On preservation of dissipation inequalities under sampling

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Abstract

We show that if we first design a controller for a continuous-time nonlinear plant with disturbances so that it achieves a certain dissipation inequality for the continuous-time closed-loop system and then implement it as a sampled-data controller using a sampler and zero order hold, then the dissipation inequality will be preserved for the exact discrete-time model of the sampled-data closed-loop system in a semiglobal practical sense (the sampling period is the parameter that we can adjust). Moreover, a similar statement is proved for open-loop systems, where controls are considered as free variables. Two different forms of dissipation inequalities are considered for the exact discrete-time models: the "weak" form and the "strong" form.

1 Introduction

An important approach to the design of sampled-data controllers, which we refer to below as the continuous-time design (CTD) method, consists of a two-step design procedure. In the first step, a continuous-time controller is designed for the continuous-time plant using some of the continuous-time controller design techniques. At this stage, the sampling is completely ignored. In the second step, the controller is implemented using a sampler and zero order hold with sufficiently fast sampling (see, for instance [1, 7, 8, 11, 13]). The available results in the literature on the CTD method addressed the design of stabilizing control laws for linear [1] and nonlinear [7, 8, 13] plants, L_p stabilizing controllers for linear systems [1] and input-to-state stabilizing (ISS) controllers for nonlinear systems [11] (for more details on ISS, see [9]).

Although the above results appear to be very diverse, there are two common ideas in all of them: first, the continuous-time controller is designed for the continuous-time plant so that it assigns a certain type of a dissipation inequality to the continuous-time closed-loop system; second, it is shown using arguments that rely on continuity of solutions of differential equations that the property which follows from the dissipation inequality is preserved in some weaker sense (in general semiglobal practical) for the closed-loop sampled-data system for sufficiently fast sampling.

It is the purpose of this paper to unify and generalize the known results in the literature, by considering the preservation of general dissipation inequalities under sampling for a rather general class of nonlinear systems (for more details on different dissipation inequalities, see [2, 3, 4, 5, 9, 11, 12] and references therein). Our results are applicable to either full or partial (output) static state feedback and to any property that can be formulated in terms of dissipation inequalities, such as stability, L_p stability, passivity, input-to-state stability, etc. Hence, in this paper we present a rather general and unified framework for the CTD method.

The paper is organized as follows. In Section 2 we present some notation, definitions and preliminaries. The main results of the paper are stated and proved in Section 3. In Section 4 we apply our results to preservation of ISS and passivity under sampling.

2 Preliminaries

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and is unbounded. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class-$\mathcal{K}$ for each $\tau \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$.

Consider the continuous-time nonlinear plant:

$$\dot{x} = f(x, u, d_c, d_s), \quad (1)$$

$$y = h(x, u, d_c, d_s), \quad (2)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the state, control input and the output of the system and $d_c \in \mathbb{R}^{n_c}$ and $d_s \in \mathbb{R}^{n_s}$ are disturbance inputs to the system. It is assumed that f is locally Lipschitz, h is continuous and that $f(0,0,0,0) = 0$ and $h(0,0,0,0) = 0$.

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Besides the continuous-time model (1), (2), we would consider the exact discrete-time model for (1), (2) when some of the variables in function f of (1), (2) are sampled or assumed piecewise constant. More precisely, let $T > 0$ be a sampling period and suppose that u and d_s in f in (1) are constant during the sampling intervals, so that $u(t) = u(kT) =: u(k)$ and $d_s(t) = d_s(kT) =: d_s(k), \forall t \in [kT, (k+1)T), \forall k \geq 0$, and y is measured only at sampling instants kT, $k \geq 0$. Given a function $d(t)$, we use the following notation $d[t_1, t_2] := \{d(t) : t \in [t_1, t_2]\ \}$. If $t_1 = kT, t_2 = (k+1)T$, we use the shorter notation $d[k]$, with norm $\|d[k]\|_\infty = \sup_{\tau \in [kT,(k+1)T]} |d(\tau)|$. The exact discrete-time model for the system (1), (2) is obtained by integrating the initial value problem

$$\dot{x}(t) = f(x(t), u(k), d_c(t), d_s(k)), \qquad (3)$$

with given $d_s(k)$, $d_c[k]$, $u(k)$ and $x_0 = x(k)$, over the sampling interval $[kT, (k+1)T]$. Let $x(t)$ denotes the solution of the initial value problem (3) with given $d_s(k)$, $d_c[k]$, $u(k)$ and $x_0 = x(k)$. Then, we can write the exact discrete-time model for (1), (2) as:

$$\begin{aligned} x(k+1) &= x(k) + \\ &\quad \int_{kT}^{(k+1)T} f(x(\tau), u(k), d_c(\tau), d_s(k)) d\tau \\ &:= F_T(x(k), u(k), d_c[k], d_s(k)) \ , \qquad (4) \\ y(k) &= h(x(k), u(k), d_c(k), d_s(k)) \ . \end{aligned}$$

The sampling period T is assumed to be a design parameter which can be arbitrarily assigned. In practice, the sampling period T is fixed and our results could be used to determine if it is suitably small. We emphasize that F_T in (4) is not known in most cases.

Definition 2.1 *The system (1), (2) is said to be (V, w)-dissipative if there exist a continuously differentiable function V, called the storage function, and a continuous function $w : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_s} \to \mathbb{R}$, called the dissipation rate, such that for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m, d_c \in \mathbb{R}^{n_c}, d_s \in \mathbb{R}^{n_s}$ the following holds:*

$$\frac{\partial V}{\partial x} f(x, u, d_c, d_s) \leq w(x, u, d_c, d_s). \qquad (5)$$

■

3 Main results

In this section we state and prove the main results on preservation of dissipation inequalities under sampling. The first and second main results (Theorem 3.1 and 3.2) are concerned, respectively, with the weak and strong forms of dissipation inequalities for the discrete-time model. A number of corollaries are stated without proof. We denote $d_c := d_c(0)$ and use it in the sequel.

Theorem 3.1 *(Weak form of dissipation) If the system (1), (2) is (V, w)-dissipative, then given any 6-tuple of strictly positive real numbers $(\Delta_x, \Delta_u, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s}, \nu)$, there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and all $|x| \leq \Delta_x$, $|u| \leq \Delta_u$, $|d_s| \leq \Delta_{d_s}$ and functions $d_c(t)$ that are Lipschitz and satisfy $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ and $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$, the following holds for the exact discrete-time model (4) of the system (1), (2):*

$$\begin{aligned} \frac{\Delta V}{T} &:= \frac{V(F_T(x, u, d_c[0], d_s)) - V(x)}{T} \\ &\leq w(x, u, d_c, d_s) + \nu \ . \end{aligned} \qquad (6)$$

■

Proof of Theorem 3.1: Suppose that the continuous time system (1), (2) is (V, w)-dissipative, that is for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m, d_c \in \mathbb{R}^{n_c}, d_s \in \mathbb{R}^{n_s}$ the inequality (5) holds. Let a 6-tuple of strictly positive real numbers (Δ_x, Δ_u, Δ_{d_c}, $\Delta_{\dot{d}_c}$, Δ_{d_s}, ν) be given. Let $R := \Delta_x + 1$, let $L > 0$ be the Lipschitz constant of $f(x, u, d_c, d_s)$ for all $|x| \leq R$, $|u| \leq \Delta_u$, $|d_c| \leq \Delta_{d_c}$, $|d_s| \leq \Delta_{d_s}$ and let $b > 0$ be a number that satisfies $\max\left\{\left|\frac{\partial V}{\partial x}\right|, |f(x, u, d_c, d_s)|\right\} \leq b$ for all $|x| \leq R$, $|u| \leq \Delta_u$, $|d_c| \leq \Delta_{d_c}$, $|d_s| \leq \Delta_{d_s}$.

Since f is bounded by b for $|x| \leq R$, $|u| \leq \Delta_u$, $|d_c| \leq \Delta_{d_c}$, $|d_s| \leq \Delta_{d_s}$, there exists a number T_1^* (in particular, we can take $T_1^* = b^{-1}$), such that for all $|x| \leq \Delta_x$, $|u| \leq \Delta_u$, $|d_s| \leq \Delta_{d_s}$, $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, and $T \in (0, T_1^*)$ the solution $x(t)$ of the initial value problem:

$$\dot{x}(t) = f(x(t), u, d_c(t), d_s),$$

with given d_s, $d_c(t)$, u and $x_0 = x$, exists and $|x(t)| \leq R, \forall t \in [0, T]$, which implies $|x(T)| \leq R$. Let $T_1^* = b^{-1}$ and consider arbitrary $|x| \leq \Delta_x$, $|u| \leq \Delta_u$, $|d_s| \leq \Delta_{d_s}$, $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$. By adding and subtracting $V(x + T f(x, u, d_c, d_s))/T$ and $\left.\frac{\partial V}{\partial x}\right|_x f(x, u, d_c, d_s)$

to $\Delta V/T$, we can write

$$\frac{\Delta V}{T} = \underbrace{\frac{\partial V}{\partial x}\bigg|_x f(x,u,d_c,d_s)}_{\mathbf{1}}$$

$$+ \underbrace{\frac{1}{T}\{V(F_T) - V(x + Tf(x,u,d_c,d_s))\}}_{\mathbf{2}}$$

$$+ \underbrace{\frac{1}{T}\{V(x + Tf(x,u,d_c,d_s)) - V(x)}_{\mathbf{3}\cdots}$$

$$\underbrace{-T\,\frac{\partial V}{\partial x}\bigg|_x f(x,u,d_c,d_s)\bigg\}}_{\cdots\mathbf{3}} . \tag{7}$$

We show now that we can further reduce T so that $\frac{\Delta V}{T}$ in (7) can be bounded as presented in (6), by considering each term on the right-hand side of (7) separately.

Term **1**: It follows from (V,w)-dissipativity of the continuous time system (1), (2) that:

$$\frac{\partial V}{\partial x}\bigg|_x f(x,u,d_c,d_s) \le w(x,u,d_c,d_s). \tag{8}$$

Term **2**: Using the Mean Value Theorem and local Lipschitz property of f we can write for all $|x| \le R$, $|u| \le \Delta_u$, $|d_c| \le \Delta_{d_c}$, $|d_s| \le \Delta_{d_s}$:

$$\begin{aligned}\frac{1}{T}\left|\frac{\partial V}{\partial x}\bigg|_{x_1}\right| \int_0^T &|f(x(\tau),u,d_c(\tau),d_s) \\ &-f(x,u,d_c,d_s)|\,d\tau \\ &\le \frac{bL}{T}\int_0^T (|x(\tau)-x| + |d_c(\tau)-d_c|)\,d\tau,\end{aligned} \tag{9}$$

where we have used the fact that $x_1 = x + \theta\int_0^T f(x(\tau),u,d_c(\tau),d_s)d\tau + (1-\theta)Tf(x,u,d_c,d_s)$, where $\theta \in (0,1)$, and since $\max\{|x(T)|, |x + Tf(x,u,d_c,d_s)|\} \le R$, then $|x_1| \le R$ which finally yields that $\left|\frac{\partial V}{\partial x}\big|_{x_1}\right| \le b$. Since f is locally Lipschitz, we can further write for all $|x| \le R$, $|u| \le \Delta_u$, $|d_c| \le \Delta_{d_c}$, $|d_s| \le \Delta_{d_s}$:

$$\begin{aligned}&|x(\tau) - x| \\ &\le (\Delta_x + \Delta_u + \Delta_{d_c} + \Delta_{d_s})[\exp(L\tau) - 1],\end{aligned} \tag{10}$$

$\forall \tau \in [0,T]$ and hence we can write

$$\begin{aligned}&\frac{1}{T}\int_0^T |x(\tau) - x|\,d\tau \\ &\le (\Delta_x + \Delta_u + \Delta_{d_c} + \Delta_{d_s})\frac{\exp(LT) - LT - 1}{LT} \\ &\le (\Delta_x + \Delta_u + \Delta_{d_c} + \Delta_{d_s})KT\ ,\end{aligned} \tag{11}$$

$\forall T \in (0, T_1^*)$, for some $K > 0$. Since $d_c(t)$ is globally Lipschitz, we have that $|d_c(t) - d_c| \le \Delta_{\dot{d}_c} t$. Hence

$$\frac{1}{T}\int_0^T |d_c(\tau) - d_c|\,d\tau \le \frac{\Delta_{\dot{d}_c}T}{2}\ . \tag{12}$$

Using (11) and (12), for all $T \in (0, T_1^*)$, we can bound Term **2** by:

$$\underbrace{bL\left[(\Delta_x + \Delta_u + \Delta_{d_c} + \Delta_{d_s})K + \frac{\Delta_{\dot{d}_c}}{2}\right]}_{M_1} T\ .$$

Hence, if we choose $T_2^* := \min\{T_1^*, \nu/(2M_1)\}$, then Term **2** is bounded by $\nu/2$ for all $T \in (0, T_2^*)$.

Term **3**: Using the Mean Value Theorem and definition of b, we can write:

$$\begin{aligned}\frac{1}{T}\bigg|V(x + Tf(x,u,d_c,d_s)) - V(x) - T\,\frac{\partial V}{\partial x}\bigg|_x \\ \cdot f(x,u,d_c,d_s)\bigg| \le b\left|\frac{\partial V}{\partial x}\bigg|_{x_2} - \frac{\partial V}{\partial x}\bigg|_x\right|\ ,\end{aligned} \tag{13}$$

where $x_2 = x + T\theta f(x,u,d_c,d_s)$ for some $\theta \in (0,1)$. Finally, since the gradient of V is continuous, it is uniformly continuous on compact sets, and since $|x_2 - x| \le T\theta\,|f(x,u,d_c,d_s)| \le Tb$, it follows that given any $\epsilon > 0$ there exists $\hat{T} > 0$ such that $\left|\frac{\partial V}{\partial x}\big|_{x_2} - \frac{\partial V}{\partial x}\big|_x\right| \le \epsilon$, $\forall T \in (0,\hat{T})$, $|x| \le R$, $|u| \le \Delta_u$, $|d_c| \le \Delta_{d_c}$ and $|d_s| \le \Delta_{d_s}$. Choose $\epsilon := \nu/(2b)$ and let this $\hat{T}$ fix. Choose $T_3^* = \min\{T_1^*, \hat{T}\}$ and then we have that for all $T \in (0, T_3^*)$ Term **3** is bounded by $\nu/2$.

Choosing $T^* := \min\{T_2^*, T_3^*\}$ and combining the bounds for Terms **1**-**3** completes the proof. ∎

Under slightly stronger conditions we can prove a stronger result that is useful in some situations:

Lemma 3.1 *If the system (1), (2) is (V,w)-dissipative, with $\frac{\partial V}{\partial x}$ being locally Lipschitz and $\frac{\partial V}{\partial x}(0) = 0$, then given any quintuple of strictly positive real numbers (Δ_x, Δ_u, Δ_{d_c}, $\Delta_{\dot{d}_c}$, Δ_{d_s}), there exist $T^* > 0$ and positive constants K_1, K_2, K_3, K_4, K_5 such that for all $T \in (0, T^*)$ and all $|x| \le \Delta_x$, $|u| \le \Delta_u$, $|d_s| \le \Delta_{d_s}$ and functions $d_c(t)$ that are Lipschitz and satisfy $\|d_c[0]\|_\infty \le \Delta_{d_c}$, $\left\|\dot{d}_c[0]\right\|_\infty \le \Delta_{\dot{d}_c}$, we have:*

$$\begin{aligned}&\frac{V(F_T(x,u,d_c[0],d_s)) - V(x)}{T} \le w(x,u,d_c,d_s) \\ &+ T\big(K_1|x|^2 + K_2|u|^2 + K_3|d_s|^2 + K_4\|d_c[0]\|_\infty^2 \\ &+ K_5\left\|\dot{d}_c[0]\right\|_\infty^2\big)\ .\end{aligned} \tag{14}$$

∎

It is of utmost importance to state and prove Theorem 3.1 for the case when a feedback controller:

$$u = u(x, d_c, d_s) \tag{15}$$

is applied to the system (1), (2). It is assumed below that the feedback (15) is bounded on compact sets. Note that this general form of feedback covers both, the full state ($u = u(x)$) and output ($u = u(y)$) static feedback. Note also, that the dissipation rate for the closed-loop system (1), (2) and (15) in the definition of (V, w)-dissipativity can be taken as $w = w(x, d_c, d_s)$. Direct consequences of Theorem 3.1 and Lemma 3.1 are given below:

Corollary 3.1 *If the system (1), (2), (15) is (V, w)-dissipative, then given any quintuple of strictly positive real numbers $(\Delta_x, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s}, \nu)$, there exists $T^* > 0$ such that $\forall T \in (0, T^*)$ and all $|x| \le \Delta_x$, $|d_s| \le \Delta_{d_s}$ and all functions $d_c(t)$ that are Lipschitz and satisfy $\|d_c[0]\|_\infty \le \Delta_{d_c}$, $\left\|\dot{d}_c[0]\right\|_\infty \le \Delta_{\dot{d}_c}$, the following holds for the discrete-time model of the closed-loop system (1), (2), (15):*

$$\frac{V(F_T(x, u(x, d_c, d_s), d_c[0], d_s)) - V(x)}{T} \le w(x, d_c, d_s) + \nu \ . \tag{16}$$

■

Corollary 3.2 *If the system (1), (2), (15) is (V, w)-dissipative, with $\frac{\partial V}{\partial x}$ and $u(x, d_c, d_s)$ in (15) being locally Lipschitz and $\frac{\partial V}{\partial x}(0) = 0$ and $u(0, 0, 0) = 0$, then given any quadruple of strictly positive real numbers $(\Delta_x, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s})$, there exists $T^* > 0$ and positive constants K_1, K_2, K_3, K_4 such that $\forall T \in (0, T^*)$ and all $|x| \le \Delta_x$, $|d_s| \le \Delta_{d_s}$ and functions $d_c(t)$ that are Lipschitz and satisfy $\|d_c[0]\|_\infty \le \Delta_{d_c}$ and $\left\|\dot{d}_c[0]\right\|_\infty \le \Delta_{\dot{d}_c}$, the closed-loop discrete-time model for system (1), (2) and (15) satisfies:*

$$\frac{V(F_T(x, u(x, d_c, d_s), d_c[0], d_s)) - V(x)}{T} \le w(x, d_c, d_s) + T\Bigg(K_1|x|^2 + K_2|d_s|^2 + K_3\,\|d_c[0]\|_\infty^2 + K_4\left\|\dot{d}_c[0]\right\|_\infty^2\Bigg) \ . \tag{17}$$

■

In the following theorem we use the strong form of dissipation inequality for the exact discrete-time model. This result is much more natural to use in the situations when the disturbances d_c are not globally Lipschitz (see the ISS application in the next section and Example 3.1).

Theorem 3.2 *(Strong form of dissipation) If the system (1), (2) is (V, w)-dissipative, then given any quintuple of strictly positive real numbers $(\Delta_x, \Delta_u, \Delta_{d_c}, \Delta_{d_s}, \nu)$ there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and all $|x| \le \Delta_x$, $|u| \le \Delta_u$, $\|d_c[0]\|_\infty \le \Delta_{d_c}$, and $|d_s| \le \Delta_{d_s}$ the following holds for the system (4):*

$$\frac{V(F_T(x, u, d_c[0], d_s)) - V(x)}{T} \le \frac{1}{T}\int_0^T w(x, u, d_c(\tau), d_s)d\tau + \nu \ . \tag{18}$$

■

In order to state the next two results we need to consider controllers of the following form:

$$u = u(x, d_s) \ . \tag{19}$$

Corollary 3.3 *If the system (1), (2), (19) is (V, w)-dissipative, then given any quadruple of strictly positive real numbers $(\Delta_x, \Delta_{d_c}, \Delta_{d_s}, \nu)$, there exists $T^* > 0$ such that $\forall T \in (0, T^*)$ and all $|x| \le \Delta_x$, $\|d_c[0]\|_\infty \le \Delta_{d_c}$, and $|d_s| \le \Delta_{d_s}$ the following holds for the closed-loop discrete-time model of the system (1), (2) and (19):*

$$\frac{V(F_T(x, u(x, d_s), d_c[0], d_s)) - V(x)}{T} \le \frac{1}{T}\int_0^T w(x, d_c(\tau), d_s)d\tau \ + \ \nu \ . \tag{20}$$

■

Remark 3.1 *A natural question is whether we can state and prove Corollary 3.3 when (19) is replaced by (15). The following example shows that this is not possible in general. Consider the system $\dot{x} = u$, and $u = -d_c$, where $d_c(0) = 0$ and $d_c(t) = 1$, $\forall t > 0$. Using $V(x) = x$, such that $\frac{\partial V}{\partial x}(-d_c) = -d_c$ and $w(x, d_c, d_s) = -d_c$. Since u is sampled and $d_c(0) = 0$, we have that $x(t) = 0, \forall t \in [0, T]$ and so $\Delta V/T = 0$. On the other hand $\int_0^T w(d_c(\tau))d\tau = -T$. Hence, if (20) was true, then we would obtain obtain $0 \le -1 + \nu$, which is not true for small ν.* ■

Corollary 3.4 *If the system (1), (2), (19) is (V, w)-dissipative, with $\frac{\partial V}{\partial x}$ and $u(x, d_s)$ in (19) being locally Lipschitz and $\frac{\partial V}{\partial x}(0) = 0$, $u(0, 0) = 0$, then given any triple of strictly positive real numbers $(\Delta_x, \Delta_{d_c}, \Delta_{d_s})$, there exists $T^* > 0$ and* *positive constants K_1, K_2, K_3 such that for all $T \in (0, T^*)$ and all $|x| \leq \Delta_x$, $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, and $|d_s| \leq \Delta_{d_s}$ the closed-loop discrete-time model for the system (1), (2) and (19) satisfies:*

$$\frac{V(F_T(x, u(x, d_s), d_c[0], d_s)) - V(x)}{T} \leq \frac{1}{T}\int_0^T w(x, d_c(\tau), d_s) d\tau + T\left(K_1|x|^2 + K_2|d_s|^2 + K_3\|d_c[0]\|_\infty^2\right) . \quad (21)$$

■

It is interesting to investigate whether the condition on the derivative d_c in Theorem 3.1 is necessary to prove the dissipation inequality for the discrete-time system. The following example shows that indeed the condition is necessary if we want to state a general result.

Example 3.1 *Consider the continuous time system:*

$$\dot{x} = u(x) + d_c = -x + d_c, \quad (22)$$

where $x, d_c \in \mathbb{R}$. Using the storage function $V = \frac{1}{2}x^2$, the derivative of V is $\dot{V} = -x^2 + xd_c \leq -\frac{1}{2}x^2 + \frac{1}{2}d_c^2$, and (22) is ISS. We will show that if $d_c(t) = \cos\left(\frac{t+2T}{T}\right)$ the claim of Theorem 3.1 does not hold since

$$\left\|\dot{d}_c\right\|_\infty = \left\|-\frac{1}{T}\sin\left(\frac{t+2T}{T}\right)\right\|_\infty = \frac{1}{T}, \quad (23)$$

which goes to infinity as $T \to 0$. Assume that $u(x)$ in (22) is piecewise constant for the sampled-data system. So, the discrete-time model of the sampled-data system is

$$x(k+1) = (1-T)x(k) + \int_{kT}^{(k+1)T} \cos\left(\frac{\tau + 2T}{T}\right) d\tau \ , \quad (24)$$

and hence the exact discrete-time model is given by:

$$x(k+1) = (1-T)x(k) + T\left[\sin(k+3) - \sin(k+2)\right], \quad \forall k \geq 0. \quad (25)$$

Suppose that for any given Δ_x, Δ_{d_c} and ν, there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and $k \geq 0$ with $|x| \leq \Delta_x$ and $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ we have

$$\frac{\Delta V}{T} \leq -\frac{1}{2}x^2 + \frac{1}{2}d_c^2 + \nu. \quad (26)$$

We show by contradiction that the claim is not true for our case. Direct computations yield:

$$\frac{\Delta V}{T} = \frac{\left((1-T)x + T\left[\sin(3) - \sin(2)\right]\right)^2 - x^2}{2T} = -x^2 + x\left[\sin(3) - \sin(2)\right] + O(T). \quad (27)$$

Let $\tilde{x} = -0.5$, (and hence $\Delta_x = 0.5$, $\Delta_{d_c} = 1$). By combining (26) and (27) we conclude that there should exist $T^* > 0$ such that $\forall T \in (0, T^*)$ we obtain:

$$-\frac{1}{2}\tilde{x}^2 + \tilde{x}\left[\sin(3) - \sin(2)\right] - \frac{1}{2}\cos(2)^2 - \nu + O(T) \leq 0 \ , \quad (28)$$

and since there exists $\nu^* > 0$ such that $-\frac{1}{2}\tilde{x}^2 + \tilde{x}\left[\sin(3) - \sin(2)\right] - \frac{1}{2}\cos(2)^2 = \nu^*$ we can rewrite (28) as $\nu^* - \nu + O(T) \leq 0$, which is a contradiction (it does not hold for $\nu \in (0, \nu^*)$).
Therefore, for $\Delta_x = 0.5$, $\Delta_{d_c} = 1$, $\nu < \nu^*$, there exists no $T^* > 0$, such that $\forall T \in (0, T^*)$ the condition (26) holds. Note that the chosen $d_c(t)$ does not satisfy the condition $\left\|\dot{d}_c\right\|_\infty \leq \Delta_{\dot{d}_c}$ for some fixed $\Delta_{\dot{d}_c} > 0$, which is evident from (23). Hence, in this case we can not apply Theorem 3.1. ▲

4 Applications

We present now two applications of our results: input-to-state stability and passivity.

4.1 Input-to-state stability: Let us suppose that the system

$$\dot{x}(t) = \tilde{f}(x(t), u(t), d_c(t)) \quad (29)$$

can be rendered ISS using the locally Lipschitz static state feedback

$$u = u(x), \quad (30)$$

in the following sense:

Definition 4.1 *The system $\dot{x} = f(x, d_c)$ is input-to-state stable if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that the solutions of the system satisfy $|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|d_c\|_\infty)$, $\forall x(0), d_c \in \mathcal{L}_\infty, \forall t \geq 0$.* ■

Suppose also that the feedback needs to be implemented using a sampler and zero order hold, that is:

$$u(t) = u(x(k)) \quad t \in [kT, (k+1)T), \ k \geq 0 \quad (31)$$

The following result was proved in [11] and it can be reproved in a different way, using Corollary 3.3, converse ISS Theorem 1 in [10], Lemma 4 of [4] and results in Section 4.3 in [6]:

Corollary 4.1 *If the continuous time system (29), (30) is ISS, then there exist* $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}$ *such that given any triple of strictly positive numbers* $(\Delta_x, \Delta_{d_c}, \nu)$*, there exists* $T^* > 0$ *such that* $\forall T \in (0, T^*)$*,* $|x(t_0)| \leq \Delta_x$*,* $\|d_c\|_\infty \leq \Delta_{d_c}$*, the solutions of the sampled-data system (29), (31) satisfy:*

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(\|d_c\|_\infty) + \nu, \quad (32)$$

$\forall t \geq t_0 \geq 0$. ∎

4.2 Passivity: Consider the continuous time system with outputs

$$\dot{x} = f(x, u), \quad y = h(x, u), \quad (33)$$

where $x \in \mathbb{R}^n, y, u \in \mathbb{R}^m$ and assume that the system is passive, that is (V, w)-dissipative, where $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $w = y^T u$. We can apply either results of Theorem 3.1 or 3.2 since u is a piecewise constant input, to obtain that the discrete-time model satisfies for any $(\Delta_x, \Delta_u, \nu)$ there exists $T^* > 0$ such that $\forall T \in (0, T^*)$, $|x| \leq \Delta_x$, $|u| \leq \Delta_u$ we have:

$$\frac{\Delta V}{T} \leq y^T u + \nu. \quad (34)$$

In stability and ISS applications, adding ν in the dissipation inequality deteriorated the property, but the deterioration was gradual. However, in (34) ν acts as an infinite energy storage (finite power source) and hence it contradicts the definition of a passive system as one that can not generate power internally. As a result, conditions which guarantee that ν in (34) can be set to zero are very important. These conditions are spelled out in the next corollary:

Corollary 4.2 *Suppose that the system (33) is strictly input and state passive in the following sense: the strorage function has gradient* $\frac{\partial V}{\partial x}$ *that is locally Lipschitz and zero at zero and the dissipation rate can be taken as* $w(x, y, u) = y^T u - \psi_1(x) - \psi_2(u)$*, where* ψ_1 *and* ψ_2 *are positive definite functions that are locally quadratic. Then given any pair of strictly positive numbers* (Δ_x, Δ_u) *there exists* $T^* > 0$ *such that for all* $T \in (0, T^*)$*,* $|x| \leq \Delta_x$*,* $|u| \leq \Delta_u$ *we have:*

$$\frac{\Delta V}{T} \leq y^T u - \frac{1}{2}\psi_1(x) - \frac{1}{2}\psi_2(u). \quad (35)$$

∎

We emphasize that the above approach can be used for more general properties than passivity to cancel ν in the dissipation inequality for the discrete-time system.

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