impedance tomgraphy. An effective non linear boundary condition that reduces the knowledge of corrosion effects to that of a function defined on the corroded boundary has been derived by using a multiscale analysis expansion in [25]. In the simplest linear case, the parameter characterizing the damage caused by corrosion is a Robin exchange coefficient, the direct problem to be solved being a Laplace equation. Sticking to this simple model, our purpose here is to set up a numerical algorithm based on constructive approximation, using analytic functions tools. Alternative algorithms have been investigated by several authors, using quasi-reversibility or least squares approaches see [13, 16, 19, 21, 22].

We first recall an identifiability result from [13], ensuring that the unknown Robin coefficient is uniquely determined in a proper class from the knowledge of the Neumann data (prescribed current flux) and of additional Dirichlet data (measured voltage potential), on a suitable proper part of the boundary.

We then approach constructively the issue of determining the unknown coefficient from the available boundary data. The point here is that the above problem amounts to that of recovering an analytic function from its trace on a proper subset of the boundary of its analyticity domain. In order to ensure stability and robustness properties of the recovery procedure, we are led to turn this interpolation issue into an approximation one in Hardy classes (normed spaces of analytic functions). After conformally mapping the involved domain into the unit disk $\mathbb{D}$ of the complex plane, we handle these problems using complex analysis tools and analytic approximation results [3, 5, 12]. The main difficulty one has to confront in processing such an approach is that related to instabilities that characterize data completion problems of this kind, i.e. the solution of Cauchy problems for elliptic operators, which are well known since Hadamard in the early twenties to be severely ill-posed. As a matter of fact, the given data may be fit as closely as desired on the part of the boundary where they have been prescribed, provided that hectic behaviours are tolerated on the remaining part of the boundary. Setting a bound on the data to be recovered is thus a feasible way, proposed in the bounded extremal extension approach [5], in order to avoid the extended solution to blow up away from the prescription part of the boundary. However, extending the function would be hardly enough since, our purpose being to recover a Robin coefficient from extended data, accuracy is not only needed on the function itself, but on its normal derivative as well. This compels us to consider higher order methods, based on the same extension process applied to the derivatives of the function to be extended. A complete version of the present work, including proofs of the stated results and numerical experiments, is in preparation [14].

This approach extends for the reconstruction of lacking data in cracks recovery : in such a case, the data to recover are not harmonic, and meromorphic extension is therefore used instead of the analytic one [6]. The issues remain however the same, especially that related to the stabilization of the reconstruction process, a task to which some part of the available data needs to be devoted.

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2 Statement and properties of the problem

2.1 Notations

Let D be a bounded domain of $\mathbb{R}^2$ (or $\mathbb{C}$) with Jordan closed boundary $T = T$. The k^{th} derivative with respect to the ambient real or complex variable of some function ϕ will be written as $\phi^{(k)}$, $k \geq 0$, with the usual convention $\phi' = \phi^{(1)}$. For $n \geq 0$ and $0 \leq \beta \leq 1$, we note $\mathcal{C}^{n,\beta}(\overline{D})$ for the space of functions f on $\overline{D}$ whose derivatives $f^{(k)}$ are of Hölder class with order β for $0 \leq k \leq n$. We put $\mathcal{C}^{n,0} = \mathcal{C}^n$. Also, D is said to have a $\mathcal{C}^{n,\beta}$ boundary if T admits a $\mathcal{C}^{n,\beta}$ parametrization [23]. The Lebesgue measure on T will be noted μ in general. However, for $T = \mathbb{T}$, the unit circle, we shall write $d\mu = d\theta$. For any connected open subset $E \subset T$, let χ_E be the characteristic function of E; traces on E of both functions and spaces will be indicated by $_{|E}$. The Hilbert space $L^2(E)$ of square summable functions with respect to μ on E is equipped with the classical norm and inner product, that we simply write $\| \, \|_E$ and $(\, , \,)_E$, respectively. For $s > 0$, the Sobolev Hilbert space $W^{s,2}(D)$ equipped with $\| \, \|_{s,D}$ is classically defined, see e.g. [15]. Whenever $n \in \mathbb{N}$, the norm on $W^{n,2}(E)$ is the usual one. For $E \subsetneq T$, $\mathcal{C}_0^n(E)$ is the subset of $\mathcal{C}^n(E)$ consisting of functions f that vanish at ∂E together with their derivatives $f^{(k)}$, $k = 0, \cdots, n-1$. The set $W_0^{n,2}(E)$ stands for the $W^{n,2}(E)$ closure of $\mathcal{C}_0^n(E)$. We also use the Sobolev Banach space $W^{n,\infty}(E)$ of functions belonging to $L^\infty(E)$ together with their derivatives up to order n.

As to Hardy spaces of the unit disk $\mathbb{D}$, we consider the Hilbertian framework where $H^2 = H^2(\mathbb{D})$ can be viewed as the space of functions analytic in $\mathbb{D}$ that are square-summable on circles of radius less than 1 centered at 0. It is a consequence of this definition that traces on the unit circle $\mathbb{T}$ of H^2 functions belong to $L^2(\mathbb{T})$, and in this sense $H^2 \subset L^2(\mathbb{T})$ inherits the inner product and can be described as the subspace of $L^2(\mathbb{T})$ consisting in functions whose Fourier coefficients of negative order do vanish:

$$H^2 = \{g(z) = \sum_{p \geq 0} a_p \, z^p \ , \ \sum_{p \geq 0} |a_p|^2 < \infty\} \, ,$$

see e.g. [17, 18, 24] for definitions and properties of Hardy spaces. A further equivalent definition of H^2 is that it is the space of complex valued functions whose real and imaginary parts are both harmonic in $\mathbb{D}$ and such that their L^2 norm on circles of radius $r < 1$ remains bounded as $r \to 1$. Note that H^2 strictly contains the functions that are analytic and uniformly bounded in $\mathbb{D}$. Denote by $P_{H^2} : L^2(T) \to H^2$ the orthogonal (analytic) projection. We finally let $\mathcal{H}^{n,2}$ be the Hardy-Sobolev Hilbert space: $\mathcal{H}^{n,2} = H^2 \cap W^{n,2}(\mathbb{T})$.

2.2 A Robin inverse problem

Let D be a simply connected bounded domain of $\mathbb{R}^2$ with boundary T, a $\mathcal{C}^{1,\beta}$ Jordan curve, for $\beta \in (0,1)$. Let then γ, K be two nonempty open subsets of T, satisfying $T = \overline{\gamma} \cup \overline{K}$. We address the following inverse problem:

being given a prescribed flux $\phi \not\equiv 0$ together with measurements u_m on K, find a function φ on γ such that the solution u of

$$\text{(PR)} \quad \left\{ \begin{array}{rcll} \Delta u & = & 0 & \text{in} \quad D, \\ \dfrac{\partial u}{\partial n} & = & \phi & \text{on} \quad K, \\ \dfrac{\partial u}{\partial n} + \varphi u & = & 0 & \text{on} \quad \gamma, \end{array} \right.$$

also satisfies $u_{|_K} = u_m$.

In the sequel, we assume that *both the measurement part* $K \subset T$ *and the "Robin" part* $\gamma = T \backslash K$ *have positive Lebesgue measure and finitely many components*, the simplest case being the one where K is an arc of T.

2.3 Identifiability and smoothness

That the above inverse problem is well-posed (meaning that its solution is unique) as soon as the additional measurements are available on any set K of positive measure, is a result proved in [13] for classes of continuous Robin coefficients.Intending to work out higher order methods leads us to restrict the class of admissible Robin coefficients. Let $c, c' > 0$, and $\mathcal{K}$ be a nonempty connected subset of γ the boundary of which does not intersect that of γ. Define then:

$$\Phi_{ad}^n = \{\varphi \in \mathcal{C}_0^n(\overline{\gamma}) / \ |\varphi^{(k)}| \leq c' \ , \ 0 \leq k \leq n \ , \ \varphi \geq c\chi_{\mathcal{K}}\}.$$

Theorem 1 ([13, Thm 1, Lem. 1]) *Let* $\phi \in L^2(K)$ *and for* $i = 1, 2$, *let* $\varphi_i \in \Phi_{ad}^0$. *Let* $u_i \in \mathcal{C}^0(\overline{D})$ *be the unique solution of* (PR) *associated to* $\varphi = \varphi_i$. *It holds that, if* $u_{1|_K} = u_{2|_K}$, *then* $\varphi_1 = \varphi_2$ *on* $\overline{\gamma}$.

The following regularity result holds:

Theorem 2 *If* $\phi \in W_0^{n,2}(K)$ *and* $\varphi \in \Phi_{ad}^n$, *then the solution* u_φ *to* (PR) *belongs to* $W^{n+\frac{3}{2},2}(D) \subset \mathcal{C}^{n,\frac{1}{2}}(\overline{D})$.

3 Identification by analytic approximation

Up to conformal mapping, problem (PR) can be expressed in the unit disk $\mathbb{D}$, in order to work with Hardy classes in one of their classical framework. Indeed, whenever D possesses a $C^{n,\beta}$ boundary for some $\beta \in (0,1)$, the Kellogg-Warschawski theorem [23, thms 3.5, 3.6] implies that there exists a conformal mapping from the unit disk $\mathbb{D}$ into D having a C^n extension to $\overline{\mathbb{D}}$. *We thus assume that* $D = \mathbb{D}$ *and* $T = \mathbb{T} = \overline{\gamma} \cup \overline{K}$. The inverse problem (PR) we are concerned with now takes place in the unit disk where it can be constructively solved using interpolation / approximation results in the Hardy space H^2.

3.1 From harmonic functions to Hardy classes

Back to problem (PR), we assume now that $\phi \in L^2(K)$ and $\varphi \in \Phi^0_{ad} \subset \mathcal{C}^0(\overline{\gamma})$. It then follows from theorem 2 that $u \in W^{1,2}(T) \subset \mathcal{C}^{0,1/2}(T)$. From the knowledge of the flux $\phi \in L^2(K)$ and of the temperature $u_m \in W^{1,2}(K) \subset \mathcal{C}^{0,1/2}(K)$ in system (PR), we can in principle build the trace on $K \subset \Gamma_N$ of a function analytic in D. This holds because u is harmonic in D and Cauchy-Riemann equations ensure that, if ω is a harmonic function in D satisfying

$$\frac{\partial \omega}{\partial \theta} = \frac{\partial u}{\partial n} \text{ on } T\,,$$

(ω is the harmonic conjugate of u), then $f = u + i\,\omega$ is an analytic function in D. Thus, if we note $\int \phi d\theta$ for some primitive of ϕ on K, then the function:

$$f = u_m + i \int \phi d\theta \text{ on } K,$$

is actually the trace on K of a (unique) function g analytic in D: $f = g_{|K}$. Moreover, smoothness preserving properties of the harmonic conjugation operator, namely Privalov's theorem or the Carleson-Jacobs one, implies that g also belongs to the Hölder class $C^{0,1/2}(T)$, see [2, 10, 18]. In particular, it belongs to the Hardy space H^2, see the definition in section 2.1. By the way, this comes directly from the fact that harmonic conjugation is bounded as an operator on $L^2(T)$, as asserted by M. Riesz theorem. That g actually belongs to $\mathcal{H}^{1,2}$ is a consequence of lemma 1 below.

Our aim is then to recover g in the whole of D, or at least on $\gamma = T \setminus \overline{K}$, from the knowledge of its trace f on K. Indeed, this function would solve for (PR) since

$$u = \operatorname{Re} g \text{ in } D,$$

and then

$$\varphi \;=\; -\frac{(\operatorname{Im} g)'}{\operatorname{Re} g} \text{ on } \gamma,$$

where the above equality should be properly understood (non tangential limits of the right hand side). Now, φ is the expected solution to (PR) on γ.

Moreover, we want the recovery procedure to be convergent in order to carry some stability and robustness properties. Our concern here is that, in practice, one may not know exactly u_m nor f on K: for example, pointwise values of u_m might only be available through experimental devices that necessarily induce noises and perturbations of the measurements. Also, the knowledge of f requires that of some "primitive of the flux", which is to be computed numerically. A more realistic problem is thus to approximately and robustly recover g in the whole of D from the knowledge of a perturbation f_ε of its trace f on $K \subset T$, where ε is a small parameter that stands for a (deterministic) measure of the perturbation. However, classical analytic interpolation or extrapolation results from data on part of the boundary (Carleman integrals, for example) do not possess any stability properties on their own and are not suitable to ensure robustness with respect to perturbations, as shows the next proposition from [7]. This is the reason why we need a constrained approximation framework. Now, despite these recovery / approximation questions should be approached in uniform norm (see e.g. [7, 8]), it is simpler to handle them in the Hardy space H^2 which possesses a Hilbertian structure. Also, for various reasons, robustness properties are easier to ensure there. We recall the following uniqueness and density results.

Proposition 1 ([17, 18, 24]) *Let K be an nonempty subset of $\mathbb{T}$ such that $\mu(K) > 0$ and let $g \in H^2$ verifying $g_{|_K} = 0$; then, $g \equiv 0$ on the whole unit disk $\mathbb{D}$.*

Proposition 2 ([7]) *Let $K \subset T$ such that K and $T \setminus K$ have positive Lebesgue measure.*
(i) $H^2_{|_K}$ is dense in $L^2(K)$.
(ii) Let $h \in L^2(K)$ and suppose (g_n) is a sequence of H^2 converging to h in $L^2(K)$. If h is not the trace of an H^2 function, then $\|g_n\|_{T\setminus K} \to \infty$.

In view of proposition 2, we see that as soon as a perturbation is involved in the measurements on K and prevents the data from being truly the trace of an H^2 function by loose of its analyticity property, any H^2 interpolating procedure from K *only* will exhibit a wild behavior outside K. A remedy for this unstable behavior is to state our recovery problem as a (best) constrained approximation issue which is a bounded generalization on subsets of T of classical (dual) extremal problems in H^2.

3.2 Best H^2 approximation and (BEP)

We now explain how to robustly recover an H^2 function on the whole $\overline{D}$ from approximate measures of its boundary values on K. Bounded extremal problems (BEP) in H^2 are as follows.

Given $h \in L^2(K)$ and $M > 0$,

$$\text{(BEP)} \left\{ \begin{array}{l} \text{find } g_0 = g_0(h, M) \in H^2, \; \|g_0\|_{\mathbb{T}\setminus K} \leq M\,, \\ \|h - g_0\|_K \;=\; \displaystyle\min_{\substack{g \in H^2 \\ \|g\|_{\mathbb{T}\setminus K} \leq M}} \|h - g\|_K\,. \end{array} \right.$$

Under the norm constraint, this problem becomes well-posed. Existence and uniqueness of solutions to (BEP) are established in [3, 5] as well as a constructive formula to compute g_0. Denote by $\mathcal{T}$ the Toeplitz operator with symbol $\chi_{\mathbb{T}\setminus K}$

$$\begin{array}{rccl} \mathcal{T} : & H^2 & \to & H^2 \\ & g & \mapsto & P_{H^2}\, \chi_{\mathbb{T}\setminus K}\, g\,. \end{array}$$

The operator $\mathcal{T}$ is a semi-infinite Toeplitz matrix: $\mathcal{T}_{k,l} = \mathcal{T}_{k-l}\,,\ k, l \geq 0$. Whenever $\mathbb{T}\setminus K$ coincides with the arc $(e^{-i\theta_0}, e^{i\theta_0})$, it holds that

$$\mathcal{T}_{k,l} = \frac{\sin(k-l)\theta_0}{\pi(k-l)}\,,\ \text{for}\ k \neq l\,,\ \mathcal{T}_{l,l} = \frac{\theta_0}{\pi}\,.$$

Hence,

$$g_0 = g_0(\lambda) = (I + \lambda\,\mathcal{T})^{-1}\, P_{H^2}\, \chi_K\, h\,,$$

for the unique value of the (Lagrange type) parameter $\lambda > -1$ such that $\|g_0\|_{\mathbb{T}\setminus K} = M$, excepted if h already belongs to the approximant class ($h \in H^2_{|K}$, $\|h\|_{\mathbb{T}\setminus K} \leq M$) in which case $g_0 = h$ corresponds to $\lambda = -1$. The behavior with respect to λ of the error $e(\lambda)$ and of the constraint $M(\lambda)$ defined by:

$$e(\lambda) = \|h - g_0(\lambda)\|^2_K\ , \quad M(\lambda) = \|g_0(\lambda)\|_{\mathbb{T}\setminus K}\,,$$

is smooth and monotonous. In particular, as $\lambda \searrow -1$,

$$e(\lambda) \searrow 0 \text{ and } \left\{ \begin{array}{ll} M(\lambda) \nearrow \infty\,, & \text{if } h \notin H^2_{|K}\,, \\ M(\lambda) \nearrow \|h\|_{\mathbb{T}\setminus K}\,, & \text{if } h \in H^2_{|K}\,. \end{array} \right.$$

Although these formula remain implicit (the parameter λ being involved in place of the norm constraint M), they give rise to a convergent algorithm that allow to build $g_0(h, M)$.

Whenever $h \in H^2_{|K}$, the above approximation process on K gives rise to interpolation on the whole T. In this case, the "true" constraint $M_0 = \|h\|_{T\setminus K}$ can be numerically computed (see [14]). Continuity properties of solutions to bounded extremal problems with respect to the data h and M are also available from [12]. For $h \in H^2_{|K}$, we still denote by h its unique H^2 extension to the whole unit disk $\mathbb{D}$ (see proposition 1). Let:

$$\mathcal{D}_- = \{(h, M) \in H^2_{|K} \times \mathbb{R}^*_+ \ /\ \|h\|_{T\setminus K} < M\}.$$

Let $\mathcal{E}$ the approximation or extension operator, which maps $h \in L^2(K)$ and $M \in \mathbb{R}^*_+$ onto the unique associated solution $g_0(h, M) = g_0$ to problem (BEP):

$$\begin{array}{rccc} \mathcal{E}: & L^2(K) \times \mathbb{R}^*_+ & \longrightarrow & H^2 \\ & (h, M) & \longmapsto & g_0. \end{array}$$

Theorem 3 *The map $\mathcal{E}$ is continuous on $\left(L^2(K) \times \mathbb{R}^*_+\right) \setminus \mathcal{D}_-$, but only weakly continuous on $L^2(K) \times \mathbb{R}^*_+$.*

In order to get actual convergence properties of our reconstruction scheme, we need a stronger continuity result. Let $\mathcal{E}_n$ be the map

$$\mathcal{E}_n : W^{n,2}(K) \times \mathbb{R}^*_+ \longrightarrow \mathcal{H}^{n,2}$$

defined by

$$[\mathcal{E}_n(h, M)]^{(n)} = \mathcal{E}(h^{(n)}, M)\,,$$

and $\mathcal{E}^{(k)}_n(h, M) = h^{(k)}(x_0)$, $0 \leq k \leq n-1$, for some fixed $x_0 \in K$ (all derivatives are taken with respect to the arclength θ on T). Note that $\mathcal{E}_0 = \mathcal{E}$. The function $\mathcal{E}_n(h, M) \in \mathcal{H}^{n,2}$ is actually strongly linked to the solution g_n of the following $(\text{BEP})_n$ problem in Hardy-Sobolev classes [4]:

given $h \in W^{n,2}(K)$ and $M > 0$,

$$(\text{BEP})_n \left\{ \begin{array}{l} \text{find } g_n \in \mathcal{H}^{n,2},\ \|g_n\|_{n,T\setminus K} \leq M\,, \\ \|h - g_n\|_{n,K} = \min\limits_{\substack{g \in \mathcal{H}^{n,2} \\ \|g\|_{n,T\setminus K} \leq M}} \|h - g\|_{n,K}. \end{array} \right.$$

Related to $(\text{BEP})_n$, the following continuity result holds as a corollary from theorem 3.

Theorem 4 *The map $\mathcal{E}_n$ is weakly continuous $W^{n,2}(K) \times \mathbb{R}^*_+ \longrightarrow \mathcal{H}^{n,2}$, for $n \geq 0$; however, it is strongly continuous as a $W^{n,2}(K) \times \mathbb{R}^*_+ \longrightarrow \mathcal{H}^{n-1,2}$ map, for $n \geq 1$. Its restriction to K is continuous $W^{n,2}(K) \times \mathbb{R}^*_+ \longrightarrow W^{n,2}(K)$.*

Let $\phi \in W^{n,2}_0(K)$, $(\varphi_p)_{p\in\mathbb{N}}$ be a sequence of Φ^n_{ad}, and $u_{m,p} = u_{p|K} \in W^{n+1,2}(K)$ the associated measurement data in (PR). We claim that $f_p = u_{m,p} + i \int \phi \in W^{n+1,2}(K)$. Indeed:

Lemma 1 *If $\phi \in W^{n,2}_0(K)$ and $\varphi \in \Phi^n_{ad}$, then the associated solution to* (PR) *is given by $u = \operatorname{Re} g$ for an analytic function $g \in \mathcal{H}^{n+1,2}$. In particular, $g_{|K} \in W^{n+1,2}(K)$.*

3.3 Convergence in Hardy-Sobolev classes

The proof of the stability theorem 6 strongly relies on the following estimate from [9, lem. 4.2].

Theorem 5 ([9]) *Assume that $\mu(K) = 2\pi\rho$, $0 < \rho < 1$. Any $g \in \mathcal{H}^{1,2}$ such that $\|g\|_1 \leq 1$ and $\|g\|_K \leq e^{-1/\rho}$ also satisfies*

$$\|g\| \leq \varepsilon(\|g\|_K)\,,$$

for some function $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ that goes to 0 at 0:

$$\varepsilon(x) \leq C\, \frac{1 + \log\left(\rho\, \log 1/x\right)}{\rho\, \log 1/x}\,,$$

for some $C > 0$.

Note that the above conclusion is false in the general case where $g \in H^2$ only, since one may exhibit a sequence of H^2 functions (g_n) such that $\|g_n\|_K \to 0$ while $\|g_n\|_{T\setminus K} = 1$. Also, if we assume that $\|g_n\| \to 0$ on the whole T, this can generally happen arbitrarily slowly on $T \setminus K$ even if the decay rate is prescribed on K. Observe also that the above estimate concerning ε may not be sharp. In particular, it is likely that $\varepsilon(x)$ goes to zero faster than $|\log x|^{-1}$.

Corollary 1 *For $n \geq 0$, any $g \in \mathcal{H}^{n+1,2}$ such that $\|g\|_{n+1} \leq 1$ and $\|g\|_{n,K} \leq e^{-1/\rho}$ for $0 < \rho = \mu(K)/2\pi < 1$, also satisfies*

$$\|g\|_n \leq \sqrt{n+1}\, \varepsilon(\|g\|_{n,K}) \, .$$

4 Stability properties of (PR)

We are now in position to establish the following stability property of (PR), whose proof strongly uses Hardy spaces and functional analysis tools from section 3.

Theorem 6 *Assume T to be $\mathcal{C}^{n+1,\beta}$ smooth, $\beta \in (0,1)$, $n \geq 0$, and let $K \subset \partial D$ with $0 < \rho = \mu(K)/2\pi < 1$. Let $\phi \in W_0^{n,2}(K)$, $\varphi_1, \varphi_2 \in \Phi^n_{ad}$, and u_1, u_2 be the associated solutions to problem* (PR)*. Then, $u_1, u_2 \in W^{n+1,2}(T)$ and for $n \geq 1$, if $\|u_1 - u_2\|_{n,K} \leq e^{-1/\rho}$,*

$$\|\varphi_1 - \varphi_2\|_{n-1,\gamma} \leq \varepsilon_n(\|u_1 - u_2\|_{n,K}) \, ,$$

for some function $\varepsilon_n : \mathbb{R}_+ \to \mathbb{R}_+$ that goes to 0 at 0:

$$\varepsilon_n(x) \leq C \sqrt{n+1}\, \frac{1 + \log(\rho \log 1/x)}{\rho \log 1/x} \, ,$$

for some $C > 0$ and $x \leq e^{-1/\rho}$. Also (for $n = 0$), it holds that $\varphi_1 - \varphi_2$ weakly converges to 0 in $L^2(\gamma)$ as $\|u_1 - u_2\|_K \to 0$.

As a consequence, we get the following uniform convergence result.

Corollary 2 *Under the assumptions and with the notations of theorem 6, it holds that, for $n \geq 2$,*

$$\|\varphi_1 - \varphi_2\|_{W^{n-2,\infty}(\gamma)} \leq \varepsilon_n(\|u_1 - u_2\|_{n,K}) \, .$$

5 Computation algorithms

In view of the results of sections 3 and 4 , we are able to provide a constructive scheme to solve for (PR). Under some smoothness assumptions, this procedure is convergent and robust with respect to measurement noises. This furnishes an original and efficient method that permits to determine the Robin coefficient. Assume that we are given data $\phi \not\equiv 0$ and f on $K \subset T$. When $D = \mathbb{D}$, the successive steps of a reconstruction algorithm for φ on γ are the following. If $D \neq \mathbb{D}$, a conformal transformation is required as preliminary and final steps.

1. Solve (BEP) with $h = f$ on K and a suitable constraint $M > 0$ on $T \setminus K$. This gives $g = g_0(h, M)$ on $\overline{D}$ (in terms of a number of its Fourier coefficients, for example).

2. Compute

$$\varphi = -\frac{(\operatorname{Im} g)'}{\operatorname{Re} g} \quad \text{on } \gamma \, .$$

This is the expected solution to (PR).However, the data u_m and ϕ are actually available from either numerical or experimental measurements. This is to the effect that the associated analytic function f is known on K up to some perturbation only, say as a function f_ϵ that will probably not keep carrying analytic properties. An efficient algorithm would thus allow to recover from f_ϵ a function φ_ϵ close to the solution φ to (PR). But the continuity results of theorem 3 make it unlikely to get such an accuracy by the above "0 order" approach, since the convergence of (BEP) solutions only holds is a weak sense, and since moreover no convergence on the derivatives is available. In order to improve these convergence results, it seems therefore natural to seek directly for an approximation of the nth derivative - instead of the function - by solving an appropriate $(\text{BEP})_n$ problem, and to reconstruct afterwards the function itself from its computed derivatives. Provided that the domain and the data are smooth enough, this is ensured by theorem 4 when the above step 1 is replaced by 1_n:

1_n. Solve $(\text{BEP})_n$ with $h = f$ on K and a suitable constraint $M > 0$ on $T \setminus K$. This gives $g = g_n(h, M)$ on $\overline{D}$.

These convergence results are properly given below.

Proposition 3 *Assume T to be $\mathcal{C}^{n+1,\beta}$ smooth, $\beta \in (0,1)$, $n \geq 0$. For data $u_m \in W^{n+1,2}(K)$, $\phi \in W_0^{n,2}(K)$ and whenever $\varphi \in \Phi^n_{ad}$, problem* (PR) *is robustly solved by the steps 1_{n+1} and 2 of the above algorithm.*

More precisely, assume that we are given noisy data $u_m + \delta_u$ and $\phi + \delta_\phi$ with $\delta_u, \delta_\phi \in L^2(K)$, $\|\delta_u\|_K$, $\|\delta_\phi\|_K \leq \delta$ for some $\delta > 0$. Steps 1_{n+1}, 2 above can be used to compute a function φ_δ that approaches the solution φ to (PR) *as follows. For $n \geq 1$,*

$$\|\varphi_\delta - \varphi\|_{n-1,\gamma} \to 0 \quad \textit{when} \quad \delta \to 0 \, .$$

For $n = 0$, it holds that $\varphi_\delta - \varphi$ converges weakly to 0 in $L^2(\gamma)$ as $\delta \to 0$.

Remark 1 Consider the particular case of smooth perturbations $\delta_u \in W^{n+1,2}(K)$, $\delta_\phi \in W^{n,2}(K)$ verifying $\|\delta_u\|_{n,K}$, $\|\delta_\phi\|_{n-1,K} \leq \delta$. Although not completely realistic (if data obtained from numerical computations should be smooth enough, this will not be the case in general for experimental measurements), it illustrates the fact that the convergence properties of the above algorithm are strictly in accordance with the stability properties of (PR), see theorem 4.

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