Output-Feedback Control of Stochastic Strict-Feedback Systems under an Exponential Cost Criterion [1]

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Abstract

We consider a class of single-input single-output stochastic nonlinear systems in strict-feedback form with a risk-sensitive cost criterion and with only the output of the system being available for feedback. We design an output-feedback controller under which the closed-loop signals maintain an arbitrarily small average risk-sensitive cost. Moreover, all closed-loop signals remain bounded in probability, and under certain conditions we obtain asymptotic stability in probability.

Keywords: Stochastic adaptive control, strict-feedback systems, risk-sensitive identification.

1 Introduction

A recent research topic has been the design of feedback control laws to achieve stabilization or tracking for nonlinear systems where the additive uncertainties are random. In the literature, the most common mathematical model used for such systems consists of a set of stochastic differential equations interpreted in the Itô sense; see [7]. The design techniques available for robust control of nonlinear deterministic systems cannot necessarily be used for stochastic systems, mainly because of the presence of the the extra quadratic variation terms resulting from the Itô differentiation rule (see [6]). Moreover, different notions of stability and performance indices need to be used to determine the usefulness of the feedback controllers in a stochastic setup. The most natural stochastic counterparts of the 'deterministic' concepts such as boundedness and (asymptotic) stability can be found in [3], whereas a more recent concept "noise-to-state stability", where the word "noise" refers to the intensity of the additive random noise, can be found in the recent book [4], which has several chapters on stabilization of stochastic nonlinear systems. Again, in the context of noise-to-state stability, the papers [1] and [2] have developed adaptive controllers, equipped with state and output information respectively, for stochastic strict feedback systems, where the adaptive nature of the controllers is related to the unknown intensity of the additive random disturbances. A concept more relevant to our work, however, is the "risk-sensitive cost criterion" in which not only the mean value but also the variance of an integral cost is penalized; see [11] and [8] in the linear context. In the nonlinear context, we cite [9] as the key reference that has presented a control design with full state measurements that results in closed-loop signals achieving an arbitrarily small average risk-sensitive cost. In this paper, we extend the results of [9] to the case where only output is available for feedback. By employing the backstepping design technique on the output and the estimates of the unmeasured states, we develop an output feedback controller that guarantees an arbitrarily small average risk-sensitive cost. Furthermore, the controller achieves boundedness of the closed-loop signals (in probability), and asymptotic regulation of the closed-loop signals (again in probability) under certain conditions, which are delineated in the paper.

Finally, we should stress that since we deal with a stochastic **nonlinear** output-feedback control problem (under an exponential cost criterion), there is no separation result that would apply here. Therefore, an optimum risk-sensitive controller where the cost function contains a priori fixed weights on the states and the control action does not seem to be within reach. In fact, such an optimum controller could be infinite-dimensional. Our approach leads to a suboptimal yet computable finite-dimensional output-feedback controller. In our risk-sensitive cost, there is almost total freedom in choosing the weights on the state variables, and the level of the average risk-sensitive cost that can be achieved. We do not include, however, a cost on the control action, as is the case in many backstepping designs, and as a result the controller could be a high-gain one especially when the state variables are heavily penalized and a small average risk-sensitive cost is aimed. Therefore, the designer should choose the parameters, such as the design functions and the average risk-sensitive cost, judiciously to trade off between a good transient response of the states and the control action.

In the sequel, we first introduce some notation that is used throughout the paper; see Section 2. Then, in Section 3, we define our objectives in more precise terms. Following this, we introduce, in Section 4, the estimator used in the

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paper. In Section 5, we present the controller design steps with output measurements, which use the estimator of Section 4. In Section 6, some simulation results are included to demonstrate the effectiveness of the proposed control design. Finally, the paper ends with the concluding remarks of Section 7.

2 Notation and Terminology

- For any positive integer m, $\mathcal{R}^m$ denotes the m dimensional Euclidean space.
- $|.|$ denotes the standard Euclidean norm.
- $x_{[i]} := [x_1, ..., x_i]^T$, where x_j is a scalar $\forall j \in \{1, ..., i\}$, and the notational artifact $x_{[0]}$ is ignored.
- $Tr[.]$ denotes the trace of a square matrix.
- $0_{m\times n}$ denotes a matrix of dimension $m \times n$, with all entries zero.
- I_m denotes the m-dimensional identity matrix.
- A continuous function $f(x) : [0,\infty) \mapsto [0,\infty)$ is said to be class $\mathcal{K}$ if f is increasing and $f(0) = 0$.
- A function $f(x) : \mathcal{R}^n \mapsto [0,\infty)$ is said to be positive definite if $f(0_{n\times 1}) = 0$ and $x \neq 0_{n\times 1} \Rightarrow f(x) > 0$.

3 Problem Formulation

We start with a strict-feedback nonlinear system described by the following Itô differential equations:

$$\begin{aligned} dx_i &= [x_{i+1} + f_i(y)]dt + h_i^T(y)dw, \; i = 1, ..., n-1 \\ dx_n &= [b(y)u + f_n(y)]dt + h_n^T(y)dw \qquad (1) \\ y &= x_1 \end{aligned}$$

where $x := [x_1, ..., x_n]^T$ is the n-dimensional state vector, u is the scalar control input, y is the scalar output, w is the $\mathcal{R}^q$-valued standard Wiener process, and the initial condition $x(0)$ is fixed. The nonlinear functions f_i, h_i, b, and $1/b$ are known, and sufficiently smooth. Our goal is to design an output-feedback controller such that the closed-loop signals are bounded in probability, and the following risk-sensitive performance inequality holds:

$$\begin{aligned} &\frac{2}{\theta} \ln E \exp\left[\frac{\theta}{2}\int_0^T [y^2 + l(t,x)]dt\right] \\ &\quad \leq \gamma(x(0), x(0) - \bar{x}_0) + RT, \quad \forall T \geq 0 \qquad (2) \end{aligned}$$

where $\theta > 0$ is an arbitrary risk-sensitivity parameter, $R > 0$ is an arbitrary constant representing the average risk-sensitive cost, $l(t,x) \geq 0$ is some weight function, $\gamma(.,.)$ is some positive definite function, and $\bar{x}_0$ is an initial estimate for $x(0)$. We further require that when $h_i(0) = 0_{q\times 1}$, $\forall i \in \{1, .., n\}$, R could be picked as zero and the closed-loop signals asymptotically approach origin in probability. Although we will not be able to pick the weight function $l(t,x)$ arbitrarily, the design procedure will give us a great deal of flexibility in shaping this function. We finally make the following assumptions, which are required to hold throughout the paper.

Assumption 1. $f_i(0) = 0$, $\forall i \in \{1, ..., n\}$.

Assumption 2. $D(y)D(y)^T \leq M_D I_n$ (in the matrix sense) for some $M_D > 0$, $\forall y \in \mathcal{R}$, where $D(y) := [h_1(y), ..., h_n(y)]^T$.

The first assumption is made to ensure that the origin is the equilibrium point of the system in the absence of random noise, whereas the second technical assumption is necessary to bound the variance of the filtering error by appropriate terms, which will be clear later.

4 The Estimator

Since $[x_2, ..., x_n]^T$ is not measured, it has to be estimated using the available on-line information. For this purpose, we rewrite the plant dynamics (1) as:

$$\begin{aligned} dx &= [Ax + F(y) + B(y)u]dt + Ddw \\ y &= C^T x \end{aligned}$$

where

$$\begin{aligned} A &= \begin{bmatrix} 0_{n-1\times 1} & I_{n-1} \\ 0_{1\times 1} & 0_{1\times n-1} \end{bmatrix}, \; F = \begin{bmatrix} f_1(y) \\ \vdots \\ f_n(y) \end{bmatrix}, \\ B &= \begin{bmatrix} 0_{n-1\times 1} \\ b(y) \end{bmatrix}, \; C = [1 \; 0_{1\times n-1}]^T. \end{aligned}$$

To estimate the unmeasured states, we introduce the following estimator, which also generates for convenience an estimate for the measured state x_1:

$$\begin{aligned} d\hat{x} &= [A\hat{x} + F(y) + B(y)u - k(y - C^T\hat{x})]dt, \qquad (3) \\ &\quad \hat{x}(0) = \bar{x}_0. \end{aligned}$$

We first note that the gain vector $k := [k_1, ..., k_n]^T$ can be picked in such a way that there exists a positive definite matrix P that satisfies the following generalized algebraic Riccati inequality:

$$PA_0 + A_0^T P + \frac{\theta}{2}(n+1)M_D PP \leq -(2+n)I_n,$$

where $A_0 := A + kC^T$. Let k be such a vector. Then, the estimation error $\tilde{x}$, defined as $\tilde{x} := x - \hat{x}$, satisfies:

$$d\tilde{x} = A_0\tilde{x}dt + Ddw.$$

For future reference, we introduce $W := \bar{\Gamma}\tilde{x}^T P\tilde{x}/2$, where we pick $\bar{\Gamma}$ as

$$\bar{\Gamma} := \min\left\{1, \frac{R/(2n)}{Tr[PD(0)D^T(0)]}\right\}, \text{ if } Tr[PD(0)D^T(0)] > 0,$$

and as $\bar{\Gamma} := 1$ otherwise. The Itô differential of W satisfies:

$$\begin{aligned} dW &= \frac{\bar{\Gamma}}{2}\left[\tilde{x}^T(PA_0 + A_0^T P)\tilde{x}dt + 2\tilde{x}^T PDdw + Tr[PDD^T]dt\right] \\ &\leq -\bar{\Gamma}(1+n/2)|\tilde{x}|^2 dt + \bar{\Gamma}\tilde{x}^T PDdw \\ &\quad -\frac{\bar{\Gamma}}{4}(n+1)\theta M_D \tilde{x}^T PP\tilde{x}dt + (R/2n)dt + x_1^2\bar{\Gamma}\psi_{hv}dt \end{aligned}$$

where the positive valued function $\psi_{hv}(x_1)$ is such that

$$Tr[PD(x_1)D(x_1)^T] \leq 2Tr[PD(0)D^T(0)] + 2x_1^2\psi_{hv}(x_1).$$

We note that the estimator (3) is not the only choice to achieve our objective (2). In fact, to achieve the same objective (2), we can use a state estimator which itself is risk-sensitive optimal (but more complex than (3)) or a reduced-order estimator whose order is one less than that of (3). Nonetheless, we will prefer the full-order estimator (3) because of its simplicity, and construct, in the next section, an output-feedback controller which will use the state estimates generated by (3).

5 Controller Design

In this section, we present design steps of an output-feedback controller that achieves an arbitrarily small positive average risk-sensitive cost for plant (1), by employing the backstepping design methodology.

Step 1: We start by defining $z_1 := x_1$, whose Itô differential is

$$dz_1 = (x_2 + \bar{f}_1)dt + \bar{h}_1^T dw \tag{4}$$

where the functions with an over-bar symbol denote the equivalents of those without an over-bar symbol in terms of the new variable z_1. This notation will be used throughout to denote the equivalent forms of the functions in terms of $z_1, \ldots, z_n$. We write (4) as:

$$dz_1 = (\hat{x}_2 + r_1 + p_1^T\tilde{x})dt + s_1^T dw$$

where

$$r_1(z_1) := \bar{f}_1(z_1), \;\; p_1 := [0,1,0,..,0]^T, \;\; s_1(z_1) := \bar{h}_1(z_1).$$

We next introduce $V_1 := \Gamma_1 z_1^2/2$, where we pick Γ_1 as $\Gamma_1 = R/(2n|s_1(0)|^2)$, if $|s_1(0)|^2 > 0$, and as $\Gamma_1 = 1$ otherwise. The Itô differential of V_1 satisfies:

$$\begin{aligned} dV_1 &= \Gamma_1 z_1[(\hat{x}_2 + r_1 + p_1^T\tilde{x})dt + s_1^T dw] + \frac{\Gamma_1|s_1|^2}{2}dt \\ &\leq \Gamma_1 z_1[\hat{x}_2 + r_1 + \Gamma_1 z_1|p_1|^2/(2\bar{\Gamma}) + z_1\psi_{1v}]dt \\ &\quad +\Gamma_1 z_1 s_1^T dw + (R/(2n))dt + (\bar{\Gamma}|\tilde{x}|^2/2)dt \end{aligned} \tag{5}$$

where the inequality in (5) follows from the fact that the function $|s_1|^2$ can be bounded as: $|s_1|^2 \leq 2|s_1(0)|^2 + 2z_1^2\psi_{1v}$ for some smooth function $\psi_{1v}(z_1) \geq 0$. We now define the error term z_2 between $\hat{x}_2$ and its desired value, $-\alpha_1(z_1)$, as: $z_2 := \hat{x}_2 + \alpha_1(z_1)$, where

$$\begin{aligned} \alpha_1 \;&:=\; r_1 + \frac{\Gamma_1 z_1|p_1|^2}{2\Gamma} + z_1\psi_{1v} + \frac{z_1\bar{\Gamma}\bar{\psi}_{hv}}{\Gamma_1} + (n-1)z_1/\Gamma_1 \\ &\quad +(n+1)\theta\Gamma_1 z_1|s_1|^2/4 + (\beta_1+1)z_1/\Gamma_1, \text{ and} \end{aligned}$$

$\beta_1(z_1) \geq 0$ is some design function. With this, we can write (5) in terms of z_2 as:

$$\begin{aligned} dV_1 \;&\leq\; -z_1^2(1+\bar{\Gamma}\bar{\psi}_{hv})dt + \Gamma_1 z_1[(z_2 - (n-1)z_1/\Gamma_1 \\ &\quad -\beta_1 z_1/\Gamma_1 - (n+1)\theta\Gamma_1 z_1|\sigma_1|^2/4)dt + \sigma_1^T dw] \\ &\quad +(R/(2n))dt + (\bar{\Gamma}|\tilde{x}|^2/2)dt \end{aligned}$$

where $\sigma_1(z_1) := s_1(z_1)$, which completes the first step.

Step 2: The Itô differential of z_2 is

$$\begin{aligned} dz_2 \;&=\; (\hat{x}_3 + \bar{f}_2 - k_2C^T\tilde{x})dt + \frac{\partial\alpha_1}{\partial z_1}[(z_2 - \alpha_1 + r_1 \\ &\quad +p_1^T\tilde{x})dt + s_1^T dw] + \frac{1}{2}\frac{\partial^2\alpha_1}{\partial z_1^2}|s_1|^2 dt \end{aligned}$$

which could be written as:

$$dz_2 = (\hat{x}_3 + r_2 + p_2^T\tilde{x})dt + s_2^T dw \tag{6}$$

where

$$\begin{aligned} r_2(z_{[2]}) \;&:=\; \bar{f}_2 + \frac{\partial\alpha_1}{\partial z_1}[z_2 - \alpha_1 + r_1] + \frac{1}{2}\frac{\partial^2\alpha_1}{\partial z_1^2}|s_1|^2, \\ p_2(z_1) \;&:=\; -k_2C + \frac{\partial\alpha_1}{\partial z_1}p_1, \quad s_2(z_1) := \frac{\partial\alpha_1}{\partial z_1}s_1. \end{aligned}$$

Since $s_2(z_1)$ is a smooth function, we can find a smooth function $\psi_{2v}(z_1) \geq 0$ such that $|s_2|^2 \leq 2|s_2(0)|^2 + 2z_1^2\psi_{2v}$. Let us now define the positive definite function $V_2 := V_1 + \Xi_2 z_2^2/2$ where $\Xi_2(z_1) := \Gamma_2/(1+\Gamma_2\psi_{2v}(z_1))$, and $\Gamma_2 > 0$ is picked as $\Gamma_2 = R/(2n|s_2(0)|^2)$, if $|s_2(0)|^2 > 0$, and as $\Gamma_2 = 1$ otherwise. The Itô differential of V_2 satisfies:

$$\begin{aligned} dV_2 \;&=\; dV_1 + \Xi_2 z_2[(\hat{x}_3 + r_2 + p_2^T\tilde{x})dt + s_2^T dw] \\ &\quad +\frac{z_2^2}{2}\frac{\partial\Xi_2}{\partial z_1}[(z_2 - \alpha_1 + r_1 + p_1^T\tilde{x})dt + s_1^T dw] \qquad (7) \\ &\quad +\frac{\Xi_2|s_2|^2}{2}dt + \frac{z_2^2}{4}\frac{\partial^2\Xi_2}{\partial z_1^2}|s_1|^2 dt + z_2\frac{\partial\Xi_2}{\partial z_1}s_1^T s_2 dt \\ &\leq\; dV_1 + \Xi_2 z_2[(\hat{x}_3 + m_2)dt + \sigma_2^T dw] \\ &\quad +(\bar{\Gamma}|\tilde{x}|^2/2)dt + z_1^2 dt + (R/(2n))dt \qquad (8) \end{aligned}$$

where

$$\begin{aligned} m_2(z_{[2]}) &:= r_2 + \frac{z_2}{2\Xi_2}\frac{\partial \Xi_2}{\partial z_1}(z_2-\alpha_1+r_1) \\ &+\frac{z_2}{4\Xi_2}\frac{\partial^2\Xi_2}{\partial z_1^2}|s_1|^2+\frac{1}{\Xi_2}\frac{\partial \Xi_2}{\partial z_1}s_1^T s_2 \\ &+\frac{z_2}{2\Xi_2\Gamma}\left|\Xi_2 p_2+\frac{z_2}{2}\frac{\partial \Xi_2}{\partial z_1}p_1\right|^2 \\ \sigma_2(z_{[2]}) &:= s_2+\frac{z_2}{2\Xi_2}\frac{\partial \Xi_2}{\partial z_1}s_1. \end{aligned}$$

We define the error term z_3 between $\hat{x}_3$ and its desired value, $-\alpha_2(z_{[2]})$, as: $z_3 := \hat{x}_3+\alpha_2(z_{[2]})$, where

$$\begin{aligned} \alpha_2 &:= m_2-m_{20}+(n+1)\theta\Xi_2 z_2|\sigma_2|^2/4 \\ &+(\beta_2 z_2+(n-2)z_2+\frac{\Xi_2^2 m_{20}^2 z_2}{2R/n}+\Gamma_1 z_1)/\Xi_2, \end{aligned}$$

$m_{20} := m_2(0_{2\times 1})$, and $\beta_2(z_{[2]}) \geq 0$ is some design function. We should mention that the term m_{20} on the right-hand-side of the definition of α_2 above is included to cancel out the bias, so as to make the origin an equilibrium point of the closed-loop system in the absence of random noise. With this, we can write (8) in terms of z_3 as:

$$\begin{aligned} dV_2 &\leq -z_1^2(1+\bar{\Gamma}\bar{\psi}_{hv})dt+\Xi_2 z_2 z_3 dt+\sum_{i=1}^{2}[(-\beta_i z_i^2 \\ &-(n+1)\Xi_i^2\theta z_i^2|\sigma_i|^2/4)dt+\Xi_i z_i\sigma_i^T dw] \\ &+2\bar{\Gamma}(|\tilde{x}|^2/2)dt-(n-2)|z_{[2]}|^2 dt \\ &+(3R/(2n))dt \end{aligned}$$

where $\Xi_1 := \Gamma_1$, which completes the second step.

Step k ($k=3,\ldots,n-1$): Assume the following structure from the previous step:

$$\begin{aligned} z_1 &= x_1, \text{ and } z_i=\hat{x}_i+\alpha_{i-1}(z_{[i-1]}) \text{ such that} \\ &\quad \alpha_{i-1}(z_{[i-1]}=0_{i-1\times 1})=0, \quad i=2,\ldots,k \\ dz_i &= (z_{i+1}-\alpha_i+r_i+p_i^T\tilde{x})dt+s_i^T dw, \\ &\quad i=1,\ldots,k-1 \\ V_{k-1} &= \sum_{i=1}^{k-1}\Xi_i z_i^2/2 \text{ such that } \Xi_i(z_{[i-1]})>0 \text{ is a} \\ &\quad \text{smooth function } i=1,\ldots,k-1 \\ dV_{k-1} &= [-z_1^2(1+\bar{\Gamma}\bar{\psi}_{hv})+\Xi_{k-1}z_{k-1}z_k]dt \\ &+\sum_{i=1}^{k-1}\left[-((n+1)\theta\Xi_i^2|\sigma_i|^2/4+\beta_i)z_i^2 dt\right. \\ &\left.+z_i\Xi_i\sigma_i^T dw\right]+(k-1)(\bar{\Gamma}|\tilde{x}|^2/2)dt \\ &-(n-k+1)|z_{[k-1]}|^2 dt+(R(k-3/2)/n)dt. \end{aligned}$$

From this, we obtain the Itô differential of z_k as:

$$\begin{aligned} dz_k &= (\hat{x}_{k+1}+\bar{f}_k-k_k C^T\tilde{x})dt+\sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial z_i}[(z_{i+1} \\ &-\alpha_i+r_i+p_i^T\tilde{x})dt+s_i^T dw] \\ &+\frac{1}{2}\sum_{i,j\in\{1,\ldots,k-1\}}\frac{\partial^2\alpha_{k-1}}{\partial z_i\partial z_j}s_i^T s_j dt \end{aligned}$$

which can be written as:

$$dz_k = (\hat{x}_{k+1}+r_k+p_k^T\tilde{x})dt+s_k^T dw \tag{9}$$

where

$$\begin{aligned} r_k(z_{[k]}) &:= \bar{f}_k+\sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial z_i}[z_{i+1}-\alpha_i+r_i] \\ &+\frac{1}{2}\sum_{i,j\in\{1,\ldots,k-1\}}\frac{\partial^2\alpha_{k-1}}{\partial z_i\partial z_j}s_i^T s_j \\ p_k(z_{[k-1]}) &:= -k_k C+\sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial z_i}p_i, \\ s_k(z_{[k-1]}) &:= \sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial z_i}s_i. \end{aligned}$$

Since $s_k(z_{[k-1]})$ is a smooth function, it can be bounded as $|s_k|^2 \leq 2|s_k(0)|^2+2|z_{[k-1]}|^2\psi_{kv}$, where $\psi_{kv}(z_{[k-1]}) \geq 0$ is also a smooth function. Let us now define the positive definite function $V_k := V_{k-1}+\Xi_k z_k^2/2$ where $\Xi_k(z_{[k-1]}) := \Gamma_k/(1+\Gamma_k\psi_{kv}(z_{[k-1]}))$, and $\Gamma_k > 0$ is picked as $\Gamma_k = R/(2n|s_k(0)|^2)$, if $|s_k(0)|^2>0$, and as $\Gamma_k=1$ otherwise. The Itô differential of V_k satisfies:

$$\begin{aligned} dV_k &= dV_{k-1}+\Xi_k z_k[(\hat{x}_{k+1}+r_k+p_k^T\tilde{x})dt+s_k^T dw] \\ &+\frac{z_k^2}{2}\sum_{i=1}^{k-1}\frac{\partial\Xi_k}{\partial z_i}[(z_{i+1}-\alpha_i+r_i+p_i^T\tilde{x})dt \\ &+s_i^T dw]+\frac{\Xi_k|s_k|^2}{2}dt+z_k\sum_{i=1}^{k-1}\frac{\partial\Xi_k}{\partial z_i}s_i^T s_k dt \\ &+\frac{z_k^2}{4}\sum_{i,j\in\{1,\ldots,k-1\}}\frac{\partial^2\Xi_k}{\partial z_i\partial z_j}s_i^T s_j dt \\ &\leq dV_{k-1}+\Xi_k z_k[(\hat{x}_{k+1}+m_k)dt+\sigma_k^T dw] \\ &+(\bar{\Gamma}|\tilde{x}|^2/2)dt+|z_{[k-1]}|^2 dt+(R/(2n))dt \end{aligned} \tag{10}$$

where

$$\begin{aligned} m_k(z_{[k]}) &:= r_k+\frac{z_k}{2\Xi_k}\sum_{i=1}^{k-1}\frac{\partial\Xi_k}{\partial z_i}(z_{i+1}-\alpha_i+r_i) \\ &+\frac{z_k}{4\Xi_k}\sum_{i,j\in\{1,\ldots,k-1\}}\frac{\partial^2\Xi_k}{\partial z_i\partial z_j}s_i^T s_j \\ &+\frac{1}{\Xi_k}\sum_{i=1}^{k-1}\frac{\partial\Xi_k}{\partial z_i}s_i^T s_k \\ &+\frac{z_k}{2\Xi_k\Gamma}\left|\Xi_k p_k+\frac{z_k}{2}\sum_{i=1}^{k-1}\frac{\partial\Xi_k}{\partial z_i}p_i\right|^2 \\ \sigma_k(z_{[k]}) &:= s_k+\frac{z_k}{2\Xi_k}\sum_{i=1}^{k-1}\frac{\partial\Xi_k}{\partial z_i}s_i. \end{aligned}$$

We define the error term z_{k+1} between $\hat{x}_{k+1}$ and its desired value, $-\alpha_k(z_{[k]})$, as: $z_{k+1} := \hat{x}_{k+1} + \alpha_k(z_{[k]})$, where

$$\begin{aligned}\alpha_k \quad := \quad & m_k - m_{k0} + (n+1)\theta\Xi_k z_k|\sigma_k|^2/4 + \beta_k z_k/\Xi_k \\ & + \frac{((n-k)+\Xi_k^2 m_{k0}^2/(2(R/n)))z_k + \Xi_{k-1}z_{k-1}}{\Xi_k},\end{aligned} \tag{11}$$

$m_{k0} := m_k(0_{k\times 1})$, and $\beta_k(z_{[k]}) \geq 0$ is some design function. With this, we can write (10) in terms of z_{k+1} as:

$$\begin{aligned}dV_k \quad \leq \quad & -z_1^2(1+\bar{\Gamma}\bar{\psi}_{hv})dt + \Xi_k z_k z_{k+1} dt \\ & + \sum_{i=1}^{k}[(-\beta_i z_i^2 - (n+1)\Xi_i^2\theta z_i^2|\sigma_i|^2/4)dt \\ & + \Xi_i z_i \sigma_i^T dw] + [k\bar{\Gamma}|\tilde{x}|^2/2 - (n-k)|z_{[k]}|^2]dt \\ & + ((k-1/2)R/n)dt.\end{aligned}$$

Since all of the relevant definitions and results of Step k are consistent with the induction hypothesis, we conclude that the induction hypothesis holds true for all $k \in \{1, ..., n-1\}$.

Step n: We note that the results of Step k hold true also for $k = n$ if we set $\hat{x}_{n+1} = \bar{b}u$, where u is the actual control input. Thus, we can make $z_{n+1} = 0$, by picking the control input as:

$$u = -\frac{\alpha_n(z_{[n]})}{b(z_1)} \tag{12}$$

where α_n is obtained by setting $k = n$ in (11). Thus, the control input (12) renders the Itô differential of the smooth positive definite function $V = \sum_{i=1}^{n}\Xi_i z_i^2/2 + W$, where W is as defined in the previous section, as:

$$\begin{aligned}dV \quad \leq \quad & -z_1^2 dt - \sum_{i=1}^{n}\beta_i z_i^2 dt - \bar{\Gamma}|\tilde{x}|^2 dt + R dt \\ & + \left[\sum_{i=1}^{n}\Xi_i z_i \sigma_i^T + \bar{\Gamma}\tilde{x}^T PD\right] dw \\ & - \frac{(n+1)\theta}{4}\left[\sum_{i=1}^{n}\Xi_i^2 z_i^2|\sigma_i|^2 + \bar{\Gamma}M_D\tilde{x}^T PP\tilde{x}\right] dt \\ \leq \quad & -z_1^2 dt - \sum_{i=1}^{n}\beta_i z_i^2 dt - \bar{\Gamma}|\tilde{x}|^2 dt + \sigma^T dw \\ & -(\theta|\sigma|^2)/4dt + Rdt\end{aligned} \tag{13}$$

where $\sigma := \sum_{i=1}^{n}\Xi_i z_i \sigma_i + \bar{\Gamma}D^T P\tilde{x}$. It is straight-forward to show that $h_i(0) = 0_{q\times 1}$, $\forall i \in \{1, .., n\} \Rightarrow |s_i(0)|^2$, $\forall i \in \{1, .., n\}$, and if this is the case R could be chosen exactly equal to zero to achieve a zero average risk-sensitive cost. This now brings us to the following result, using a relationship established in [9] between (13) and risk-sensitive cost.

Theorem 1. *Consider the nonlinear system described by (1) under Assumptions 1 and 2, and pick the design functions β_i such that they are uniformly bounded away from zero, i.e., $\beta_i \geq k_\beta$ $\forall i \in \{1, ..., n\}$ for some $k_\beta > 0$, Then, for any given risk-sensitivity parameter $\theta > 0$ and desired average risk-sensitive cost $R > 0$ (which could be chosen as zero, i.e., $R = 0$, when $h_i(0) = 0_{q\times 1}$, $\forall i \in \{1, .., n\}$), the designed controller (12) achieves:*

1.

$$\begin{aligned}& \frac{2}{\theta}\ln E\exp\frac{\theta}{2}\left[V(T) + \int_0^T [z_1^2 + \sum_{i=1}^{n}\beta_i z_i^2 + \bar{\Gamma}|\tilde{x}|^2]dt\right] \\ & \quad \leq V(0) + RT, \ \forall T \geq 0.\end{aligned}$$

2. The closed-loop signals are bounded in probability, i.e.,

$$\lim_{c\to\infty}\sup_{t\geq 0} P\{|\xi(t)| > c\} = 0,$$

where

$$\xi(t) := [z_1(t), ..., z_n(t), \tilde{x}_1(t), ..., \tilde{x}_n(t)]^T.$$

3. If $h_i(0) = 0_{q\times 1}$, $\forall i \in \{1, .., n\}$, then the equilibrium point $\xi = 0_{2n\times 1}$ of the closed-loop system is globally and asymptotically stable, i.e., $\forall \varepsilon > 0$ there exists a class K function $\bar{\gamma}(.)$ such that

$$P\{|\xi(t)| \leq \bar{\gamma}(|\xi(0)|)\} \geq 1 - \varepsilon, \quad \forall t \geq 0, \ \forall \xi(0) \in R^{2n},$$

$$\text{and } P\left\{\lim_{t\to\infty}|\xi(t)| = 0\right\} = 1.$$

Proof. From (13), we write: $\forall T \geq 0$,

$$\begin{aligned}& V(T) + \int_0^T [z_1^2 + \sum_{i=1}^{n}\beta_i z_i^2 + \bar{\Gamma}|\tilde{x}|^2]dt \\ & \quad \leq V(0) + RT + \int_0^T \sigma^T dw - \frac{\theta}{4}\int_0^T |\sigma|^2 dt.\end{aligned}$$

This implies:

$$\begin{aligned}& \frac{2}{\theta}\ln E\exp\frac{\theta}{2}\left[V(T) + \int_0^T [z_1^2 + \sum_{i=1}^{n}\beta_i z_i^2 + \bar{\Gamma}|\tilde{x}|^2]dt\right] \\ & \quad \leq \frac{2}{\theta}\ln E\exp\frac{\theta}{2}\left[\int_0^T \sigma^T dw - \frac{\theta}{4}\int_0^T |\sigma|^2 dt\right] \\ & \quad + V(0) + RT,\end{aligned}$$

for all $T \geq 0$. From Theorem 11 in the Appendix of [9], we conclude that

$$E\exp\frac{\theta}{2}\left[\int_0^T \sigma^T dw - \frac{\theta}{4}\int_0^T |\sigma|^2 dt\right] \leq 1, \quad \forall T \geq 0$$

which proves the first part. The second and third parts directly follow from Theorem 5 of [9] and Theorem 3.2 of [4], respectively.

6 An Illustrative Example

To illustrate the results of Section 5, a second-order system of the form (1) is considered, where:

$$\begin{aligned} f_1 &= x_1^2, \quad f_2 = x_1 \cos(x_1), \quad b = 1 \\ h_1 &= [2x_1/(1+x_1^2), 0]^T, \; h_2 = [2\tanh(x_1), 2\sin(x_1)]^T. \end{aligned}$$

The design parameters are picked as: $\beta_1 = 0.01$, $\beta_2 = 0.01$, and $k = [-3, -2]^T$. To illustrate the effect of the risk-sensitivity parameter, the simulation is performed for two different values of θ: $\theta = 0$ and $\theta = 1.5$. Figures 1 and 2 show the sample paths of the states and the control action under the same random disturbance for $\theta = 0$ and $\theta = 1.5$, respectively. Clearly, in both cases, the controller asymptotically stabilizes the system. In the risk-sensitive case, i.e., $\theta = 1.5$, the controller becomes a high gain controller and the state variables experience less fluctuations than the ones in the risk-neutral case, i.e., $\theta = 0$. This illustrates the trade-off between the transient performance of the system and the available control action via the risk-sensitivity parameter θ.

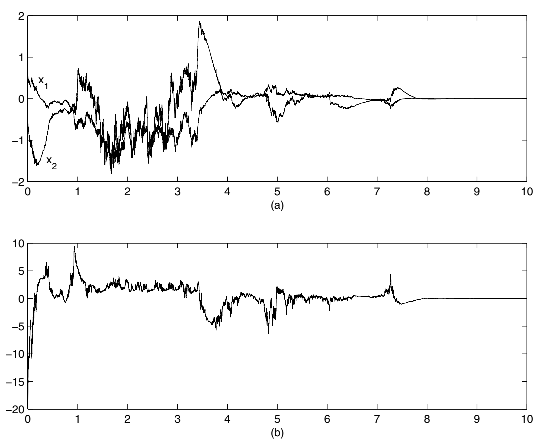

Figure 1: ($\theta = 0$): (a) $x_1(t)$, $x_2(t)$ (b) $u(t)$.

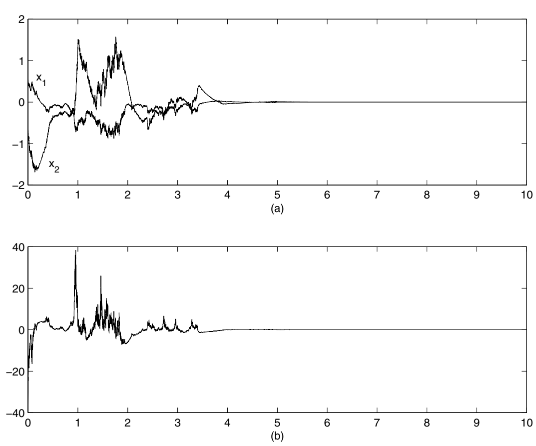

Figure 2: ($\theta = 1.5$): (a) $x_1(t)$, $x_2(t)$ (b) $u(t)$.

7 Conclusions

We have presented an output-feedback controller design for stochastic strict-feedback systems. The controller design procedure involves the estimation of unmeasured states, the backstepping design methodology, and the use of radially unbounded functions. The resulting controller achieves an arbitrarily small average risk-sensitive cost at the expense of increased control effort, as stated in Theorem 1. Also, the closed-loop signals remain bounded in probability, and if the functions multiplying the noise terms vanish at the origin, then the closed-loop signals asymptotically converge to zero (again in probability). The paper has also included some simulation results to numerically demonstrate the effectiveness of the proposed controller. One extension of this work would be to obtain the counterparts of these results for the case where the system model includes additional linearly parameterized uncertainty.

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