

Global regulation and local robust stabilization of chained systems

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Abstract

In the present work a discontinuous control law for chained systems of order n that yields global asymptotic regulation and local exponential stability in the sense of Lyapunov is proposed. As a consequence of the Lyapunov stability property the new control law assures robustness against the effect of (small) measurement noise and of external disturbances. Simulation results complement the theoretical developments.

1 Introduction

The problem of asymptotic stabilization or regulation of nonholonomic systems has been widely studied in recent years. A possible stabilization *tool* has been introduced in [2], where discontinuous¹ control laws for chained systems has been designed using the so-called σ -process [1]. A similar idea has been developed independently in [20] and [15]. This methodology has been further used to cope with robustness issues and with more general classes of systems. In particular, the extension of this approach to high-order chained forms has been reported in [13] and some robustness considerations have been studied in [21]. Finally, the same ideas have been used in [11] to develop discontinuous control laws for a large class of nonlinear systems.

¹The term discontinuous is here used in a special sense, see [2] for detail.

Despite the substantial research effort, some fundamental problems still remain open. In particular, all the control laws presented in the above mentioned papers (see also reference therein) are non-robust against measurement noise. This means that for any initial condition \bar{x} the trajectory of the closed loop system with the nominal control $u(x)$ converges exponentially to zero, whereas, for any non-zero ε , the trajectory of the system with the perturbed control law $u(x+\varepsilon)$ is in general not close to the nominal trajectory and may diverge. Moreover, the control laws presented in these papers do not yield Lyapunov stability, but only exponential attractivity, and not all initial conditions are *driven* to the origin. This implies, in particular, that the presence of arbitrarily small exogenous disturbances may generate unbounded trajectory, see *e.g.* [4].

In the present work we partially address these issues. More precisely, we propose a simple modification of the control law considered in [2] for chained systems which yields global asymptotic regulation, local exponential stability in the sense of Lyapunov and local robustness against measurement errors and exogenous disturbances.

We focus our attention on n -dimensional chained systems with two controls (see [17] for detail), *i.e.* systems described by equations of the form

$$\begin{aligned}\dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= x_2 u_1, \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1.\end{aligned}\tag{1}$$

As observed in several research papers, system (1) is a

classical example of a system which is completely controllable but not asymptotically stabilizable by smooth (or even continuous) static or dynamic feedback laws. For this reason the problem of designing *stabilizing* control laws has attracted a lot of interest, see [2-12, 14, 16-19, 21] and references therein for further detail.

2 Preliminary results

In this section we discuss a few preliminary facts which are instrumental to prove the main results of the paper. Consider the system (1), the control law

$$\begin{aligned} u_1 &= u_{1A}(x) = -x_1 \\ u_2 &= u_{2A}(x) = p_2 x_2 + p_3 \frac{x_3}{x_1} + p_4 \frac{x_4}{x_1^2} + \cdots \\ &\quad \cdots + p_n \frac{x_n}{x_1^{n-2}}, \end{aligned} \quad (2)$$

with $\left(\frac{x_i}{x_1^{i-2}} \right) \Big|_{(0,0)} = 0$, and the matrix

$$A = \begin{bmatrix} p_2 & p_3 & p_4 & p_5 & \cdots & p_{n-1} & p_n \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & n-2 \end{bmatrix}. \quad (3)$$

It has been proved in [2] that the closed loop system (1)-(2) admits a unique forward solution for any initial condition $x(0)$ such that $x_1(0) \neq 0$, and that if

$$\sigma(A) \subset \mathbb{C}^- \quad (4)$$

such a solution converges *exponentially* to zero. However, more structure can be exposed as detailed in the following two simple facts.

Lemma 1 *Consider the system (1) with initial condition $x(0)$ such that $x_1(0) \neq 0$ and the control law (2). Let p_i be such that*

$$\sigma(A) \cap \mathbb{R}^- \neq \emptyset. \quad (5)$$

Then there exist real numbers $\alpha_1, \alpha_2, \dots, \alpha_{n-2}$ and a positive number λ such that the variable

$$z = x_2 + \alpha_1 \frac{x_3}{x_1} + \alpha_2 \frac{x_4}{x_1^2} + \cdots + \alpha_{n-2} \frac{x_n}{x_1^{n-2}} \quad (6)$$

is such that $\dot{z} = -\lambda z$, i.e. the manifold $z = 0$ is invariant and (exponentially) attractive. Moreover, if $x(0)$ is such that $|z(0)| \leq k$ then, for any $t > 0$, $|z(t)| < k$, i.e. the region $|z| \leq k$ is positively invariant.

Lemma 2 *Consider the matrix (3) and assume that the p_i 's are such that $\sigma(A) \subset \mathbb{C}^-$. Then $p_2 < 0$.*

3 A new discontinuous controller

One of the main results of the paper is summarized in the following statements, which provide a simple and natural modification of the control law presented in [2]. It must be noted that, despite the simplicity of the modification, this new control law possesses very appealing properties, as detailed in the next section.

Proposition 1 *Consider the system (1) and the control law*

$$\begin{aligned} u_1 &= \begin{cases} u_{1A}(x) = -x_1 & |z| \leq k \\ u_{1B}(x) = \text{sign}(x_1) & |z| > k \end{cases} \\ u_2 &= \begin{cases} u_{2A}(x) = p_2 x_2 + p_3 \frac{x_3}{x_1} + \cdots \\ \quad \cdots + p_4 \frac{x_4}{x_1^2} + p_n \frac{x_n}{x_1^{n-2}} & |z| \leq k \\ u_{2B}(x) = -\mu x_2 & |z| > k \end{cases} \end{aligned} \quad (7)$$

with the p_i such that conditions (4) and (5) hold, $\mu > 0$, $k > 0$, z defined as in (6), and ²

$$\text{sign}(x_1) = \begin{cases} 1 & x_1 \geq 0 \\ -1 & x_1 < 0 \end{cases}. \quad (8)$$

Then the closed loop system (1)-(7) admits a unique forward solution for any initial condition $x(0)$ and such a solution converges exponentially to zero.

Proof: The prove can be arranged in two cases: i) $|z(0)| < k$ and ii) $|z(0)| \geq k$. In the first case the control law (2) is used for all $t \geq 0$; hence, as discussed in Section 2 the state converges exponentially to zero. In the second case, the control law

$$\begin{aligned} u_1 &= u_{1B} = \text{sign}(x_1) \\ u_2 &= u_{2B} = -\mu x_2 \end{aligned} \quad (9)$$

is initially active. Therefore, a direct integration of the closed loop system (1)-(9) yields

$$\left\{ \begin{aligned} x_1(t) &= x_{10} + t, \\ x_2(t) &= x_{20} e^{-\mu t}, \\ x_3(t) &= x_{30} + \frac{x_{20}}{\mu} (1 - e^{-\mu t}), \\ &\vdots \\ x_n(t) &= \sum_{j=3}^n \frac{x_{j0} t^{n-j}}{(n-j)!} + \\ &\quad + \sum_{j=4}^n \frac{(-1)^{j-4} x_{20} t^{n-j+1}}{\mu^{j-3} (n-j+1)!} + \\ &\quad + \frac{(-1)^{n-3} x_{20}}{\mu^{n-2}} (1 - e^{-\mu t}), \end{aligned} \right. \quad (10)$$

²Note the somewhat non-standard definition of the sign function.

when $x_1(0) \geq 0$, and

$$\left\{ \begin{array}{l} x_1(t) = x_{10} - t, \\ x_2(t) = x_{20}e^{-\mu t}, \\ x_3(t) = x_{30} - \frac{x_{20}}{\mu}(1 - e^{-\mu t}), \\ \vdots \\ x_n(t) = \sum_{j=3}^n \frac{(-1)^{n-j} x_{j0} t^{n-j}}{(n-j)!} + \\ \quad + \sum_{j=4}^n \frac{(-1)^{n-j} x_{20} t^{n-j+1}}{\mu^{j-3} (n-j+1)!} - \\ \quad - \frac{x_{20}}{\mu^{n-2}}(1 - e^{-\mu t}), \end{array} \right. \quad (11)$$

when $x_1(0) < 0$.

It is simple to verify that in both cases $\lim_{t \rightarrow \infty} z(t) = 0$, so there exists a finite time $t^* \geq 0$ such that $|z(t^*)| < k$. Therefore, in t^* there will be a switch from the controller (9) to the controller (2). For the properties of $z(t)$, the control law (2) will work for all $t \geq t^*$, which allows to conclude, also in this second case, the exponential convergence. ■

Remark 1 The control law (7) is obviously discontinuous. However, the nature of the discontinuity is different from the nature of the discontinuity of the control law in [2], because the control law (7) provides a bounded control action for any bounded x , and this is not the case for the control law in [2]. Despite this substantial difference the control law discussed in this work inherits some of the properties of the control laws in [2]. In particular, the closed loop trajectories are *natural*, and no oscillatory behaviour is observed.

Remark 2 It is worth noting that if $x_1(0) \neq 0$ (resp. $x_2(0) \neq 0$ and $|x_2(0)| < k$) and $x_i(0) = 0$ for $i = 2, 3, \dots, n$ (resp. $x_i(0) = 0$ for $i = 1, 3, \dots, n$) then $x_1(t) = x_1(0) \exp^{-t}$ and $x_i(t) = 0$ for $i = 2, \dots, n$ (resp. $x_2(t) = x_2(0) \exp^{-p_2 t}$ and $x_i(t) = 0$ for $i = 1, 3, \dots, n$) and for all $t \geq 0$, i.e. the x_1 -axis is invariant and internally stable (resp. the x_2 -axis is locally invariant and internally stable).

Proposition 2 Consider the closed loop system (1)-(7) with the p_i such that conditions (4) and (5) hold, $\mu > 0$ and $k > 0$.

Then there exists a function $V(x)$ differentiable almost every where, with $V(0) = 0$ and $V(x) > 0$ for every nonzero x , such that along the trajectories of the closed loop system one has (almost everywhere)

$$\dot{V} = -(x_1^2 + x_2^2 + \dots + x_n^2) = -\|x\|^2 < 0, \forall x \neq 0$$

i.e. the closed loop system (1)-(7) is locally asymptotically stable in the sense of Lyapunov.

Proof: It has been proved in Proposition 1 that the closed loop system trajectories are exponentially stable. Hence, the function

$$V(x(t)) = V(x(0)) + \int_0^t (-\|x(\tau)\|^2) d\tau$$

is such that $V(0) = 0$ and $V(x) > 0, \forall x \neq 0$. It should be noted that the function $\dot{V}(x)$ is a continuous function and it is almost every where \mathcal{C}^1 . ■

Remark 3 The properties summarized in Proposition 2 is instrumental to carry out a satisfactory robustness analysis. It must be observed that this property is not possessed by the control laws presented in [7, 2, 15, 4, 11, 21], which are therefore potentially less robust to model errors, measurement noise and exogenous disturbances.

Remark 4 The function $V(x)$ in Proposition 2 is positive definite, but it is not obvious that it is radially unbounded. Therefore it is not possible to infer global asymptotic stability in the sense of Lyapunov of the closed loop system (1)-(7).

4 Robustness results

As observed in the previous section the control law (7) is such that the closed loop system (1)-(7) is locally asymptotically stable in the sense of Lyapunov and the origin is globally attractive. This properties can be exploited to prove some interesting robustness properties. In particular, we study robustness of the closed loop system against measurement noise and external disturbances.

Proposition 3 Consider the system (1) and the control law (7) in which x is substituted by $^3 x + \varepsilon$, i.e. the control law

$$u_1 = \begin{cases} u_{1A}(x + \varepsilon) = -(x_1 + \varepsilon_1) & |z| \leq k \\ u_{1B}(x + \varepsilon) = \text{sign}(x_1 + \varepsilon_1); & |z| > k \end{cases}$$

$$u_2 = \begin{cases} u_{2A}(x + \varepsilon) = p_2(x_2 + \varepsilon_2) + \\ \quad + p_3 \frac{x_3 + \varepsilon_3}{x_1 + \varepsilon_1} + \dots \\ \quad \dots + p_n \frac{x_n + \varepsilon_n}{(x_1 + \varepsilon_1)^{n-2}} & |z| \leq k \\ u_{2B}(x + \varepsilon) = -\mu(x_2 + \varepsilon_2); & |z| > k \end{cases} \quad (12)$$

with the p_i such that conditions (4) and (5) hold, $\mu > 0$, $k > 0$ and z defined as in (6).

³This is the case if the state measurements are affected by noise.

Then there exists a neighborhood of the origin $\Omega \subset \mathbb{R}^n$ such that for any neighborhood of the origin $\Omega_1 \subset \Omega$ there exists a positive $\varepsilon_{\Omega, \Omega_1}$ such that if $\|\varepsilon\|_{\infty} < \varepsilon_{\Omega, \Omega_1}$ any trajectory of the closed loop system (1)-(12) starting in Ω enters Ω_1 in some finite time T and stays in Ω_1 for all $t \geq T$.

Proof: The closed loop system (1)-(12) can be synthetically rewritten as

$$\dot{x} = g(x)u(x + \varepsilon). \quad (13)$$

Moreover, along the trajectory of the closed loop system one has

$$\begin{aligned} \dot{V} &= V_x(g(x)u(x + \varepsilon)) \\ &= V_x(g(x)u(x)) + V_x \varepsilon \frac{\partial(g(x)u(x + \varepsilon))}{\partial \varepsilon} \Big|_{\varepsilon=0} \\ &= -\|x\|^2 + \varepsilon \varphi(x). \end{aligned} \quad (14)$$

Letting

$$M = \max_{x \in \Omega} \varphi(x)$$

and

$$M_1 = \max_{x \in \Omega_1} \|x\|^2,$$

in Ω/Ω_1 one has $\dot{V} \leq -M_1 + \varepsilon M$. Therefore, for all ε such that

$$\|\varepsilon\|_{\infty} < \frac{M_1}{M} = \varepsilon_{\Omega, \Omega_1},$$

the Lyapunov function is such that $\dot{V} < 0$ in Ω/Ω_1 . As a consequence the trajectory of the closed loop system (13) starting in Ω , converges to Ω_1 , when

$$\|\varepsilon\|_{\infty} < \varepsilon_{\Omega, \Omega_1}. \quad \blacksquare$$

Remark 5 The previous proposition shows that the effect of small measurement noise on the close loop trajectories is small, at least if the initial condition is sufficiently close to the origin.

Proposition 4 Consider the system

$$\begin{aligned} \dot{x}_1 &= u_1 + d_1, \\ \dot{x}_2 &= u_2 + d_2, \\ \dot{x}_3 &= x_2 u_1 + d_3, \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1 + d_n \end{aligned} \quad (15)$$

in closed loop with the control laws (7) with the p_i such that conditions (4) and (5) hold, $\mu > 0$ and $k > 0$.

Then there exists a neighborhood of the origin $\Omega \subset \mathbb{R}^n$ such that for any neighborhood of the origin $\Omega_1 \subset \Omega$ one

can find a positive number K_{Ω, Ω_1} such that the condition

$$\|d\|_{\infty} < K_{\Omega, \Omega_1}$$

implies that any trajectory of the closed loop system (15)-(7) starting in Ω enters Ω_1 in some finite time T and stays in Ω_1 for all $t \geq T$.

Proof: Along the trajectory of the closed loop system one has

$$\begin{aligned} \dot{V} &= V_x(g(x)u(x) + d) = V_x(g(x)u(x)) + V_x d \\ &= -\|x\|^2 + d\varphi(x), \end{aligned} \quad (16)$$

hence the proof is similar to that of Proposition 3. \blacksquare

Remark 6 It is worth stressing that the control law (7) is able to (locally) counteract (small) exogenous disturbances. As a result, the effect of small constant disturbances remains small. Note that, as detailed in [4, 21], most of the existing control laws for chained systems are unable to yield this property.

5 Simulations

For illustrative purposes some simple simulations, for a five dimensional chained form, are included. In all simulations we set the parameters p_2, p_3, p_4 , and p_5 such that the matrix A defined in (3) has eigenvalues $(-1, -2, -3, -4)$ and we select $\mu = 1$.

Figure 1 displays the state history of the nominal chained system (1) in closed loop with the control law (7) from an initial condition with $x_1(0) \neq 0$; whereas the state history of the closed loop system (1)-(7) when $x_1(0) = 0$ is showed in Figure 2. Finally, Figures 3 and 4 display the state histories in presence of measurement noise and external disturbances.

6 Conclusion

A novel discontinuous state feedback control law for chained systems has been proposed and the properties of the resulting close loop system have been discussed in detail. It is shown that the proposed control law yields global exponential regulation and local asymptotic stability in the sense of Lyapunov. As a consequence of the Lyapunov stability property the closed loop system is robust against the effect of (small) measurement noise and of external disturbances. Simulations results complement the theoretical development.

Further study is in progress to design a control law yielding global asymptotic stability in the sense of Lyapunov and to verify the robustness of the proposed con-

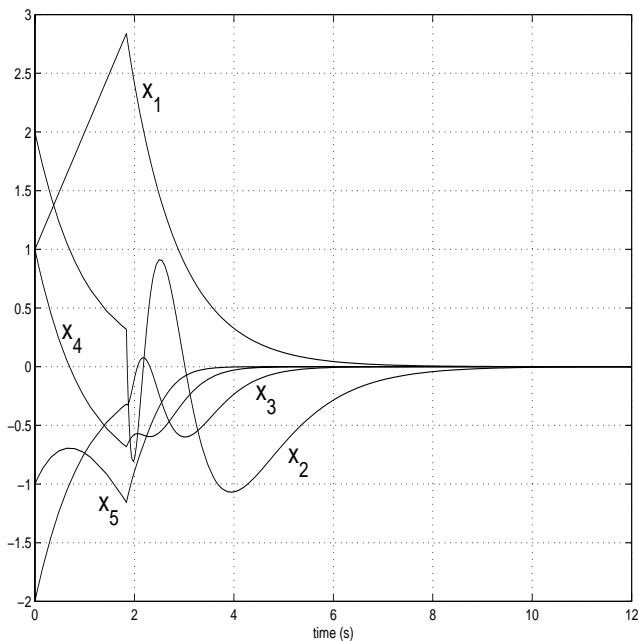


Figure 1: State histories of the closed loop system (1)-(7) with $k = 5$ and $x(0) = [1, 2, -2, 1, -1]'$.

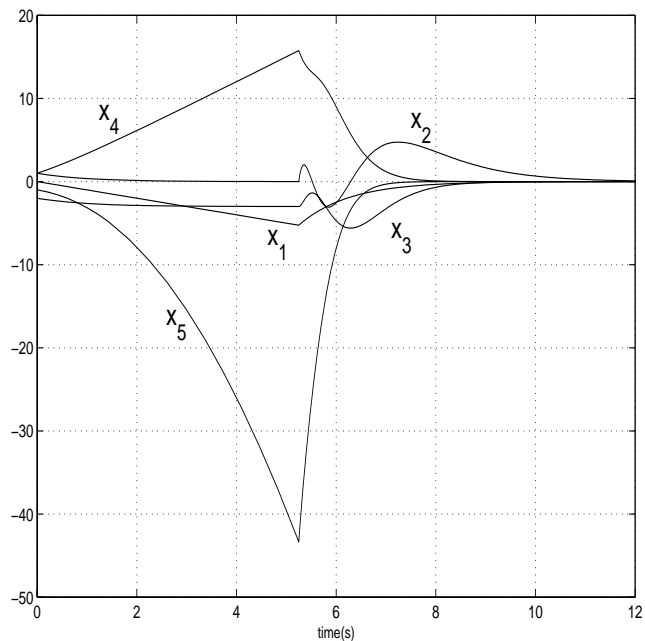


Figure 2: State histories of the closed loop system (1)-(7) with $k = 20$ and $x(0) = [0, 2, -2, 1, -1]'$.

trol law in the presence of model errors and unmodelled dynamics.

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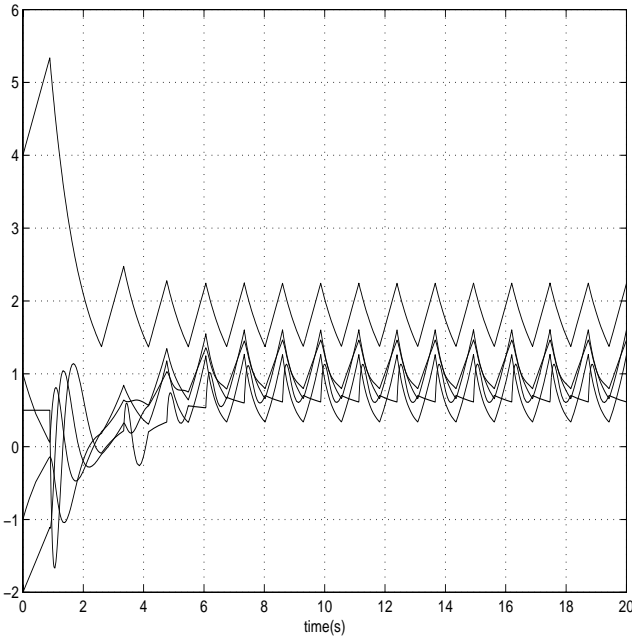


Figure 3: State histories of the closed loop system (15)-(7) with $k = 4$, $x(0) = [4, 0.5, -2, 1, -1]^T$, $d_1(t) = 0.3$, $d_2(t) = 0.2$, $d_3(t) = d_4(t) = d_5(t) = 0.1$.

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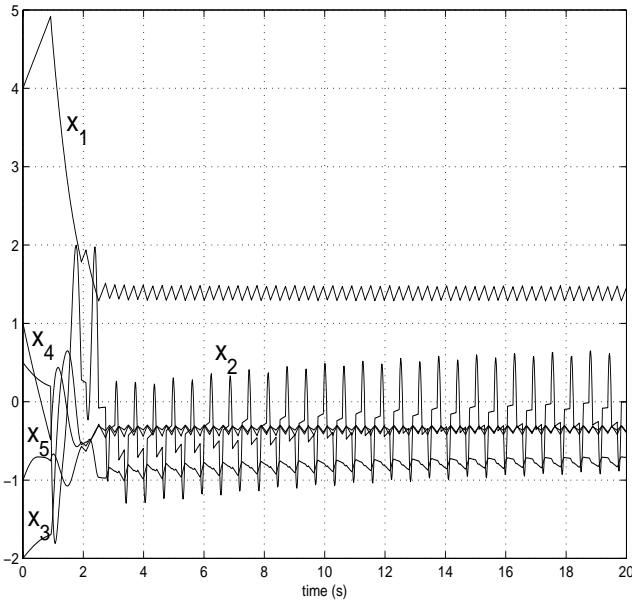


Figure 4: State histories of the closed loop system (1)-(12) with $k = 4$, $x(0) = [4, 0.5, -2, 1, -1]^T$, $\varepsilon_1 = 0.1 \sin(10t)$, $\varepsilon_2 = 0.05 \sin(5t)$, $\varepsilon_3 = 0.2$, $\varepsilon_4 = 0.15 \sin(10t)$ and $\varepsilon_5 = 0.2$.