Stabilization in probability of nonlinear stochastic systems with guaranteed cost

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Abstract

We deal with nonlinear dynamical systems, consisting of a linear nominal part perturbed by model uncertainties, nonlinearities and both additive and multiplicative random noise, modeled as a Wiener process. In particular, we study the problem of finding suitable measurement feedback control laws such that the resulting closed–loop system is stable in some probabilistic sense and a given cost functional is minimized. We give a Lyapunov–based separation result which splits the control design into a *state feedback* problem and a *filtering problem.*

1 Introduction

According to the existing literature ([15], [16], [11], [12], [13]; see also the textbooks [8] and [14]), by stability is usually meant that

- the probability that the trajectory, stemming from x_0, leaves an ϵ–ball around the origin goes to zero as x_0 tends to the origin
- the trajectory, stemming from x_0, goes asymptotically to zero almost surely.

This stability, usually known as *stability in probability*, is either *local* or *global* according to whether x_0 is in some (small) neighbourhood of the origin or, respectively, it is *any point of the state space.* In [8] Lyapunov–based conditions are given for guaranteeing stability in probability and require the solution of partial differential inequalities (PDI's). In [10]and [12] it has been proved that a step–by–step algorithm (*backstepping*) can be successfully implemented for solving globally these PDI's, whenever the state is available for feedback, while in [11] for a class of systems with output nonlinearities the problem of global output feedback stabilization in probability is solved. For deterministic systems, the complexity and the conditions for solving these PDI's can be weakened by relaxing the stability requirements of the closed–loop system. In a deterministic setting *semiglobal stabilization* was introduced in [3]and requires a local asymptotic stability of the closed–loop system plus a region of attraction containing any *a priori* given compact set of the state space. The basic ingredients for achieving semiglobal stability via output feedback are *control saturations* and *high–gain observers* ([5], [7]): large values of the observer gain guarantee that the error between the state and its estimate, generated by the observer itself, goes to zero "sufficiently fast", while input saturations rule out destabilizing effects such as *peaking* ([4]), which is a phenomenon occurring when one is trying to force some state variables to zero as fast as possible causing an impulsive–like behaviour of some others.

A first objective of our paper is to extend the notion of semiglobal stabilization to the following class of nonlinear stochastic systems

$$\begin{aligned} \dot{x} &= Ax + B_2 u + B_1 \Phi(t,u,x) + H(t,x)\dot{w} \\ y &= C_2 x + C_1 \Phi(t,u,x) + K(t,x)\dot{w} \end{aligned} \quad (1)$$

where $w \in I\!R^s$ is a Wiener process, $u \in I\!R^m$ is the control, $x \in I\!R^n$ is the state, $y \in I\!R^p$ are the measurements and $\Phi \in I\!R^r$ are model uncertainties and nonlinearities (in section 2 we will make precise in which sense $\dot{x}$ and $\dot{w}$ are meant). Moreover, we will consider families of admissible controllers

$$\begin{aligned} u &= \eta(F(k)\sigma) \\ \dot{\sigma} &= L(k)\sigma(k) + B_2 u + G(k)y, \qquad \sigma \in I\!R^n \end{aligned} \quad (2)$$

for $k \in I\!R^+$ and for some matrices $F(k)$, $L(k)$ and $G(k)$ and a C^0 function $\eta : I\!R^m \to I\!R^m$, *linear* near the origin. This is a reasonable structure for the controller since near the origin (1) behaves as its own linearization.

As a second step, we define our optimality and robustness criteria. First, we give a set of admissibility constraints, which impose precise characteristics to Φ, H and η, generally satisfied under mild assumptions. As it will be clear, these constraints lead to an optimal controller (2) in which

$$\begin{aligned} u &= F(k)\sigma \\ \dot{\sigma} &= L(k)\sigma(k) + B_2 u + G(k)y, \qquad \sigma \in I\!R^n \end{aligned} \quad (3)$$

is a *worst–case linear controller for the nominal system.*

The optimality criteria are formulated in terms of achieving either a guaranteed value or the minimum value of some cost functionals, according to whether multiplicative or additive noise is taken into account. These functionals penalize the "distance" from a reference situation for which the worst–case linear controller (3) is designed and in the linear case they reduce to a standard quadratic cost (see [13]and [14]for comparisons with other inverse optimal schemes for deterministic and stochastic nonlinear systems).

As a counterpart of the deterministic case studied in [1], we show that the problem of finding a stabilizing optimal controller can be split into two lower dimensional problems: one is related to the case in which the *state x is available for feedback* and the other to the possibility of *constructing an observer for estimating the state x.*

2 Notations and basic notions

First, we give some notations extensively used throughout the paper.

- by $\mathcal{SP}^n$ (resp. $\mathcal{SN}^n$) we denote the set of $n \times n$ positive (resp. negative) definite symmetric matrices; by $\mathcal{SSP}^n$ we denote the set of $n \times n$ positive semidefinite symmetric matrices; $I\!R^+$ denotes the set of positive real numbers and $I\!R^{\geq}$ the set of nonnegative real numbers;
- for any given set $\mathcal{S}$, we denote by $\overline{\mathcal{S}}$ its closure and by $\partial\mathcal{S}$ its boundary; moreover, given $\delta > 0$ and a set $\mathcal{S}$, by *δ–neighbourhood* of $\mathcal{S}$ we denote the set $\mathcal{S}_\delta = \{z : \inf_{y\in\mathcal{S}} \|z - y\| < \delta\}$;
- for any sequence of sets $\{\mathcal{S}_k\}$, $\liminf_{k\to\infty} \mathcal{S}_k = \cup_{k=1}^{\infty} \cap_{i\geq k} \mathcal{S}_i$ and $\limsup_{k\to\infty} \mathcal{S}_k = \cap_{k=1}^{\infty} \cup_{i\geq k} \mathcal{S}_i$. It is easy to see that if $\liminf_{k\to\infty} \mathcal{S}_k \supset \mathcal{V}$ then there exists k° such that $\mathcal{S}_k \supseteq \mathcal{V}$ for all $k \geq k^\circ$. Similarly, if $\limsup_{k\to\infty} \mathcal{S}_k \subset \mathcal{V}$ then there exists k° such that $\mathcal{S}_k \subseteq \mathcal{V}$ for all $k \geq k^\circ$.

In the remaining part of this section, we shortly recall some notions of stochastic processes, referring the reader for the basic concepts to standard textbooks ([18]). We assume that the reader is familiar with the basic notions of probability theory and stochastic processes $\{x_t, t \in I\!R\}$ on a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We denote by $\mathbf{E}\{\cdot\}$ the expectation.

By a *stochastic differential equation* we mean the following equation

$$dx_t = f(x_t, t)dt + g(x_t, t)dw_t \tag{4}$$

with initial condition $x_{t_0} = \overline{x}$, where $\{w_t, t \in I\!R\}$ is a Wiener process. The solution $\{x_t, t \in I\!R\}$, whenever it exists, is a Markov process satisfying

$$x_t = x_{t_0} + \int_{t_0}^{t} f(x_s, s)ds + \int_{t_0}^{t} g(x_s, s)dw_s \tag{5}$$

almost surely (a.s.). The last integral, whenever it is well–defined, is called *Îto integral.* It is well–known that if $f(s, t)$ and $g(s, t)$ are locally Lipschitz functions on the domain $\mathcal{B} \times [t_0, T]$, with $\mathcal{B}$ a compact set containing $\overline{x}$, then there exists an a.s. unique stochastic process x_t, sample continuous and satisfying (5) on $[t_0, \tau_{\mathcal{B},T}(t)]$, where $\tau_{\mathcal{B},T}(t) = \min(t, \tau_{\mathcal{B},T})$, $\tau_{\mathcal{B},T} = \min\{\tau_{\mathcal{B}}, T\}$ and $\tau_{\mathcal{B}}$ is the stopping time defined as the first time at which x_t reaches the boundary of $\mathcal{B}$ ([18]).

For any given process η_t we will use $\dot{\eta}_t$ instead of $d\eta_t$. Accordingly, we denote the stochastic equation (4) by

$$\dot{x}(t) = f(x(t), t) + g(x(t), t)\dot{w}(t) \tag{6}$$

and by $x(t, t_0, \overline{x})$ its solution (5) at time t starting from $\overline{x}$.

Given a C^2 function $V : I\!R^n \to I\!R$, define

$$\begin{aligned} \mathcal{L}V(x(t), t) &= \frac{\partial V}{\partial x}(x(t))f(x(t), t) \\ &\frac{1}{2}\mathbf{Tr}\{g^T(x(t), t)\frac{\partial^2 V}{\partial x^2}(x(t))g(x(t), t)\} \end{aligned} \tag{7}$$

where $x(t)$ satisfies (6).

Proposition 2.1 **[8]** *Let $\overline{x} \in \mathcal{B}$ a.s. The solution $x(t)$ of (6) satisfies on $[t_0, \tau_{\mathcal{B},T}(t)]$ the following equation*

$$\mathbf{E}\{V(x(\tau_{\mathcal{B},T}(t), t_0, \overline{x}))\} - V(\overline{x}) = \mathbf{E}\Big\{\int_{t_0}^{\tau_{\mathcal{B},T}(t)} \mathcal{L}V(x(s))ds\Big\} \tag{8}$$

3 Problem formulation

Let us consider nonlinear stochastic systems Σ of the form (1), where $x(t) \in I\!R^n$, $u(t) \in I\!R^m$, $y(t) \in I\!R^p$, $w(t)$ is an s–dimensional Wiener process and $\Phi \in I\!R^r$ represents model uncertainties and nonlinearities. Moreover, for each sequence of real positive extended numbers $\{\Delta(k)\}$, with $k \in I\!R^+$, we define the class of *candidate controllers* $\{\mathcal{C}(k)\}$

$$\begin{aligned} u &= \eta(F(k)\sigma) \\ \dot{\sigma} &= L(k)\sigma(k) + B_2 u + G(k)y, \qquad \sigma \in I\!R^n \end{aligned} \tag{9}$$

with

$$L(k) = A + \frac{1}{\gamma^2(k)} B_1 B_1^T P_{SF}(k) - G(k)C_2 \tag{10}$$

for some sequences of matrices $\{F(k)\}$ and $\{G(k)\}$, positive numbers $\{\gamma(k)\}$ and symmetric positive definite matrices$\{P_{SF}(k)\}$ and $\eta : I\!R^m \to I\!R^m$ is any C^0 function such that

$$\|\eta(s)\| \leq \Delta(k), \qquad \forall s \tag{11}$$

$$\eta(s) = s, \qquad \|s\| \leq s_0 \tag{12}$$

for some $s_0 > 0$. In other words, any candidate controller is the composition of a *linear controller* with a *static nonlinearity* η, which is *bounded by* $\Delta(k)$ (unbounded if $\Delta(k) = \infty$) and *it is the identity function near the origin.* While η is designed in such a way to counteract the destabilizing effects due to large values of $G(k)$ (peaking), $\Delta(k)$ accounts for possible limitations on the control u (as an example, saturations of the control actuators).

For the analysis of stability of the closed–loop system (1)–(9), we also define a class of *candidate Lyapunov functions* $\{V_k^e\}$

$$V_k^e(x,\sigma) = \|x\|^2_{P_{SF}(k)} + \varphi(\|x-\sigma\|^2_{P_m(k)}) \qquad (13)$$

where $\{P_m(k)\}$ is a sequence in $\mathcal{SP}^n$ and $\varphi : I\!R^{\geq} \to I\!R$ is any (at least) C^2, positive definite and proper function such that

$$\frac{\partial^2\varphi}{\partial s^2}(s) \leq 0 < \frac{\partial\varphi}{\partial s}(s) \leq 1 \qquad (14)$$

for all $s \geq 0$. Conditions (14) imply that over any compact set containing the origin any *candidate Lyapunov function is bounded from below and above by a quadratic function* and are is instrumental in enlarging the region of attraction of the closed–loop system.

Next, we define some *admissibility constraints* for the noise coefficients H and K and for the uncertainty term Φ. For, define the following compact sets

$$\begin{aligned} \Omega(k) &= \{x \in I\!R^n : \|x\|^2_{P_{SF}(k)} \leq k\} \\ \mathcal{U}_{\Delta(k)} &= \{u \in I\!R^m : \|u\| \leq \Delta(k)\} \end{aligned} \qquad (15)$$

Let $\{E(k)\}$, $\{R_1(k)\}$ and $\{c_1(k)\}$ be sequences in $\mathcal{SSP}^n$, $\mathcal{SP}^m$ and $I\!R^{\geq}$, respectively. Define

$$\begin{aligned} &\mathcal{P}_1(t,u,x,e,k) = \widetilde{\mathcal{P}}_1(t,u,x,e,k) \\ &+\gamma^2\Big\|\Phi(t,u,x) - \frac{1}{\gamma^2(k)}\Big[B_1^T P_{SF}(k)x \\ &+\frac{\partial\varphi}{\partial s}\Big|_{s=\|e\|^2_{P_m(k)}}(B_1 - G(k)C_1)^T P_m(k)e)\Big]\Big\|^2 \\ &\widetilde{\mathcal{P}}_1(t,u,x,e,k) = -\gamma^2(k)\|\Phi(t,u,x)\|^2 + \|x\|^2_{E(k)} \\ &+\|u\|^2_{R_1(k)} + c_1(k) \end{aligned} \qquad (16)$$

and let $\mathcal{F}(k)$ be the class of C^0 functions $\Phi : I\!R \times I\!R^m \times I\!R^n \to I\!R^r$ such that $\widetilde{\mathcal{P}}_1(t,u,x,e,k) \geq 0$ for all $t \geq 0$, $u \in \mathcal{U}_{\Delta(k)}$, $x \in \Omega(k)$ and $e \in I\!R^n$.

Let $\{\widehat{H}_j(k)\}$, $j = 1,\ldots,s$, be a sequence in $I\!R^{n\times n}$ and $\{c_2(k)\}$ a sequence in $I\!R^{\geq}$. Define

$$\begin{aligned} &\mathcal{P}_2(t,u,x,e,k) = -\mathbf{Tr}\{H^T(t,x)P_{SF}(k)H(t,x))\} \\ &+\sum_{j=1}^{s} x^T\widehat{H}_j^T(k)P_{SF}(k)\widehat{H}_j(k)x + c_2(k) \end{aligned} \qquad (17)$$

Note that $\mathcal{P}_2$ penalizes the distance of $\mathbf{Tr}\{H^T(t,x)P_{SF}(k)H(t,x))\}$ from a sum of quadratic functions.

Let $\mathcal{H}(k)$ be the class of C^0 functions $H : I\!R \times I\!R^n \to I\!R^{n\times s}$ such that $\mathcal{P}_2(t,u,x,e,k) \geq 0$, for all $t \geq 0$, $u \in \mathcal{U}_{\Delta(k)}$, $x \in \Omega(k)$ and $e \in I\!R^n$. Since $\Omega(k)$ and $\mathcal{U}_{\Delta(k)}$ are compact sets, the admissibility constraints on $\widetilde{\mathcal{P}}_1$ and $\mathcal{P}_2$ can be always met whenever Φ and H are *locally Lipschitz*, uniformly w.r.t. t.

Let $\{Q_m(k)\}$ and $\{c_3(k)\}$ be sequences in $\mathcal{SP}^n$ and $I\!R^{\geq}$ respectively and let

$$M(t,x,k) \;=\; H(t,x) - G(k)K(t,x) \qquad (18)$$

Define $\mathcal{K}(k)$ and $\mathcal{D}(k)$ as the class of C^0 functions $K : I\!R \times I\!R^n \to I\!R^{p\times s}$ and, respectively, the class of pairs of C^0 functions $\eta : I\!R^m \to I\!R^m$ and $\varphi : I\!R^{\geq} \to I\!R$ satisfying (11), (12) and (14) and such that

$$\begin{aligned} &\mathcal{P}_3(t,u,x,e,k) = -\|\eta(F(k)(x-e)) - F(k)x\|^2_{R_1(k)} \\ &+\|x\|^2_{Q_{SF}(k)} + c_3(k) \\ &+\frac{\partial\varphi}{\partial s}\Big|_{s=\|e\|^2_{P_m(k)}}\Big[\|F(k)e\|^2_{R_1(k)} + \|e\|^2_{Q_m(k)} \\ &-\mathbf{Tr}\{M^T(t,x,k)P_m(k)M(t,x,k))\}\Big] > 0 \end{aligned} \qquad (19)$$

for all $(x,e) \in (\Omega(k) \times I\!R^n)\backslash(0,0)$ and $\mathcal{P}_3(t,u,x,e,k)$ *is lower bounded by a quadratic positive definite function of* x *and* e *near the origin.* This assumption implies that the argument of the cost functionals we define later is *locally lower bounded by a quadratic positive definite function of* x *and* e (up to constant terms). The function η is typically designed for avoiding the peaking phenomenon while φ is instrumental in enlarging the region of attraction of the closed–loop system.

Finally, let

$$\begin{aligned} &\mathcal{P}_4(t,u,x,e,k) = \frac{1}{\gamma^2(k)}\frac{\partial\varphi}{\partial s}\Big|_{s=\|e\|^2_{P_m(k)}} \cdot \\ &\cdot\Big[(1 - \frac{\partial\varphi}{\partial s}\Big|_{s=\|e\|^2_{P_m(k)}})\|(B_1 - G(k)C_1)^T P_m(k)e\|^2 \\ &+\|G^T(k)P_m(k)e\|^2_{R_2(k)-C_1C_1^T}\Big] \\ &-\frac{\partial^2\varphi}{\partial s^2}\Big|_{s=\|e\|^2_{P_m(k)}}\|M^T(t,x,k)P_m(k)e\|^2 \end{aligned} \qquad (20)$$

for some sequence $\{R_2(k)\}$ in $\mathcal{SP}^p$ such that $R_2(k) \geq C_1C_1^T$. Note that, by (14), (20) is nonnegative for all $e \in I\!R^n$ and if $C_1C_1^T$ is nonsingular then we can take directly $R_2(k) = C_1C_1^T$. Note also that $\mathcal{P}_4$ penalizes the distance from the situation for which φ is linear (i.e. quadratic Lyapunov functions) and $R_2(k) = C_1C_1^T$.

In what follows, we will refer to $\Phi \in \mathcal{F}(k)$, $H \in \mathcal{H}(k)$, $K \in \mathcal{K}(k)$ and $(\eta,\varphi) \in \mathcal{D}(k)$ as *admissible functions.* Moreover, any choice of $\{P_{SF}(k)\},\{P_m(k)\}$,

$\{Q_{SF}(k)\}, \{Q_m(k)\}, \{R_1(k)\}, \{R_2(k)\}, \{\gamma(k)\}, \{E(k)\}, \{c_j(k)\}$, $j = 1,2,3$, $\{\widehat{H}_j(k)\}$, $j = 1,\ldots,s$, for which $\Phi \in \mathcal{F}(k)$, $H \in \mathcal{H}(k)$ and $K \in \mathcal{K}(k)$ will be referred to as *admissible parametrization.*

Denote by $x_k^e(t) = \text{col}(x_k(t), \sigma_k(t))$ the trajectories of $\Sigma \circ \mathcal{C}(k)$ at time $t \geq t_0$ stemming from the initial condition $x_0^e = \text{col}(x_0, \sigma_0)$ and let $e_k(t) = x_k(t) - \sigma_k(t)$. We introduce two sequences of cost functionals $\{J_h(k)\}$, $h = 1,2$, defined as follows

$$J_1(k) \quad = \quad \lim_{T\to\infty} \frac{1}{T-t_0}\int_{t_0}^{T} \mathbf{E}\Big\{\sum_{j=1}^{4}\mathcal{P}_j\Big\}dt \qquad (21)$$

and

$$J_2(k) \quad = \quad \lim_{T\to\infty} \int_{t_0}^{T} \mathbf{E}\Big\{\sum_{j=1}^{4}\mathcal{P}_j\Big\}dt \qquad (22)$$

Note that $J_h(k) \geq 0, h = 1,2$, for any $\Phi \in \mathcal{F}(k)$, $H \in \mathcal{H}(k)$, $K \in \mathcal{K}(k)$ and $(\eta,\varphi) \in \mathcal{D}(k)$. While $J_1(k)$ is more suitable in the case of both *additive and multiplicative noise*, $J_2(k)$ is not suitable for the case of *additive noise*, since the constant $c_j(k) \neq 0$ for at least one j would cause $J_2(k)$ to diverge.

The aim of this paper is to study under which conditions it is possible to modify the behaviour of (1) in such a way that $J_1(k)$ achieves a guaranteed value (resp. $J_2(k)$ achieves its minimum) and to obtain stability in some "stochastic" sense. To make the last point precise, let us give the following definition ([2]).

Definition 3.1 *Let $\alpha,\beta \in [0,1)$ and $\Omega^e, \mathcal{B}^e \subset I\!R^{2n}$ be compact sets containing the origin. The system (1) is said to be $(\Omega^e,\mathcal{B}^e,\alpha,\beta)$–stabilizable in probability ($(\Omega^e,\mathcal{B}^e,\alpha,\beta)$–SP) if there exist a sequence of candidate control laws $\{\mathcal{C}(k)\}$, a sequence of compact sets $\{\Omega^e(k)\}$, $\Omega^e(k) \subset I\!R^{2n}$, and open sets $\{\mathcal{B}^e(k)\}$, $\mathcal{B}^e(k) \subset I\!R^{2n}$, both containing the origin, such that*

(i) $\liminf_{k\to\infty}\Omega^e(k) \supset \Omega^e \supset \mathcal{B}^e \supseteq \limsup_{k\to\infty}\mathcal{B}^e(k)$;

(ii) for each $\delta > 0$

$$\liminf_{k\to\infty}\mathbf{P}\{\forall x_0^e \in \overline{\mathcal{B}}^e(k),\ \forall\Phi\in\mathcal{F}\ : x_k^e(t) \in \overline{\mathcal{B}}^e_\delta \ \ \forall t \geq t_0\} \geq 1-\beta \qquad (23)$$

(iii) for each $\delta > 0$

$$\liminf_{k\to\infty}\mathbf{P}\{\forall x_0^e \in \Omega^e\backslash\mathcal{B}^e(k),\ \forall\Phi\in\mathcal{F}: x_k^e(t) \in \Omega^e(k)\ \forall t \geq t_0 \text{ and } x_k^e(t) \in \overline{\mathcal{B}}^e_\delta\} \ \forall t \in [\tau_{I\!R^{2n}\backslash\mathcal{B}^e(k)},\infty) \geq (1-\alpha)(1-\beta) \qquad (24)$$

If, besides

$$\liminf_{k\to\infty}\Omega^e(k) \supset \Omega^e$$
$$\liminf_{k\to\infty}\mathbf{P}\{\forall x_0^e \in \Omega^e,\ \forall\Phi\in\mathcal{F}: x_k^e(t) \in \Omega^e(k)\ \forall t \geq t_0\ \} \geq 1-\alpha \qquad (25)$$

the following holds:

- *conditionally to the event $x_k^e(t) \in \Omega^e(k)$,*

$$\mathbf{E}\{\|x_k^e(t)\|^2\} \leq \lambda(k)\|x_0^e\|^2 e^{-\nu(k)(t-t_0)} \qquad (26)$$

  for all $t \geq t_0$ and for some sequences of positive numbers $\{\lambda(k)\}$ and $\{\nu(k)\}$

then we will say that *(1) is (Ω^e,α)–stabilizable in quadratic mean.*

The set Ω^e gives the *guaranteed* region of attraction of $\Sigma \circ \mathcal{C}(k)$, while $\mathcal{B}^e$ represents the *guaranteed target set.* The numbers α and β are given *risk margins: the first one quantifies the risk of leaving the compact set $\Omega^e(k)$ with initial condition in Ω^e rather than getting close to the target, while the second one gives a risk margin for remaining close to the target.*

This definition of stability in probability recovers, according to the different choices of $\mathcal{B}^e$, Ω^e, α and β, the classical definitions given in ([8]) and [3].

All the above remarks can be straightforwardly extended to the definition of stability in quadratic mean.

We are ready to formulate our problems.

Problem I: Nonlinear Stabilization in Probability with Guaranteed Cost. Let $\Phi \in \mathcal{F}(k)$, $H \in \mathcal{H}(k)$, $K \in \mathcal{K}(k)$, $\mathcal{B}^e \subset \Omega^e$ be compact sets of $I\!R^{2n}$, $\alpha,\beta \in [0,1)$, $x_0^e \in \Omega^e$ and $\{\Delta(k)\}$ and $\{\varpi(k)\}$ given sequences in $(0,\infty]$ and $I\!R^{\geq}$, respectively. Find an admissible parametrization and $(\eta,\varphi) \in \mathcal{D}(k)$ such that

- *(Guaranteed cost)* along the trajectories $x_k^e(t)$ of the closed–loop systems $\Sigma \circ \mathcal{C}(k)$,

$$\liminf_{k\to\infty}\mathbf{Pr}\{J_1(k) \leq \varpi(k)\} \geq (1-\alpha) \qquad (27)$$

- *(Stability)* $\Sigma\circ\mathcal{C}(k)$ is $(\Omega^e,\mathcal{B}^e,\alpha,\beta)$–stable in probability.

Problem II: Nonlinear Stabilization in Quadratic Mean with Optimality. Let $\Phi \in \mathcal{F}(k)$, $H \in \mathcal{H}(k)$, $K \in \mathcal{K}(k)$, $\mathcal{B}^e \subset \Omega^e$ be compact sets of $I\!R^{2n}$, $\alpha,\beta \in [0,1)$, $x_0^e \in \Omega^e$ and $\{\Delta(k)\}$ a given sequence in $(0,\infty]$. Find an admissible parametrization and $(\eta,\varphi) \in \mathcal{D}(k)$ such that

- *(Optimality)* along the trajectories $x_k^e(t)$ of the closed–loop systems $\Sigma \circ \mathcal{C}(k)$,

$$\liminf_{k\to\infty}\mathbf{Pr}\{J_2(k) \leq \widetilde{J}_2(k)\} \geq (1-\alpha) \qquad (28)$$

where $\widetilde{J}_2(k)$ is the value of $J_2(k)$ corresponding to any other admissible parametrization;

- *(Stability)* $\Sigma \circ \mathcal{C}(k)$ is (Ω, α)–stable in quadratic–mean.

4 Main results

Let $H(t,x) = (\, H_1(t,x) \quad \cdots \quad H_s(t,x)\,)$, $K(t,x) = (\, K_1(t,x) \quad \cdots \quad K_s(t,x)\,)$ and, without loss of generality, assume that $B_1 C_1^T = 0$.

Theorem 4.1 *Assume that there exist an admissible parametrization and $(\eta, \varphi) \in \mathcal{D}(k)$ such that*

- **(state feedback (SF))**

$$\begin{aligned} & A^T P_{SF}(k) + P_{SF}(k) A \\ & + \frac{1}{\gamma^2(k)} P_{SF}(k) B_1 B_1^T P_{SF}(k) + E(k) \\ & - F^T(k) R_1(k) F(k) + \sum_{j=1}^{s} \widehat{H}_j^T(k) P_{SF}(k) \widehat{H}_j(k) \\ & = -Q_{SF}(k) \end{aligned} \quad (29)$$

where $F(k) = -R_1^{-1}(k) B_2^T P_{SF}(k)$;

- **(output injection (OI))**

$$\begin{aligned} & P_m(k)\Big(A + \frac{1}{\gamma^2(k)} B_1 B_1^T P_{SF}(k)\Big) \\ & + \Big(A + \frac{1}{\gamma^2(k)} B_1 B_1^T P_{SF}(k)\Big)^T P_m(k) \\ & + F^T(k) R_1 F(k) + \frac{1}{\gamma^2(k)} P_m(k) B_1 B_1^T P_m(k) \\ & - \gamma^2(k) C_2^T R_2^{-1}(k) C_2 = -Q_m(k) \end{aligned} \quad (30)$$

- **(risk margins (RM))** *if*

$$\Omega^e(k) \;=\; \Big\{(x,\sigma) \in I\!R^n \times I\!R^n : V_k^e(x,\sigma) \le k\Big\} \quad (31)$$

and $\{\mathcal{B}^e(k)\}$, a sequence of open sets of $I\!R^{2n}$, are such that

$$\limsup_{k\to\infty} \mathcal{B}^e(k) \subseteq \mathcal{B}^e \subset \Omega^e \subset \liminf_{k\to\infty} \Omega^e(k) \quad (32)$$

then for each $\delta > 0$

$$\sup_{(x,\sigma)\in\Omega^e} \limsup_{k\to\infty} \frac{V_k^e(x,\sigma)}{k} \le \alpha \quad (33)$$

$$\limsup_{k\to\infty} \sup_{(x,\sigma)\in\partial\mathcal{B}^e(k)} \frac{V_k^e(x,\sigma)}{\inf_{(s_1,s_2)\in I\!R^{2n}\setminus \mathcal{B}_\delta^e} V_k^e(s_1,s_2)} \le \beta \quad (34)$$

and

$$\sum_{j=1}^{4} \mathcal{P}_j(t, \eta(F(k)\sigma), x, x-\sigma, k) - \sum_{j=1}^{3} c_j(k) \ge Q_k^e(x,\sigma) \quad (35)$$

for all t and $(x,\sigma) \in \Omega^e(k)\setminus\mathcal{B}^e(k)$ and for some sequence of locally quadratic C^0 positive definite functions $\{Q_k^e\}$.

Under the above assumptions, the controller (9) *with $F(k)$ as above and $G(k) = \gamma^2(k) P_m^{-1}(k) C_2^T R_2^{-1}(k)$ solves problem I with $\overline{\omega}(k) = \sum_{j=1}^3 c_j(k)$. If, in addition, $c_j(k) = 0$ for all $j = 1,2,3$, and $K(t,x) = 0$ for all t, x and $j = 1,\dots,r$, the same controller* (9) *solves problem II.*

Proof. (sketch) Throughout the proof, unless otherwise stated, we will omit k and the arguments of Φ, K and H. Moreover, we can assume $k \ge k^*$, where k^* is such that $\Omega^e(k) \supseteq \Omega^e \supset \mathcal{B}^e(k)$ for all $k \ge k^*$ (this is always possible by (32)).

Let V_{SF}, V_k^e, M and e as in section 4, $\widetilde{\Phi} = \Phi - \frac{1}{\gamma^2} B_1^T P_{SF} x$ and $u = \eta(F\sigma)$. We have

$$\begin{aligned} & \mathcal{L}\varphi + \frac{\partial\varphi}{\partial s}\Big|_{s=\|e\|^2_{P_m}} \|Fe\|^2_{R_1} - \gamma^2 \|\widetilde{\Phi}\|^2 \\ & = -\|u - Fx\|^2_{R_1} + \|x\|^2_{P_{SF}} + \frac{\partial\varphi}{\partial s}\Big|_{s=\|e\|^2_{P_m}} \|Fe\|^2_{R_1} \\ & -\mathcal{P}_1 + \widetilde{\mathcal{P}}_1 - \mathcal{P}_3 - \mathcal{P}_4 + c_3 \end{aligned} \quad (36)$$

and

$$\begin{aligned} & \mathcal{L}V_{SF} = \|u - Fx\|^2_{R_1} - \gamma^2 \|\widetilde{\Phi}\|^2 - \|x\|^2_{Q_{SF}} \\ & -\widetilde{\mathcal{P}}_1 - \mathcal{P}_2 + c_1 + c_2 \end{aligned} \quad (37)$$

Summing up together (36) and (37), we conclude that

$$\mathcal{L}V_k^e + \sum_{j=1}^{4} \mathcal{P}_j = \overline{\omega} \quad (38)$$

We are left with proving the following facts:

(a) $J_1(k) \le \overline{\omega}(k)$ (resp. $J_2(k)$ achieves its minimum), conditionally to the event $x_k^e(t) \in \Omega^e(k)$ for all $t \ge t_0$;

(b) $\liminf_{k\to\infty} \mathbf{Pr}\{(x_k^e(t) \in \Omega^e(k)\} \ge 1 - \alpha$ for all $t \ge t_0$;

(c) $\Sigma \circ \mathcal{C}(k)$ is $(\Omega^e, \mathcal{B}^e, \alpha, \beta)$ stable in probability ($\Sigma \circ \mathcal{C}(k)$ is (Ω^e, α) stable in quadratic mean, respectively).

First, we prove (a). By (38) and (8) for each $T > t_0$

$$\begin{aligned} & \frac{1}{T - t_0} \int_{t_0}^{T} \mathbf{E}\{\sum_{j=1}^{4} \mathcal{P}_j(s, u, x_k^e(s), k)\} ds \\ & = \frac{1}{T - t_0}\Big(V_k^e(x_0^e) - \mathbf{E}\{V_k^e(x_k^e(T)\}\Big) + \overline{\omega}(k) \end{aligned} \quad (39)$$

for all $x_0^e \in \Omega^e(k)$. From (39), letting $T \to \infty$ and since $V_k^e \geq 0$, we obtain $0 \leq J_1(k) = \overline{\omega}(k)$. If, in addition, $c_j(k) = 0$ for all $j = 1, 2, 3$, and $K(t, x) = 0$ for all t, x, it is easy to see that (39) holds with $\overline{\omega}(k) = 0$. Moreover, since $x_k^e(t)$ is defined for all $t \geq t_0$, by (8)

$$\mathbf{E}\{V_k^e(x_k^e(t))\} - V_k^e(x_0^e) = \int_{t_0}^{t} \mathbf{E}\Big\{\mathcal{L}V_k^e(x_k^e(s))\Big\}ds \quad (40)$$

and, since $\mathcal{P}_3$ is lower bounded by a quadratic positive definite function of x and e on $\Omega^e(k)$, (35) holds with $\Omega^e(k)\backslash\mathcal{B}^e(k)$ replaced by $\Omega^e(k)$. Differentiating both sides of (40), since V_k^e is lower bounded by a quadratic function of x and e, we obtain for some sequences $\{\widetilde{\lambda}(k)\}, \{\widetilde{\nu}(k)\}$ of positive numbers

$$\mathbf{E}\{V_k^e(x_k^e(t))\} \leq \widetilde{\lambda}(k) V_k^e(x_0^e) e^{-\widetilde{\nu}(k)(t-t_0)} \quad (41)$$

conditionally to the event $x_k^e(t) \in \Omega^e(k)$. We conclude that $\mathbf{E}\{V_k^e(x_k^e(t))\} \to 0$ as $t \to \infty$ and $J_2(k) = V_k^e(x_0)$.

Since

$$\mathcal{L}V_k^e + \sum_{j=1}^{4} \mathcal{P}_j \geq 0 \quad (42)$$

for any other $F(k)$ and $G(k)$, we derive (a).

Finally, (b) and (c) follow as in [2]. On the other hand, if in addition, $c_j = 0$ for all j and $K(t, x) = 0$ for all t and x, from (41) we conclude the (Ω^e, α) stability in quadratic mean of $\Sigma \circ \mathcal{C}(k)$. □

Remark 4.1 (*Linear case*). Consider the class of systems (1) with $H_j(t, x) = H_j x$ and $K_j(t, x) = K_j x$ for all $j = 1, \ldots, s$, x and t and, in addition, with $\Phi(t, u, x)$ satisfying $\widetilde{\mathcal{P}}_1 \geq 0$ *for all* x, u and t. With admissible $\eta(s) = s$ (i.e. *linear controllers*) and $\varphi(s) = s$ (i.e. *quadratic* Lyapunov functions), we recover the stabilization result of [9]. Moreover, if $H_j = 0$ and $K_j = 0$ for all j (i.e. deterministic case) then the constraint on $\mathcal{P}_3$ is trivially satisfied (see [1]). □

Remark 4.2 (*Feedback linearizable systems*). We remark (without proof) that the conditions of theorem 4.1 can be satisfied with arbitrarily large region of attraction and small target set for the following class of systems

$$\begin{aligned} \dot{x} &= Ax + B(u + h(t, x)\dot{w} + \Phi(t, u, x)) \\ y &= Cx \end{aligned} \quad (43)$$

with (A, B, C) invertible with no invariant zeroes, $\Phi(t, u, x)$ and $h(t, x)$ norm–bounded from above by a locally Lipschitz function of x and u, uniformly w.r.t. t. It turns out that *control saturations* and *high gain observers* are instrumental for doing this exactly as in the case of deterministic systems ([5], [7]). □

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