Some remarks on dynamic output feedback control of non uniformly observable systems

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Abstract

This paper presents results on dynamic output feedback control for a class of systems which are in cascade form and a priori *are not* uniformly observable. These results are based on previous studies on observer design for the considered class of systems on the one hand, and similar arguments as in the case of *uniformly observable* systems on the other hand. Some further extensions on this basis are also discussed.

Keywords: Cascade systems, non uniformly observable systems, output feedback control, high gain observers.

1 Introduction and motivation

It is well-known that unlike for linear systems, separate possible designs for state feedback and state observers for a *nonlinear* system do not systematically result in some dynamic output feedback controller with the same stability properties.
However, it has been shown a few years ago how for systems with a *strong* observability property (namely *uniform observability* [1]), separate designs with *global* stability result in *semiglobal* stabilizability by dynamic output feedback [2, 3, 4], which roughly means that for any compact set, the stability can be made global w.r.t. this set. Notice that in those works, the uniform observability basically excludes singularities in the observation of the system.
In the present note, we show how by using similar techniques, namely high gain observers (e.g. as in [5]) and saturations (e.g. as in [2]), we can extend the result to cascade nonlinear systems which in general do not satisfy the same observability property, and for which the problem of observer design has been previously studied for instance in [6, 7]. The required observability property here is weaker, in the sense that it allows possible singularities. However, using convergence properties of the first subsystem, we show how this problem of singularity can be overcome.

The main ideas are presented in section 2, and possible extensions are discussed in section 3. Some conclusions end the paper in section 4.

2 Main result

As an illustrative case, let us here consider a class of interconnected systems described as follows:

$$\begin{array}{rcl} \dot{x}_1 & = & A_1x_1 + \varphi_1(x_1, u_1) \\ \dot{x}_2 & = & A_2(x_1, u_1)x_2 + \varphi_2(x_1, x_2, u_1, u_2) \\ y_1 & = & C_1x_1 \\ y_2 & = & C_2x_2 \end{array} \qquad (1)$$

with $x_i \in \mathbb{R}^{n_i}, y_i \in \mathbb{R}^{p_i}, u_i \in \mathbb{R}^{m_i}$ for $i = 1, 2$. Let $x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^n$, and x_{ij}, φ_{ij} denote the component j of x_i, φ_i, and assume that $\varphi_1, \varphi_2, A_2$ are smooth functions w.r.t. their arguments.

In [6, 7], we have studied how the possible design of a sub-observer for each sub-system assuming the state of the other subsystem is known, can result in the design of an observer for the whole system. Here, this study is prolonged by the problem of stabilization, extending previous results of [2, 3, 4].
The basic idea is that if the first subsystem can be (semiglobally) stabilized by output feedback for instance as in [3], and if the second subsystem can be stabilized by state feedback and satisfies some observability property, which here is weaker than "usual" uniform observability (in the sense that its

interconnexion with the first subsystem may induce *singularities for the observation* which are not a priori discarded), a semiglobally stabilizing dynamic output feedback controller can still be designed for the whole system provided that the first state feedback is "fast enough".
The required properties can be formulated as follows:

(H1) For any $t_1 > 0, \rho_1 > 0$, $\exists u_1 = k_1(x_1)$ and $u_2 = k_2(x_1, x_2)$ such that:

(i) $k_1(0) = k_2(0,0) = 0$;

(ii) the origin of the closed loop system is globally asymptotically stable;

(iii) $\forall t \geq t_1$, $\|x_1(t)\| \leq \rho_1$.

This means some *state feedback stabilizability with partial arbitrary rate of convergence* (namely for x_1).

(H2) $A_1 = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix}$ and $\dfrac{\partial \varphi_{1i}}{\partial x_{1j}} = 0$ for $i = 1$ to n_1, and $j = i+1$ to n_1 (*uniform observability for the subsystem in* x_1).

(H3) The following system:

$$\begin{array}{rcl} \dot{x}_2 & = & A_2(0,0)x_2 + \varphi_2(0, x_2, 0, u_2) \\ y_2 & = & C_2 x_2 \end{array} \quad (2)$$

is uniformly observable, in the sense that it can be turned into a form satisfying (H2) (which means some *"ultimate" uniform observability for the subsystem in* x_2).

Notice that (H3) does not exclude cases when x_1, u_1 can be singular for the estimation of x_2.
We then claim the following:

Theorem 2.1 *Under assumptions (H1), (H2), (H3), the system (1) can be semiglobally stabilized by dynamic output feedback.*

■

The proof can be achieved by using the same techniques as in [2, 3], that is high gain observers and saturations w.r.t. any given initial compact set:
the observer for x_1 is computed as usual [5], except that estimates are saturated w.r.t. the considered initial set Ω, i.e.:

$$\begin{array}{rcl} \dot{\hat{x}}_1 & = & A_1\hat{x}_1 + \varphi_1(sat(\hat{x}_1), u_1) - S_{\theta_1}^{-1} C_1^T (C_1 \hat{x}_1 - y_1) \\ S_{\theta_1} & = & -\frac{1}{\theta_1}(A_1^T S_{\theta_1} + S_{\theta_1} A_1 - C_1^T C_1) \end{array} \quad (3)$$

where $sat(\sigma) = \sigma$ if $\|\sigma\| < M$ and $\frac{\sigma}{\|\sigma\|}M$ otherwise, for some M depending on Ω.
On the other hand, the observer for x_2 is based on (2), by considering $z = Px_2$ such that (2) is turned into the "canonical" form of (H2) for some $\bar{A}_2$ and $\bar{\varphi}_2$:

$$\begin{array}{rcl} \dot{\hat{z}} & = & \bar{A}_2\hat{z} + \bar{\varphi}_2(P\hat{x}_2, u_2) - S_{\theta_2}^{-1} C_2^T (C_2 P^{-1}\hat{z} - y_2) \\ S_{\theta_2} & = & -\frac{1}{\theta_2}(\bar{A}_2^T S_{\theta_2} + S_{\theta_2}\bar{A}_2 - C_2^T C_2) \\ \hat{x}_2 & = & sat(P^{-1}\hat{z}). \end{array} \quad (4)$$

Then the result for the first subsystem is obtained in the same way as in [2, 3], with the additional property that x_1 can be made arbitrarily close to zero, arbitrarily fast: roughly speaking, it can be shown that for any compact set of initial conditions, one can choose M and θ_1 so as to make the estimation error $e_1 = \hat{x}_1 - x_1$ as small as desired before x_1 leaves some given compact set, and from properties of the state feedback, it can then be guaranteed that x_1 is still driven to zero when using the estimate of x_1 instead of x_1 in the feedback.
As for the second subsystem, the dynamics of the observer error $e_2 = \hat{z} - z$ are given by:

$$\begin{array}{rcl} \dot{e}_2 & = & (\bar{A}_2 - S_{\theta_2}^{-1} C_2^T C_2)e_2 \\ & & + \bar{\varphi}_2(P\hat{x}_2, u_2) - P\varphi_2(x_1, x_2, u_1, u_2) \\ & & + [\bar{A}_2 - PA_2(x_1, u_1)P^{-1}]Px_2. \end{array}$$

This equation is thus similar to that satisfied by e_1, up to some additional term which goes to zero as x_1 goes to zero, as long as x_2 is bounded.
The idea is then to appropriately tune the convergence speed of the first subsystem, so as to basically "fast enough" recover the same property for e_2 as that of e_1 mentioned above, and conclude in the same way. □

From this sketch of a proof, it can be seen that apart from state feedback stabilizability and observability properties, an important feature for the whole system to be semiglobally stabilizable by output feedback is the "arbitrarily" fast practical stabilization of the first subsystem by output feedback.
Moreover, it can be noticed that (H3)-(iii) can actually be weakened into:

$$\forall t \geq t_1, \|A_2(x_1(t), k_1(x_1(t))) - A_2(0,0)\| \leq \rho_1. \quad (5)$$

Hence we can more generally state the following:

Theorem 2.2 *Consider a system of the form:*

$$\begin{array}{rcll} \dot{x}_1 & = & f_1(x_1, u_1) & (6) \\ \dot{x}_2 & = & A_2(x_1, u_1)x_2 + \varphi_2(x_1, x_2, u_1, u_2) & (7) \\ y_1 & = & h_1(x_1) & (8) \\ y_2 & = & C_2 x_2 & (9) \end{array}$$

such that:

(I) For any $\rho_1, t_1 > 0$, there exists a dynamic output feedback $u_1(y_1, t)$ vanishing at the origin such that the origin of (6) under u_1 is globally asymptotically stable and $\forall t \geq t_1$, $\|A_2(x_1(t), u_1(y_1(t), t)) - A_2(0,0)\| \leq \rho_1$;

(II) There exists a state feedback $u_2(x_1, x_2)$ vanishing at the origin such that the origin of (6)-(7) under u_1, u_2 is globally asymptotically stable;

(III) (H3) holds.

Then the whole system is semiglobally stabilizable by dynamic output feedback.

■

As a simple example, one can consider the following system:

$$\begin{array}{rcl} \dot{\xi}_1 & = & u_1 \xi_1^2 \\ \dot{\xi}_2 & = & cos(u_1)\xi_3 + u_2 \\ \dot{\xi}_3 & = & \xi_2 - \xi_3^3 \\ y_1 & = & \xi_1 \\ y_2 & = & \xi_2 \end{array} \quad (10)$$

This system is obviously under the form (1) with $x_1 = \xi_1$ and $x_2 = (\xi_2 \ \ \xi_3)^T$.
Moreover, it can be seen that $u_1 = \frac{\pi}{2} + k\pi$ for any integer k is singular for the observation of ξ_3.
However, for any $\rho_1, t_1 > 0$, one can choose $\lambda_1 > 0$ such that (I) of theorem 2.2 is satisfied with $u_1 = -\lambda_1 y_1$. Moreover, by using this control together with $u_2 = -cos(\lambda_1 y_1)\xi_3 - \lambda_2 \xi_2 - \xi_3$ condition (II) is also satisfied, while (III) obviously holds. Thus, a semiglobally stabilizing output feedback can be designed, on the basis of some "saturated" high gain observer for the second subsystem.

Remark 2.1 *If the state was known to start out of a singular trajectory for the estimation of x_2, one could imagine to relax condition (H1)-(iii) and rather estimate x_2 fast enough before reaching any singularity. But this is restrictive on initial conditions, and this is no longer possible in the more delicate case when the equilibrium for x_1 is singular for the estimation of x_2. This case is briefly discussed in next section.*

3 Comments and extensions

In view of the proof of theorem 2.1, the formulation of theorem 2.2 and remark 2.1, the ideas of theorem 2.1 can be extended as follows:

• First, notice that if the equilibrium of the first subsystem is singular for the observation of the second subsystem, output feedback stabilization can still be achieved if the conditions are for instance modified in the following way:

(H1') For any bounded δ, and any $t_1 > 0, \rho_1 > 0$, $\exists u_1 = k_1(x_1)$ and $u_2 = k_2(x_1, x_2)$ such that:

(i) $k_1(0) = k_2(0,0) = 0$;

(ii) the origin of the system:

$$\begin{array}{rcl} \dot{x}_1 & = & A_1 x_1 + \varphi_1(x_1, u_1 + \delta) \\ \dot{x}_2 & = & A_2(x_1, u_1 + \delta)x_2 \\ & & +\varphi_2(x_1, x_2, u_1 + \delta, u_2) \\ y_1 & = & C_1 x_1 \\ y_2 & = & C_2 x_2 \end{array} \quad (11)$$

in closed loop with u_1, u_2 is globally asymptotically stable;

(iii) $\forall t \geq t_1$, $\|A_2(x_1(t), k_1(x_1(t))) - A_2(0,0)\| \leq \rho_1$.

This means some *robust state feedback stabilizability with partial arbitrary rate of convergence.*

(H2) $A_1 = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix}$ and $\dfrac{\partial \varphi_{1i}}{\partial x_{1j}} = 0$ for $i = 1$ to n_1, and $j = i+1$ to n_1 (*uniform observability for the subsystem in* x_1).

(H3') $\exists \varepsilon > 0$ such that the following system:

$$\begin{array}{rcl} \dot{x}_2 & = & A_2(0, \varepsilon)x_2 + \varphi_2(0, x_2, \varepsilon, u_2) \\ y_2 & = & C_2 x_2 \end{array} \quad (12)$$

is uniformly observable, in the sense that it can be turned into a form satisfying (H2) (which means some *disturbed ultimate uniform observability for the subsystem in* x_2).

As an illustrative example, one can again consider (10) but now with $\dot{\xi}_2 = u_1 \xi_3 + u_2$ instead of $\dot{\xi}_2 = cos(u_1)\xi_3 + u_2$.
In this case, $u_1 = -\lambda_1 \xi_1$ still drives ξ_1 to zero, but this makes u_1 become singular for the estimation of ξ_2.
However, one can check that (H1') - and clearly (H3') - are satisfied, and thus the output feedback stabilizability can still be achieved.

As a further example, one can consider the following system:

$$\begin{array}{rcl} \dot{\xi}_1 & = & \xi_2 - \xi_1 \\ \dot{\xi}_2 & = & u_1 \xi_2^2 - \xi_1 - \xi_2 \\ \dot{\xi}_3 & = & u_1 \xi_4 \\ \dot{\xi}_4 & = & u_2 \\ y_1 & = & \xi_1 \\ y_2 & = & \xi_3 \end{array} \quad (13)$$

System (13) can indeed be seen under the form (1) with $x_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$, and $x_2 = \begin{pmatrix} \xi_3 \\ \xi_4 \end{pmatrix}$. Obviously the first subsystem is globally asymptotically stable when $u_1 \equiv 0$, but this control value is clearly singular for the second subsystem.
One can yet check that for any initial condition, $u_1 = -\lambda\xi_2$ drives $x_1(t)$ to zero (use $V(x_1) = \frac{1}{2}x_1^T x_1$ as a Lyapunov function), and that with such a control, $\xi_2(t)$ can moreover be made arbitrarily small in an arbitrarily short time by appropriately tuning $\lambda > 0$ (consider the dynamics of ξ_2 together with $V_2 = \frac{1}{2}\xi_2^2$). Those properties remain unchanged if u_1 is replaced by $u_1 + \delta$ for any $\delta \neq 0$ and thus (H1') w.r.t u_1 is satisfied, together with (H3'), and obviously (H2).

Notice that in this example, the singularity for the second subsystem a priori also holds for the state feedback design, and is again overcome thanks to (H1').

• Secondly, notice that condition (H1)–(iii) of theorem 2.1 is basically required to overcome the problem of singularities in the observation of the second subsystem, by ensuring that (x_1, u_1) will not remain singular for a too long time.
It could thus be directly replaced by the condition that $(x_1(t), k_1(x_1(t))$ resulting from the first subsystem remains "sufficiently exciting" for the second subsystem, on the basis of available definitions of such regular inputs for observation problems (see e.g. [8, 6]). In this case, the observer for the second subsystem would require a time-varying gain $S^{-1}(t)C^T$ where S is obtained as the solution of a Lyapunov-like differential equation, as in Kalman-like observers [8, 6], and the conclusion would follow in the same way.

• Finally, notice that the same result holds for any system of the form:

$$\begin{array}{rcl} \dot{x}_1 &=& f_1(x_1, u_1) \\ \dot{x}_2 &=& f_2(x_2, x_1, u_1, u_2) \\ y_1 &=& h_1(x_1) \\ y_2 &=& h_2(x_2) \end{array} \quad (14)$$

where the first subsystem satisfies similar conditions to (I) of theorem 2.2, and the second subsystem admits some state observer with arbitrary rate of convergence, and some stabilizing state feedback.

4 Conclusions

In this paper we have discussed some problems arising in output feedback stabilization of nonlinear systems which are not a priori uniformly observable, and presented some conditions for possible solutions.
Some generalizations of the ideas which have been here presented are currently under investigation.

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