Semiglobal Nonlinear Output Regulation with Adaptive Internal Model [1]

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Abstract

We address the problem of output regulation for nonlinear systems driven by a linear, exosystem whose natural frequencies are not known a priori. We present a classical solution in terms of the parallel connection of a robust stabilizer and an internal model, where the latter is adaptively tuned to the device that reproduces the control required to maintain the output-zeroing condition. We obtain robust regulation (i.e. in presence of parameter uncertainties) with a semiglobal domain of convergence for a significant class of nonlinear minimum-phase system.

1 Introduction

The problem of having the output of a system asymptotically tracking prescribed trajectories and/or asymptotically rejecting unwanted disturbances, in the presence of (possibly large) *uncertainties*, is ubiquitous in control theory. In case the trajectory to be followed, or the disturbance to be rejected, are *not known*, then the control strategy must reconstruct in some way this information also, which is usually done by means of feedback schemes based on the so called "internal-model principle" [5, 7]. Internal-model based control schemes efficiently address the problem of tracking/rejecting those *families* of exogenous inputs that can be generated by a *fixed* autonomous finite-dimensional dynamical system. However, the main limitation of these schemes is that a precise model of the system that generates all exogenous inputs must be available, to be replicated in the control law. This limitation becomes immediately evident in the problem of rejecting a *sinusoidal* disturbance of unknown amplitude and phase. An internal-model based controller is able to cope with uncertainties on amplitude and phase of the exogenous sinusoid, but the frequency at which the internal-model oscillates must exactly match the frequency of the exogenous sinusoid: any mismatch in such frequencies results in a nonzero steady-state error. Motivated by the wish to remove such a limitation, we address in this paper the problem of designing an internal-model based control scheme in which the "natural frequencies" of the internal model are automatically tuned so as to match those of an exosystem which is *totally unknown* (except for an upper bound on its dimension). In this context, we provide a control scheme that is able to successfully address the problem of asymptotically tracking/rejecting any family of exogenous inputs generated by some (fixed dimensional, but otherwise completely unknown) autonomous dynamical system, for a significant class of *nonlinear systems*, in the presence of possibly *large parameter uncertainties.*

2 Problem formulation

The systems we consider in this paper are single-input, single-output finite-dimensional nonlinear systems, described by equations of the form

$$\begin{aligned} \dot{x} &= f(x,u,w,\mu) \\ y &= h(x,w,\mu)\,, \end{aligned} \tag{1}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}$, and output $y \in \mathbb{R}$. The system depends on a vector of unknown parameters $\mu \in \mathbb{R}^p$, whose values are assumed to range over a known compact set $\mathcal{P}$. Without loss of generality, $\mathcal{P}$ contains the origin in its interior. The vector field $f(x,u,w,\mu)$ and the output map $h(x,w,\mu)$ are assumed to be smooth, with $f(0,0,0,\mu) = 0$ and $h(0,0,\mu) = 0$ for every value of μ. The system is driven by an exogenous input $w \in \mathbb{R}^d$ generated by a *neutrally stable* linear time-invariant exosystem of the form

$$\dot{w} = S(\sigma)w\,. \tag{2}$$

The exosystem depends on a vector $\sigma \in \mathbb{R}^\nu$ of unknown parameters, assumed to range over a known compact

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set Σ. We denote with $e = y - q(w,\mu)$ the regulated error, being $q(w,\mu)$ a smooth function of its arguments. The control input to (1) is to be provided by an *error-feedback controller* modeled by equations of the form

$$\begin{array}{rcl} \dot{\xi} & = & \eta(\xi, e) \\ u & = & \theta(\xi)\,, \end{array} \tag{3}$$

with state $\xi \in \mathbb{R}^m$, in which $\eta(\xi,e)$ and $\theta(\xi)$ are smooth, with $\eta(0,0)=0$, $\theta(0)=0$. The problem we consider in this paper is the following: given arbitrary fixed compact sets $\mathcal{K}_x \subset \mathbb{R}^n$, $\mathcal{K}_w \subset \mathbb{R}^d$, find a controller of the form (3) and a compact set $\mathcal{K}_\xi \subset \mathbb{R}^m$, such that

1. The equilibrium $(x,\xi) = (0,0)$ of the unforced closed loop system

$$\begin{array}{rcl} \dot{x} & = & f(x,\theta(\xi),0,\mu) \\ \dot{\xi} & = & \eta(\xi, h(x,0,\mu)) \end{array} \tag{4}$$

   is asymptotically stable for every $\mu \in \mathcal{P}$, with domain of attraction containing the set $\mathcal{K}_x \times \mathcal{K}_\xi$.

2. The trajectory $(x(t),\ \xi(t))$ of the closed loop system

$$\begin{array}{rcl} \dot{w} & = & S(\sigma)w \\ \dot{x} & = & f(x,\theta(\xi),w,\mu) \\ \dot{\xi} & = & \eta(\xi, h(x,w,\mu)) \end{array} \tag{5}$$

   originating from any initial conditions $(x(0),\ \xi(0),\ w(0)) \in \mathcal{K}_x \times \mathcal{K}_\xi \times \mathcal{K}_w$ exists for all $t \geq 0$, is bounded and satisfies $\lim_{t\to\infty} e(t) = 0$, for every $\mu \in \mathcal{P}$ and every $\sigma \in \Sigma$.

We stress the fact that, as opposite to the classic formulation of the regulator problem, both in the nonlinear and the linear setting, the controller has to be designed on the basis of an uncertain exosystem. A trivial necessary condition for the existence of a solution to the above problem is the solvability for every value of $\sigma \in \Sigma$. This, in turn, implies that, given a fixed σ, there exist mappings $x = \pi_\sigma(w,\mu)$ and $u = c_\sigma(w,\mu)$, with $\pi_\sigma(0,\mu) = 0$ and $c_\sigma(0,\mu) = 0$, defined in a neighborhood of the origin, satisfying

$$\begin{array}{rcl} \dfrac{\partial \pi_\sigma(w,\mu)}{\partial w} S(\sigma) w & = & f\left(\pi_\sigma(w,\mu), c_\sigma(w,\mu), w, \mu\right) \\ 0 & = & h\left(\pi_\sigma(w,\mu), w, \mu\right) - q(w,\mu)\,, \end{array} \tag{6}$$

and such that the autonomous system with output v

$$\begin{array}{rcl} \dot{w} & = & S(\sigma)w \\ \dot{\mu} & = & 0 \\ v & = & c_\sigma(w,\mu) \end{array} \tag{7}$$

is immersed into a zero-state detectable system [3].

3 Standing assumptions

The class of systems we consider are those modeled by differential equations of the kind (1) which are globally diffeomorphic, via a possibly parameter-dependent change of coordinates, to a system of the form

$$\begin{array}{rcl} \dot{z} & = & f_0(z, x_1, w, \mu) \\ \dot{x}_1 & = & x_2 \\ & \cdots & \\ \dot{x}_r & = & f_r(z, x_1, \ldots, x_r, w, \mu) + b(\mu) u \\ y & = & x_1 \end{array} \tag{8}$$

with regulated error given by

$$e = x_1 - q(w,\mu) \tag{9}$$

where $b(\mu) \geq b_0 > 0$ for all $\mu \in \mathcal{P}$. To ensure the problem is well-posed, we need the following

Assumption 3.1 *For every $\sigma \in \Sigma$, there exists a globally defined solution $\zeta_\sigma(w,\mu)$ to the equation*

$$\frac{\partial \zeta_\sigma(w,\mu)}{\partial w} S(\sigma) w = f_0(\zeta_\sigma(w,\mu), q(w,\mu), w, \mu). \tag{10}$$

Assumption 3.1 and the triangular structure of equation (8) imply the existence of a unique, globally defined solution $(\pi_\sigma(w,\mu), c_\sigma(w,\mu))$ of the regulator equation (6) for the system (8), where $\pi_\sigma(w,\mu) = \mathrm{col}(\zeta_\sigma(w,\mu), \vartheta_\sigma(w,\mu))$. The function $c_\sigma(w,\mu)$ is assumed to satisfy the following:

Assumption 3.2 *There exist $q \in \mathbb{N}$ and a set of real numbers $a_0(\sigma), a_1(\sigma), \ldots, a_{q-1}(\sigma)$ such that the identity*

$$L^q_{S(\sigma)w} c_\sigma(w,\mu) = \sum_{i=0}^{q-1} a_i(\sigma) L^i_{S(\sigma)w} c_\sigma(w,\mu) \tag{11}$$

holds for all $(w,\mu) \in \mathbb{R}^d \times \mathcal{P}$ all $\sigma \in \Sigma$.

Assumption 3.2 implies that, for any fixed σ, there exists a mapping $\tau_\sigma(w,\mu)$ satisfying

$$\begin{array}{rcl} L_{S(\sigma)w} \tau_\sigma(w,\mu) & = & \Phi(\sigma) \tau_\sigma(w,\mu) \\ c_\sigma(w,\mu) & = & \Gamma \tau_\sigma(w,\mu)\,, \end{array} \tag{12}$$

in which $\{\Phi(\sigma), \Gamma\}$ are in companion form. As a result, (7) is immersed into the linear observable system

$$\begin{array}{rcl} \dot{\eta} & = & \Phi(\sigma)\eta \\ u & = & \Gamma\eta. \end{array} \tag{13}$$

The matrix $\Phi(\sigma)$ can be chosen so that its spectrum contains all simple eigenvalues on the imaginary axis, for any $\sigma \in \Sigma$. The spectrum of $\Phi(\sigma)$ must contain the distinct natural frequencies of the exosystem and a number of higher order harmonics generated by the nonlinearity of the plant [6, 8]. Assumptions 3.2 and **??** are satisfied if the function $c_\sigma(w,\mu)$ is a polynomial in the components of w with coefficients dependent on μ and σ, and of a degree not exceeding a fixed number, independent of μ and σ [6]. Throughout the paper, we let $\mathcal{B}_r$ and $\overline{\mathcal{B}}_r$ denote respectively the open and the closed ball of radius r around the origin of $\mathbb{R}^n$, where the number $n \in \mathbb{N}$ will be clear from the context.

4 The canonical internal model

In this section we describe a particular parameterization of the internal model (13), which will prove to be crucial to the solution of the problem. A major source of inspiration comes from [10], and precisely from the following lemma:

Lemma 4.1 ([10]) *Given any Hurwitz matrix $F \in \mathbb{R}^{q\times q}$ and any vector $G \in \mathbb{R}^q$ such that the pair (F,G) is controllable, the Sylvester equation*

$$M_\sigma \Phi(\sigma) - F M_\sigma = G\Gamma$$

has a unique solution M_σ, which is non singular.

Then, indeed

$$M_\sigma \Phi(\sigma) M_\sigma^{-1} = F + G\Psi_\sigma\,,$$

where $\Psi_\sigma = \Gamma M_\sigma^{-1}$. Therefore, $\Phi(\sigma)$ is similar to $F + G\Psi_\sigma$. Note that, since (F,G) is controllable, and G has just one column, the row vector Ψ_σ is precisely the *unique* solution to the problem of assigning to $F+G\Psi_\sigma$ the poles of $\Phi(\sigma)$. The system (7) is immersed into the system

$$\begin{array}{rcl}\dot\eta &=& (F+G\Psi_\sigma)\,\eta\\ v &=& \Psi_\sigma\eta\,.\end{array} \qquad (14)$$

We refer to (14) as the *canonical parameterization of the internal model*. Specifically, the immersion map from (7) to (14) is given by $\bar\tau_\sigma(w,\mu) = M_\sigma\tau_\sigma(w,\mu)$, which satisfies $c_\sigma(w,\mu) = \Psi_\sigma\bar\tau_\sigma(w,\mu)$. This relation plays a crucial role in the sequel, as the relation $c_\sigma(w,\mu) = \Gamma\tau_\sigma(w,\mu)$ does in the non-adaptive case.

5 Regulation via partial-state feedback

We design a controller the solves the problem of output regulation, starting from the simplifying assumption that the error e and its derivatives $e^{(1)},\ldots,e^{(r-1)}$ are available for feedback, and the parameters of the exosystem are all known. These assumptions will be removed in the next sections. We begin performing the global change of coordinates

$$\tilde z = z - \zeta_\sigma(w,\mu), \quad \tilde x = x - \vartheta_\sigma(w,\mu) \qquad (15)$$

which puts (8) in the *error system* form

$$\begin{array}{rcl}\dot{\tilde z} &=& \tilde f_0(\tilde z,\tilde x_1,w,\rho)\\ \dot{\tilde x}_1 &=& \tilde x_2\\ & \cdots & \\ \dot{\tilde x}_r &=& \tilde f_r(\tilde z,\tilde x_1,\ldots,\tilde x_r,w,\rho) + b(\mu)[u - c_\sigma(w,\mu)]\\ e &=& \tilde x_1\,,\end{array} \qquad (16)$$

where we have denoted $\rho = \mathrm{col}(\mu,\sigma)$. To simplify the notation, let $\mathcal{R} = \mathcal{P}\times\Sigma$. The further change of coordinates

$$\theta = \tilde x_r + k^{r-1}b_0\tilde x_1 + k^{r-2}b_1\tilde x_2 + \cdots + kb_{r-2}\tilde x_{r-1}\,, \quad (17)$$

where $k>0$ is a number yet to be determined, and the polynomial

$$p_\circ(\lambda) = \lambda^{r-1} + b_{r-2}\lambda^{r-2} + \cdots + b_1\lambda + b_0$$

is Hurwitz, puts system (16) into the form

$$\begin{array}{rcl}\dot{\tilde z}_{\mathrm a} &=& f_{\mathrm a}(\tilde z_{\mathrm a},w,\rho,k) + G_{\mathrm a}\theta\\ \dot\theta &=& \phi(\tilde z_{\mathrm a},\theta,w,\rho,k) + b(\mu)[u - c_\sigma(w,\mu)]\end{array} \qquad (18)$$

where $\tilde z_{\mathrm a} = \mathrm{col}(\tilde z,\tilde x_1,\ldots,\tilde x_{r-1})$. The standard controller which solves the problem of output regulation consists of the parallel connection of a stabilizer and an internal model. In view of the discussion in section 4, we choose as an internal model the system

$$\begin{array}{rcl}\dot\xi &=& (F+G\Psi_\sigma)\,\xi + N\theta\\ u_{\mathrm{im}} &=& \Psi_\sigma\xi\,,\end{array} \qquad (19)$$

where N is a vector to be determined, and let $u = u_{\mathrm{im}} - K\theta$, with $K>0$. Change coordinates as

$$\xi \mapsto \chi = \xi - \bar\tau_\sigma(w,\mu) - \frac{1}{b(\mu)}G\theta\,,$$

and let $N = -KG$. The resulting closed-loop can be viewed as the interconnection of system

$$\begin{array}{rcl}\dot\chi &=& F\chi + \frac{1}{b(\mu)}[FG\theta - G\phi(\tilde z_{\mathrm a},\theta,w,\rho,k)]\\ \dot{\tilde z}_{\mathrm a} &=& f_{\mathrm a}(\tilde z_{\mathrm a},w,\rho,k) + G_{\mathrm a}\theta\\ \dot\theta &=& \phi(\tilde z_{\mathrm a},\theta,w,\rho,k) + b(\mu)\Psi_\sigma\chi + \Psi_\sigma G\theta + b(\mu)\bar u\\ \bar y &=& \theta\,,\end{array} \qquad (20)$$

with input $\bar u$ and output $\bar y$, and the output feedback $\bar u = -K\bar y$. System (20) has a zero dynamics given by

$$\begin{array}{rcl}\dot\chi &=& F\chi - \frac{1}{b(\mu)}G\phi(\tilde z_{\mathrm a},0,w,\rho,k)\\ \dot{\tilde z}_{\mathrm a} &=& f_{\mathrm a}(\tilde z_{\mathrm a},w,\rho,k)\,.\end{array} \qquad (21)$$

The following assumption is required for the solvability of the problem:

Assumption 5.1 *There exists a smooth, positive definite function $V_0(\tilde z)$ such that*

$$\underline{\alpha}_0(\|\tilde z\|) \le V_0(\tilde z) \le \overline{\alpha}_0(\|\tilde z\|)$$

$$\frac{\partial V_0(\tilde z)}{\partial \tilde z}\tilde f_0(\tilde z,0,w,\rho) \le -\alpha_0(\|\tilde z\|)\,,$$

for all $\tilde z_{\rm a} \in \mathbb{R}^{n-r}$, all $w^\circ \in \mathcal{K}_w$ and all $\rho \in \mathcal{R}$, where $\underline{\alpha}_0(\cdot)$, $\overline{\alpha}_0(\cdot)$ and $\alpha_0(\cdot)$ are class-$\mathcal{K}_\infty$ functions satisfying

$$\underline{\alpha}_0(s) \geq \underline{a}_0 s^2, \;\; \overline{\alpha}_0(s) \leq \overline{a}_0 s^2, \;\; \alpha_0(s) \geq a_0 s^2 \,, \;\; \forall s \in [0, r_0]$$

for some positive numbers $\underline{a}_0$, $\overline{a}_0$, a_0, r_0.

It is well known that, if assumption 5.1 is satisfied, the equilibrium $\tilde z_{\rm a} = 0$ of the system

$$\dot{\tilde z}_{\rm a} = f_{\rm a}(\tilde z_{\rm a}, w(t), \rho, k)$$

is *uniformly semiglobally asymptotically stable in the parameter* k [1, 11]. For the cascade system (21) the following result holds:

Proposition 5.1 *System (21) is uniformly semiglobally asymptotically stable in the parameter k. In particular, for any $R > 0$ there exist $\bar k > 0$, $a > 0$ and a continuously differentiable Lyapunov function $U : \mathbb{R}^q \times \mathbb{R}^{n-1} \to \mathbb{R}_{\geq 0}$ such that*

- *the level set $\{U(\chi, \tilde z_{\rm a}) \leq a\}$ is compact and $\overline{\mathcal{B}}_R \subset \{U(\chi, \tilde z_{\rm a}) < a\}$*
- *the estimate $\underline{b}\|(\chi, \tilde z_{\rm a})\|^2 \leq U(\chi, \tilde z_{\rm a}) \leq \overline{b}\|(\chi, \tilde z_{\rm a})\|^2$ holds for any $(\chi, \tilde z_{\rm a}) \in \mathcal{B}_r$, for some positive numbers $\underline{b}$, $\overline{b}$, r;*
- *the estimate*
$$\frac{\partial U}{\partial \chi}[F\chi - \tfrac{1}{b(\mu)} G\phi(\tilde z_{\rm a}, 0, w, \rho, \bar k)] + \frac{\partial U}{\partial \tilde z_{\rm a}} f_{\rm a}(\tilde z_{\rm a}, w, \rho, \bar k) \leq -\beta(\|(\chi, \tilde z_{\rm a})\|)$$
holds for any $(\chi, \tilde z_{\rm a}) \in \{U(\chi, \tilde z_{\rm a}) \leq a\}$, for any $w^\circ \in \mathcal{K}_w$ and $\rho \in \mathcal{R}$, for some class-$\mathcal{K}$ function $\beta(\cdot)$ satisfying $\beta(s) \geq b\, s^2$, $\forall s \in [0, r)$, where b is some positive number.

Proposition 5.1 guarantees that the high gain feedback $\bar u = -K\theta$ uniformly asymptotically and locally exponentially stabilizes the equilibrium $(\chi, \tilde z_{\rm a}, \theta) = (0, 0, 0)$, with domain of attraction which can be enlarged to include any compact set of initial conditions. Indeed, the positive definite, locally quadratic function

$$W(\chi, \tilde z_{\rm a}, \theta) = U(\chi, \tilde z_{\rm a}) + \tfrac{1}{2}\theta^2 \,,$$

is such that a number $d > 0$ can always be found in such a way that the level set $\{(\chi, \tilde z_{\rm a}, \theta) : W(\chi, \tilde z_{\rm a}, \theta) \leq d\}$ is compact and contains an arbitrary closed ball $\overline{\mathcal{B}}_R$ in its interior. By the semiglobal backstepping lemma of Teel and Praly [11, lemma 2.2], there exists a value $K^\star$ and a class-$\mathcal{K}$ function $\lambda(\cdot)$, locally quadratic around the origin, such that, if K is chosen greater than $K^\star$, the derivative of W along trajectories of (20) satisfies

$$\tfrac{\rm d}{{\rm d}t} W(\chi(t), \tilde z_{\rm a}(t), \theta(t)) \leq -\lambda(\|(\chi(t), \tilde z_{\rm a}(t), \theta(t))\|) \,,$$

for all $(\chi(t), \tilde z_{\rm a}(t), \theta(t)) \in \{(\chi, \tilde z_{\rm a}, \theta) : W(\chi, \tilde z_{\rm a}, \theta) \leq d\}$. As a result, we achieve semiglobal robust regulation in the case σ is known.

6 Tuning the internal model

The controller designed in the previous section is based on the standard assumption that the parameters of the exosystem are exactly known. If the vector σ is not known, we need to employ a method to asymptotically estimates the frequencies of the exogenous signal as well. We derive an "adaptive" version of the above controller that "tunes" a parameterized family of internal models to the one which reproduces the signal $c_\sigma(w, \mu)$. Due to space limitation, we restrict the analysis to the case in which none of the eigenvalues of the exosystem are known in advance, and no integral action is needed in the internal model. Also, we assume that the signal $\tau_\sigma(w, \mu)$ is "sufficiently rich", in the following sense:

Assumption 6.1 *The initial condition w° of the exosystem is such that $\bar\tau_\sigma(w^\circ, \mu)$ excites all modes of $F + G\Psi_\sigma$.*

6.1 Adaptive regulation

Appealing to the principle of "certainty equivalence", we replace Ψ_σ in (19) with an estimate $\hat\Psi$, governed by an adaptation law of the kind $\frac{\rm d}{{\rm d}t}\hat\Psi = \varphi(\xi, \theta)$, to obtain

$$\begin{aligned} \dot\xi &= (F + G\hat\Psi)\xi + N\theta \\ u_{\rm im} &= \hat\Psi\xi \,. \end{aligned} \tag{22}$$

As usually done in the analysis of any adaptive control scheme, we change coordinates as $\tilde\Psi = \hat\Psi - \Psi_\sigma$ and obtain, after easy computations, the following expression for the closed loop system (compare with (20))

$$\begin{aligned} \dot\chi &= F\chi + \tfrac{1}{b(\mu)}[FG\theta - G\phi(\tilde z_{\rm a}, \theta, w, \rho, k)] \\ \dot{\tilde z}_{\rm a} &= f_{\rm a}(\tilde z_{\rm a}, w, \rho, k) + G_{\rm a}\theta \\ \dot\theta &= \phi(\tilde z_{\rm a}, \theta, w, \rho, k) + b(\mu)\Psi_\sigma\chi \\ &\quad + (\Psi_\sigma G - b(\mu)K)\theta + b(\mu)\tilde\Psi\xi \\ \tfrac{\rm d}{{\rm d}t}\tilde\Psi &= \varphi(\xi, \theta) \,. \end{aligned} \tag{23}$$

We stress that ξ is available for feedback, as it is the state of the adaptive internal model (22). Consider the following Lyapunov function candidate for (23)

$$\bar W(\chi, \tilde z_{\rm a}, \theta, \tilde\Psi) = W(\chi, \tilde z_{\rm a}, \theta) + \tfrac{1}{\gamma} b(\mu)\tilde\Psi\tilde\Psi^{\rm T} \,,$$

where $\gamma > 0$ is a design parameter. For a fixed $R > 0$, there is a number $\ell > 0$ such that the level set

$$\bar{\mathcal{S}}_\ell = \left\{(\chi, \tilde z_{\rm a}, \theta, \tilde\Psi) : \bar W(\chi, \tilde z_{\rm a}, \theta, \tilde\Psi) \leq \ell\right\}$$

is compact and contains $\overline{\mathcal{B}}_R \subset \mathbb{R}^{n+2q}$ in its interior. Then, as shown in the previous section, values of k and K can be determined such that the derivative of $\bar W$ along solutions of (23) satisfies

$$\begin{aligned} \tfrac{\rm d}{{\rm d}t}\bar W(\chi(t), \tilde z_{\rm a}(t), \theta(t), \tilde\Psi(t)) &\leq -\lambda(\|(\chi(t), \tilde z_{\rm a}(t), \theta(t))\|) \\ &\quad + b(\mu)\tilde\Psi(t)[\theta(t)\xi(t) + \tfrac{1}{\gamma}\varphi^{\rm T}(\xi(t), \theta(t))] \end{aligned}$$

for all $(\chi(t), \tilde{z}_{\rm a}(t), \theta(t), \tilde{\Psi}(t))$ in $\bar{\mathcal{S}}_\ell$. The obvious choice $\varphi(\xi,\theta) = -\gamma\theta\xi^{\rm T}$ yields

$$\tfrac{\rm d}{{\rm d}t}\bar{W}(\chi(t), \tilde{z}_{\rm a}(t), \theta(t), \tilde{\Psi}(t)) \le -\lambda(\|(\chi(t), \tilde{z}_{\rm a}(t), \theta(t))\|)\,,$$

and thus, trajectories originating within $\bar{\mathcal{S}}_\ell$ are bounded and, by the Lasalle-Yoshizawa theorem, such that

$$\lim_{t\to\infty}(\chi(t), \tilde{z}_{\rm a}(t), \theta(t)) = (0,0,0)\,,$$

which implies that asymptotic regulation is achieved.

6.2 Convergence of the estimates

The previous result states that regulation is attained and the estimates $\hat{\Psi}(t)$ are bounded signals, but whether they converge to some constant value is still debatable. Indeed, it is possible to prove that, not only $\lim_{t\to\infty}\hat{\Psi}(t)$ exists, but is equal to the true value Ψ_σ, and the convergence rate is ultimately exponential. This can be accomplished analyzing the trajectories of the closed-loop system on their ω-limit set. For the sake of simplicity, let $\mathbf{x} = \mathrm{col}(\chi, \tilde{z}_{\rm a}, \theta)$, and denote with Ω the ω-limit set of an arbitrary trajectory. Choose initial conditions in Ω, and denote with $\mathbf{x}_\infty(t)$, $\xi_\infty(t)$, $\tilde{\Psi}_\infty(t)$ the corresponding trajectories. Then, since $\mathbf{x}_\infty(t) \equiv 0$, we have $\hat{\Psi}(t) \equiv \hat{\Psi}_\infty = \mathrm{const}$, and $\xi_\infty(t) = \bar{\tau}_\sigma(w(t),\mu)$, for all $t \ge 0$. Note that ξ_∞ satisfies

$$\dot{\xi}_\infty = (F + G\hat{\Psi}_\infty)\xi_\infty\,,$$

and that, by assumption 6.1, the trajectory $\xi_\infty(t)$ contains all modes of $F + G\Psi_\sigma$. Therefore, the spectrum of $F + G\Psi_\sigma$ is contained in the spectrum of $F + G\hat{\Psi}_\infty$, and, since the cardinality of both spectra is the same, by uniqueness of Ψ_σ we conclude that necessarily $\hat{\Psi}_\infty = \Psi_\sigma$. This implies that the equilibrium $(\mathbf{x}, \tilde{\Psi}) = (0,0)$ of (23) is asymptotically stable for every initial condition $w^\circ \in \mathcal{K}_w$ and every $\rho \in \mathcal{R}$. Next, we claim that the convergence of the estimates is ultimately exponential. This can be inferred from the analysis of linear approximation of (23). It can be shown that the signal $\xi(t)$ satisfies the *persistence of excitation* condition [2], and that the higher order terms in (23) can be bounded uniformly in (w°, ρ). Therefore, Theorem 4.5 in [9] applies locally around the origin, and we conclude that there exist numbers $\delta > 0$, $\kappa_1 > 0$, $\kappa_2 > 0$, all independent of w° and ρ, and a number $T > 0$, possibly dependent on w° and ρ, such that

$$\|(\mathbf{x}(t), \tilde{\Psi}(t))\| \le \kappa_1\|(\mathbf{x}(T), \tilde{\Psi}(T))\|{\rm e}^{-\kappa_2(t-T)}\,,\quad \forall t \ge T\,,$$

for all $(\mathbf{x}^\circ, \tilde{\Psi}^\circ) \in \bar{\mathcal{S}}_\ell$.

7 Error-feedback adaptive controller

In order to realize a controller that uses information from the error signal $e = \tilde{x}_1$ only, the partial state $\tilde{x}_1, \ldots, \tilde{x}_r$ must be replaced by suitable estimates. The saturated high-speed observer of [4] can be effectively used to generate estimates of the error and its derivatives. In our case, we obtain a dynamic output feedback controller of the kind

$$\begin{array}{rcl} \dot{\eta} &=& M\eta + Le \\ \dot{\xi} &=& (F + G\hat{\Psi})\xi - KG\mathrm{sat}(l,\hat{\theta}) \\ \frac{\rm d}{{\rm d}t}\hat{\Psi} &=& -\gamma\mathrm{sat}(l,\hat{\theta})\xi^{\rm T} \\ \hat{\theta} &=& \eta_r + k^{r-1}b_0\eta + \cdots + kb_{r-2}\eta_{r-1} \\ u &=& \hat{\Psi}\xi - K\mathrm{sat}(l,\hat{\theta}) \end{array} \qquad (24)$$

where $\mathrm{sat}(\cdot,\cdot)$ is a saturation function. The following theorem summarizes the result of the paper:

Theorem 7.1 *Consider the system (8) driven by the exosystem (2). The dynamic controller (24) solves the problem of robust semiglobal regulation via feedback from the error (9).*

8 Example

As an illustrative example, consider the controlled Van der Pol equation

$$\begin{array}{rcl} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& -x_1 + \mu_1 x_2 - x_2^3 + \delta(x_1, x_2, \mu) + u \\ e &=& x_1 - w_1 \end{array}$$

in which the term $\delta(x_1, x_2, \mu) = -\mu_2\, x_1 x_2^2$ is regarded as a perturbation. The parameter vector $\mu = \mathrm{col}(\mu_1, \mu_2)$ is assumed to satisfy $\mu \in \{|\mu_1| \le 3\,,\ 0 \le \mu_2 \le 5\}$, while the exosystem is an harmonic oscillator in which the frequency σ ranges between 1 and 4 rad/s. Easy computations show that the regulator equation (6) has the following globally defined solution

$$\begin{array}{rl} \pi_{\sigma_1}(w,\mu) = & w_1 \\ \pi_{\sigma_2}(w,\mu) = & \sigma\, w_2 \\ c_\sigma(w,\mu) = & (1-\sigma^2)w_1 - \sigma\mu_1 w_2 + \sigma^2\mu_2 w_1 w_2^2 + \sigma^3 w_2^3\,. \end{array}$$

The autonomous system (7) is immersed into the four-dimensional system (13), with $\Phi(\sigma)$ given by

$$\Phi(\sigma) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9\sigma^4 & 0 & -10\sigma^2 & 0 \end{pmatrix}$$

The canonical parameterization of the internal model has been chosen such that the matrix F has eigenvalues placed at $\{-12, -10, -9, -8\}$. The nominal internal model has been designed for the case $\sigma_0 = 1$ rad/s. To improve the numerical stability of the algorithm, the triplet (F, G, Ψ_{σ_0}) has been obtained as a balanced realization from the controllability canonical form. The controller parameters have been chosen according to table 1. The initial conditions for the exosystem have been chosen as $w(0) = (2, 0)$, and the parameter vector as $\mu = (3, 4)$. The frequency of the exosystem has

$k = 0.5$	$K = 75$	$\gamma = 1$	$g = 100$
$l = 30$	$b_0 = 1$	$c_0 = 2$	$c_1 = 3$

Table 1: Controller parameters

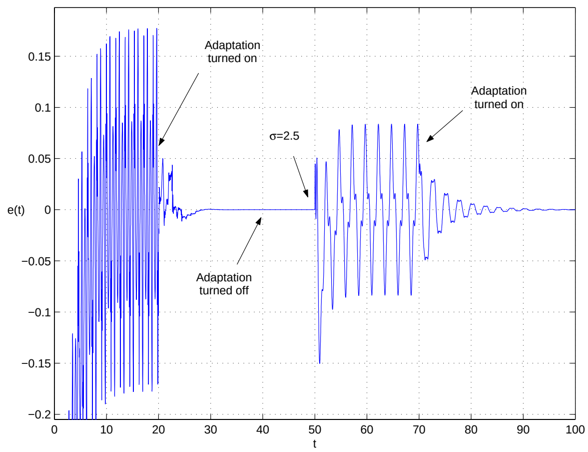

Figure 1: Regulation error $e(t)$

been chosen first as $\sigma = 3.5$ rad/s, while the internal model has been tuned for $\sigma_0 = 1$ rad/s. The adaptation is disconnected at first, and then it is turned on at $t = 20$ s. We notice a remarkable steady-state error (about 10%) until the adaptation is connected (see figure 1). Figure 2 shows the evolution of the parameter estimates $\hat{\Psi}(t)$. At $t = 40$ s, the adaptation is disconnected again. Remarkably, no error arises, which shows that the adaptive internal model has been tuned to the "right one". At time $t = 50$ s, we change the frequency of the exosystem to $\sigma = 2.5$ rad/s, resulting in a steady state error. At time $t = 70$ s, we turn the adaptation on again, and we recover zero steady state error, as the internal model is tuned again to the right frequencies.

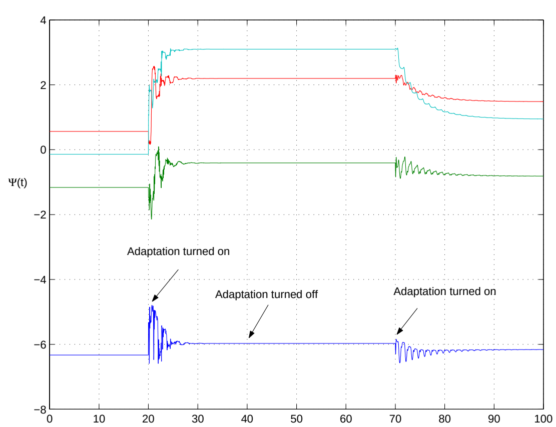

Figure 2: Parameter estimates $\hat{\Psi}(t)$

9 Conclusions

In this paper we have shown how one of the greatest limitations of output regulation theory so far, namely the requirement of complete knowledge of the frequencies of the exosystem, can be overcome by incorporating an adaptation mechanism in the internal model-based control. Inspired by the result of [10], we have solved the problem of semiglobal robust regulation with unknown frequencies of the exosystem for a large class of nonlinear systems, in the very general framework of the nonlinear regulator theory of [7]. A case study has been presented to show the effectiveness of the method, which may prove to be of great importance in control applications.

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