

Non-Smooth Stabilizers for Nonlinear Systems with Uncontrollable Unstable Linearization *

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Abstract

We prove that every chain of *odd power* integrators perturbed by a C^1 triangular vector field can be stabilized in the large via *continuous* state feedback, although it is not stabilizable, even locally, by any *smooth* state feedback. The proof is constructive and accomplished by developing a machinery—a *continuous type of adding a power integrator*—that enables one to explicitly design a C^0 *globally* stabilizing feedback law as well as a C^1 control Lyapunov function which is positive definite and proper.

1 Introduction

In this paper we consider a class of highly nonlinear systems in the lower-triangular form

$$\begin{aligned} \dot{x}_1 &= x_2^{p_1} + f_1(x_1) \\ &\vdots \\ \dot{x}_{n-1} &= x_n^{p_{n-1}} + f_{n-1}(x_1, \dots, x_{n-1}) \\ \dot{x}_n &= u^{p_n} + f_n(x_1, \dots, x_n), \end{aligned} \quad (1.1)$$

where $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the system state and the control input, p_i , $i = 1, \dots, n$, are arbitrarily *odd positive integers*, and $f_i: \mathbb{R}^i \rightarrow \mathbb{R}^1$, $i = 1, 2, \dots, n$, are C^1 functions with $f_i(0, \dots, 0) = 0$.

The main purpose of the paper is to address the questions: (i) when is there a *continuous* (i.e. C^0) state feedback control law that renders the trivial solution $x = 0$ of (1.1) *globally strongly stable* (**GSS** for short) in the sense of Kurzweil [4] (see the precise definition in Section 2)? (2) how to explicitly construct such a continuous, globally strongly stabilizing controller if there exists a one?

Our interest in these two questions is motivated by a number of papers and books [1, 3, 12, 13, 18, 20], [5]–[10], which are devoted to the problem of *local* asymptotic stabilization via *continuous* or *homogeneous* state feedback, for lower-dimensional (two and three-dimensional) systems or lower-triangular systems whose first approximation has uncontrollable, unstable modes. The best known example, which has been extensively studied in the literature, is the planar system

$$\begin{aligned} \dot{x}_1 &= x_2^3 + x_1 \\ \dot{x}_2 &= u. \end{aligned} \quad (1.2)$$

This system has an uncontrollable mode whose eigenvalue is positive, and thus the system (1.2) cannot be stabilized, even locally, by any smooth (or C^r , $r \geq 1$) state feedback control laws [2].

While *no smooth* state feedback laws can locally asymptotically stabilize the system (1.2), it has been proved in [12, 13] that there exists, however, a *locally Hölder continuous* (non-Lipschitz) stabilizing controller for (1.2). An explicit algorithm was given in [12, 13], for the construction of Hölder continuous state feedback control laws that achieve local asymptotic stability, for a class of *small-time locally controllable* (STLC for short)¹ affine systems in the plane, even if the Jacobian linearization has uncontrollable modes associated with eigenvalues whose real part is positive.

The results of [12, 13] (also see the survey paper [19]) suggest that *continuous* feedback design seems to be a natural strategy to overcome some topological obstruction such as the one illustrated by (1.2), which may occur in the case of *smooth* feedback stabilization (e.g. see [15]). This appealing idea, together with the powerful notions such as *homogeneous approximation* and *homogeneity with respect to a family of dilations*, [8, 10, 12, 13] has led to various exciting developments [1, 5, 6, 7, 3, 10, 9, 18] in the area of asymptotic stabilization of inherently nonlinear systems (e.g. (1.1) and STLC affine systems in \mathbb{R}^2) having uncontrollable linearization, via continuous or non-differentiable state feedback.

Most of the stabilization results obtained in [1, 5, 6, 8, 9, 10, 12, 13] for lower-dimensional systems are *local*, due to the use of a *homogeneous approximation*. In the higher-dimensional case, the paper [7] studied the asymptotic stabilization of lower-triangular systems (1.1). It was shown that if p_i , $1 \leq i \leq n$, are odd positive integers, (1.1) is *locally* asymptotically stabilizable by continuous state feedback. The result was proved by using Hermes' theorem on robust stability of homogeneous systems [9], and is basically an *existence* result. An important question on how to design a local continuous stabilizer remained, however, unknown. The issue was addressed later in [3], where a formula was provided for the calculation of C^0 *local* controllers. Recently, the paper [22] investigated an interesting question on how to design a C^0 globally stabilizing controller for a class of homogeneous systems. In particular, it was proved in [22], among other things, that a subclass of nonlinear systems of the form (1.1) with $f_i(x_1, \dots, x_i)$ being a suitable linear function of the state (x_1, \dots, x_i) are *homogeneous* with

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¹STLC condition used in the work [12, 13] has been studied, for instance, by Stefani [20] and Sussmann [21]. Computable sufficient conditions for STLC can be founded in [21].

respect to certain dilation, and can thus be stabilized in the large by C^0 homogeneous state feedback. The proof of the result in [22] relied crucially on a homogeneous-like Lyapunov function motivated by [7].

In a different direction, we considered in [15] the problem of *global* asymptotic stabilization via *smooth* state feedback, for a class of high-order nonlinear systems of the form (1.1). Using a design tool called *adding a power integrator*, we showed how to explicitly construct a globally stabilizing C^∞ state feedback control law for the system (1.1), under appropriate *sufficient conditions* which turn out to be somewhat *necessary* for *smooth* feedback stabilization [15].

In this work we focus our attention on the problem of *global* stabilization via *continuous* instead of *smooth* (e.g. [11, 15, 16]) state feedback. By combining the idea of the use of homogeneous-based Lyapunov functions [22] and the adding a power integrator technique [15], we develop a new, systematic design algorithm that simultaneously constructs a C^1 control Lyapunov function, which is positive definite and proper, and a C^0 state feedback control law that renders the trivial solution of the triangular system (1.1) *globally strongly stable* in the sense of Kurzweil [4]. This in turn leads to a rather surprising but important conclusion on global stabilization of nonlinear systems: *every chain of odd power integrators perturbed by a C^1 lower-triangular vector field is globally strongly stabilizable by continuous state feedback*.

The paper is organized as follows: In Section 2, we review a concept related to the general notions of stability and *strong stability* introduced by Kurzweil in the continuous framework [4]. We then introduce three technical lemmas that will be frequently used in the paper. The main result of the paper is given in Section 3, where a *continuous* state feedback control law is explicitly constructed, making the trivial solution of the lower-triangular system (1.1) *globally strongly stable*. Section 4 includes two simple yet challenging examples for which global stabilization, to the best of our knowledge, cannot be achieved by any existing *continuous* feedback control schemes but by the new strategy proposed in this work. Concluding remarks are drawn in Section 5.

2 Preliminaries

The classical Lyapunov stability theory (e.g. see [8, 14]) and the well-known concepts such as stability and asymptotic stability in the sense of Lyapunov can only be applied to a nonlinear differential equation whose solution from any initial condition is *unique*. However, a differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \quad (2.1)$$

with $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ being a *continuous*, non-Lipschitz mapping and $f(0) = 0$, may have more than one solution starting from a given initial condition (e.g. $\dot{x} = x^{2/3}$ with $x(0) = 0$), it is therefore necessary to introduce different (from Lyapunov) notions of stability and asymptotic stability in the continuous framework. In [4] Kurzweil introduced the new notions of stability for continuous systems (2.1) and established Lyapunov's second theorem as well as the converse theorem of Lyapunov on stability, *without requiring uniqueness* of the trajectory of (2.1). In what follows, we recall Kurzweil's definition on *global strong stability* which will be used in the next section.

Definition 2.1 (pp. 69, [4] or pp. 469 in [18]) The trivial solution $x = 0$ of (2.1) is *globally strongly stable (GSS)* if there are two functions $B : (0, +\infty) \rightarrow (0, +\infty)$ and $T : (0, +\infty) \times (0, +\infty) \rightarrow (0, +\infty)$ with B being increasing and $\lim_{\varepsilon \rightarrow 0} B(\varepsilon) = 0$, such that $\forall \beta > 0$ and $\forall \varepsilon > 0$, for every solution $x(t)$ of (2.1) defined on $[0, t_1)$, $0 < t_1 \leq +\infty$ with $\|x(0)\| \leq \beta$, there exists a solution $z(t)$ of (2.1) defined on $[0, +\infty)$ satisfying the following:

- (i) $z(t) = x(t)$, $t \in [0, t_1)$; (ii) $\|z(t)\| \leq B(\beta)$, $\forall t \geq 0$; (iii) $\|z(t)\| < \varepsilon$, $\forall t \geq T(\beta, \varepsilon)$.

This definition is clearly a generalization of global asymptotic stability in the sense of Lyapunov for a system of the form (2.1) that has a unique solution. With the help of the notion of GSS, Kurzweil proved [4] that the Lyapunov's second theorem remains true under the hypothesis that the function $f(x)$ is *continuous*.

Theorem 2.2 (Kurzweil) (pp. 23-24, [4]) Suppose there exists a C^1 Lyapunov function $V(x)$, which is *positive definite* and proper, such that

$$\frac{\partial V}{\partial x} f(x) < 0, \quad \forall x \in \mathbb{R}^n - \{0\}.$$

Then, the trivial solution $x = 0$ of the system (2.1) is *globally strongly stable*. ■

Obviously, Theorem 2.2 is analogous to Lyapunov's second theorem and contains the so-called global asymptotic stability as its special case when the solution is unique. Intuitively speaking, Kurzweil's GSS theorem, together with Definition 2.1, implies that every solution $x(t)$ of the continuous system (2.1) with any initial condition $x(0)$ is stable in the sense of Kurzweil and converges to zero as time goes to infinity, no matter whether the system (2.1) has a unique solution or not.

We conclude this section with three lemmas whose proofs are a direct application of Young's inequality. They will be frequently used throughout this paper.

Lemma 2.3 For $x \in \mathbb{R}, y \in \mathbb{R}, p \geq 1$ is an integer, the following inequalities hold:

$$|x + y|^p \leq 2^{p-1} |x^p + y^p|, \quad (2.2)$$

$$(|x| + |y|)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}} (|x| + |y|)^{\frac{1}{p}} \quad (2.3)$$

As a consequence, when $p \geq 1$ is an *odd* integer,

$$|x - y|^p \leq 2^{p-1} |x^p - y^p|. \quad (2.4)$$

Lemma 2.4 For any positive real numbers c, d and any real-valued function $\gamma(x, y) > 0$,

$$|x|^c |y|^d \leq \frac{c}{c+d} \gamma(x, y) |x|^{c+d} + \frac{d}{c+d} \gamma^{-\frac{c}{d}}(x, y) |y|^{c+d}. \quad (2.5)$$

Lemma 2.5 Let $a \geq 0, b \geq 0$, be real numbers and $p \geq 1, q \geq 1$, be integers. Then,

$$a^{p-1} b^q \leq a^p + b^{pq}. \quad (2.6)$$

3 Explicit Construction of C^0 GSS Controllers

In this section we present the main result of the paper, which shows that the problem of global strong stabilization (GSS)

of the triangular system (1.1) is solvable by *continuous state feedback, without requiring any extra conditions*. Moreover, a C^0 GSS controller can be explicitly constructed, using a new iterative design algorithm that combines the idea of using homogeneous-based Lyapunov functions [22] with the adding a power integrator technique [15].

Theorem 3.1 For a chain of *odd* power integrators perturbed by a C^1 lower-triangular vector field (1.1), there exists a constructive continuous controller $u = u(x)$ with $u(0) = 0$, such that the trivial solution $x = 0$ of the closed-loop system is globally strongly stable (**GSS**). ■

Remark 3.2 By assumption, the function $f_i(\cdot)$ in (1.1) is C^1 and vanishes at the origin. Then, by the Taylor expansion formula with integration remainder, there exists a smooth non-negative function $\gamma_i(x_1, \dots, x_i)$ such that

$$|f_i(x_1, \dots, x_i)| \leq (|x_1| + \dots + |x_i|)\gamma_i(\cdot), \quad i = 1, \dots, n. \quad (3.1)$$

This property will be used in the proof of Theorem 3.1. ■

Proof of Theorem 3.1. The proof is based on a *continuous type of adding a power integrator* technique which simultaneously constructs a C^1 control Lyapunov function, which is positive definite and proper, as well as a *continuous GSS feedback control law*.

Step 1. Choose the C^1 Lyapunov function

$$V_1(x_1) = \frac{x_1^{1+\frac{1}{p_1}}}{1+\frac{1}{p_1}}.$$

Using (1.1) and (3.1), we have

$$\begin{aligned} \dot{V}_1(x_1) &= x_1^{\frac{1}{p_1}} [x_2^{p_1} - x_2^{*p_1}] + x_1^{\frac{1}{p_1}} (x_2^{p_1} + f_1(x_1)) \\ &\leq x_1^{\frac{1}{p_1}} [x_2^{p_1} - x_2^{*p_1}] + x_1^{\frac{1}{p_1}} x_2^{*p_1} + x_1^{1+\frac{1}{p_1}} \gamma_1(x_1). \end{aligned}$$

Then, the *continuous* virtual controller x_2^* defined by

$$x_2^{*p_1} = -x_1(n + \gamma_1(x_1)) := -x_1\beta_1(x_1) \quad (3.2)$$

yields

$$\dot{V}_1(x_1) \leq -nx_1^{1+\frac{1}{p_1}} + x_1^{\frac{1}{p_1}} [x_2^{p_1} - x_2^{*p_1}]. \quad (3.3)$$

Inductive Step. Suppose at step $k-1$, there are a C^1 Lyapunov function $V_{k-1}(x_1, \dots, x_{k-1})$, which is positive definite and proper, and a set of C^0 virtual controllers x_1^*, \dots, x_k^* , defined by

$$\begin{aligned} x_1^* &= 0 & \xi_1 &= x_1 - x_1^*, \\ x_2^{*p_1} &= -\xi_1\beta_1(x_1) & \xi_2 &= x_2^{p_1} - x_2^{*p_1} \\ & \vdots & & \vdots \\ x_k^{*p_1 \cdots p_{k-1}} &= -\xi_{k-1}\beta_{k-1}(\cdot) & \xi_k &= x_k^{p_1 \cdots p_{k-1}} - x_k^{*p_1 \cdots p_{k-1}}, \end{aligned} \quad (3.4)$$

with $\beta_1(x_1) > 0, \dots, \beta_{k-1}(x_1, \dots, x_{k-1}) > 0$, being *smooth*, such that

$$\dot{V}_{k-1} \leq -(n-k+2) \sum_{j=1}^{k-1} \xi_j^{1+\frac{1}{p_1}} + \xi_{k-1}^{q_{k-1}} [x_k^{p_1 \cdots p_{k-1}} - x_k^{*p_1 \cdots p_{k-1}}] \quad (3.5)$$

with $q_{k-1} := 1 + \frac{1}{p_1} - \frac{1}{p_1 \cdots p_{k-2}}$. We shall show that (3.5) also holds at step k . To see how this can be done, consider the Lyapunov function

$$\begin{aligned} V_k(x_1, \dots, x_k) &= V_{k-1}(x_1, \dots, x_{k-1}) + W_k(x_1, \dots, x_k) \\ W_k(\cdot) &= \int_{x_k^*}^{x_k} (s^{p_1 \cdots p_{k-1}} - x_k^{*p_1 \cdots p_{k-1}})^{q_k} ds, \end{aligned} \quad (3.6)$$

where $q_k := 1 + \frac{1}{p_1} - \frac{1}{p_1 \cdots p_{k-1}}$. The function $V_k(x_1, \dots, x_k)$ thus defined has several important properties listed in the following two propositions whose proofs involve a tedious calculation and can be found in [17].

Proposition 1 $W_k(x_1, \dots, x_k)$ is C^1 . Moreover, for $l = 1, \dots, k-1$, the following holds

$$\begin{aligned} \frac{\partial W_k}{\partial x_k} &= \xi_k^{q_k} \\ \frac{\partial W_k}{\partial x_l} &= -q_k \frac{\partial x_k^{*p_1 \cdots p_{k-1}}}{\partial x_l} \int_{x_k^*}^{x_k} (s^{p_1 \cdots p_{k-1}} - x_k^{*p_1 \cdots p_{k-1}})^{q_k-1} ds. \end{aligned}$$

Proposition 2 $V_k(x_1, \dots, x_k)$ is C^1 , positive definite and proper.

With the aid of Proposition 1, one can deduce from (3.5) that

$$\begin{aligned} \dot{V}_k &= \dot{V}_{k-1} + \frac{\partial W_k}{\partial x_k} \dot{x}_k + \sum_{l=1}^{k-1} \frac{\partial W_k}{\partial x_l} \dot{x}_l \\ &\leq -(n-k+2) \sum_{j=1}^{k-1} \xi_j^{1+\frac{1}{p_1}} + \xi_{k-1}^{q_{k-1}} [x_k^{p_1 \cdots p_{k-1}} - x_k^{*p_1 \cdots p_{k-1}}] \\ &\quad + \xi_k^{q_k} (x_{k+1}^{p_k} + f_k(x_1, \dots, x_k)) + \sum_{l=1}^{k-1} \frac{\partial W_k}{\partial x_l} \dot{x}_l. \end{aligned} \quad (3.7)$$

To continue the proof and the construction of a C^0 GSS controller, we need to introduce additional two propositions whose proofs are also given in [17]. They are quite useful when estimating the last two terms in the inequality (3.7).

Proposition 3 There is a non-negative smooth function $\tilde{\gamma}_k(x_1, \dots, x_k)$ such that

$$|f_k(x_1, \dots, x_k)| \leq \left(|\xi_1|^{\frac{1}{p_1 \cdots p_{k-1}}} + \dots + |\xi_k|^{\frac{1}{p_1 \cdots p_{k-1}}} \right) \tilde{\gamma}_k(\cdot).$$

Proposition 4 There is a non-negative smooth function $C_{k,l}(x_1, \dots, x_k)$ such that

$$\left| \frac{\partial x_k^{*p_1 \cdots p_{k-1}}}{\partial x_l} \dot{x}_l \right| \leq (|\xi_1| + \dots + |\xi_k|) C_{k,l}(x_1, \dots, x_k).$$

Using Proposition 3 and Lemma 2.4, one obtains the following estimate

$$|\xi_k^{q_k} f_k(\cdot)| \leq \frac{\xi_1^{1+\frac{1}{p_1}} + \dots + \xi_{k-1}^{1+\frac{1}{p_1}}}{4} + \xi_k^{1+\frac{1}{p_1}} \tilde{\rho}_k(\cdot), \quad (3.8)$$

where $\tilde{\rho}_k(x_1, \dots, x_k)$ is a non-negative smooth function.

By Lemmas 2.3-2.4, it is not difficult to show that

$$\begin{aligned} & \left| \xi_{k-1}^{q_{k-1}} [x_k^{p_1 \cdots p_{k-1}} - x_k^{*p_1 \cdots p_{k-1}}] \right| \\ & \leq |\xi_{k-1}|^{q_{k-1}} 2^{1-\frac{1}{p_1 \cdots p_{k-2}}} \left| x_k^{p_1 \cdots p_{k-1}} - x_k^{*p_1 \cdots p_{k-1}} \right|^{\frac{1}{p_1 \cdots p_{k-2}}} \\ & \leq \frac{\xi_1^{1+\frac{1}{p_1}} + \dots + \xi_{k-1}^{1+\frac{1}{p_1}}}{2} + \xi_k^{1+\frac{1}{p_1}} c_k, \quad c_k > 0. \end{aligned} \quad (3.9)$$

To estimate the last term in (3.7), we observe from Proposition 1 that for $l = 1, \dots, k-1$,

$$\begin{aligned} \left| \frac{\partial W_k}{\partial x_l} \right| &\leq q_k |x_k^* - x_k| |\xi_k|^{q_k-1} \left| \frac{\partial x_k^{*p_1 \dots p_{k-1}}}{\partial x_l} \right| \\ &\leq a_k |\xi_k|^{\frac{1}{p_1}} \left| \frac{\partial x_k^{*p_1 \dots p_{k-1}}}{\partial x_l} \right|, \quad a_k > 0. \end{aligned} \quad (3.10)$$

Then, combining (3.10) and Proposition 4 yields

$$\begin{aligned} \left| \sum_{l=1}^{k-1} \frac{\partial W_k}{\partial x_l} \dot{x}_l \right| &\leq a_k |\xi_k|^{\frac{1}{p_1}} \sum_{l=1}^{k-1} \left| \frac{\partial x_k^{*p_1 \dots p_{k-1}}}{\partial x_l} \right| |\dot{x}_l| \\ &\leq a_k |\xi_k|^{\frac{1}{p_1}} (|\xi_k| + \dots + |\xi_k|) \sum_{l=1}^{k-1} C_{k,l}(\cdot) \\ &\leq \frac{1}{4} \sum_{j=1}^{k-1} \xi_j^{1+\frac{1}{p_1}} + \xi_k^{1+\frac{1}{p_1}} \bar{\rho}_k(\cdot) \end{aligned} \quad (3.11)$$

The last inequality follows from Lemma 2.4.

Substituting the estimates (3.9), (3.8) and (3.11) into (3.7), we arrive at

$$\begin{aligned} \dot{V}_k &\leq -(n-k+1)(\xi_1^{1+\frac{1}{p_1}} + \dots + \xi_{k-1}^{1+\frac{1}{p_1}}) \\ &\quad + \xi_k^{q_k} x_{k+1}^{p_k} + \xi_k^{1+\frac{1}{p_1}} (c_k + \tilde{\rho}_k(\cdot) + \bar{\rho}_k(\cdot)) \end{aligned}$$

Clearly, a *continuous* virtual controller $x_{k+1}^{*p_k}(x_1, \dots, x_k)$ of the form

$$x_{k+1}^{*p_k} = -\xi_k^{\frac{1}{p_1 \dots p_{k-1}}} (n-k+1 + c_k + \tilde{\rho}_k(\cdot) + \bar{\rho}_k(\cdot)) \quad (3.12)$$

or, equivalently,

$$x_{k+1}^* = -(\xi_k \beta_k(x_1, \dots, x_k))^{\frac{1}{p_1 \dots p_k}} \quad (3.13)$$

with $\beta_k(\cdot) := (n-k+1 + c_k + \tilde{\rho}_k(\cdot) + \bar{\rho}_k(\cdot))^{p_1 \dots p_{k-1}} > 0$ being smooth, is such that

$$\dot{V}_k \leq -(n-k+1) \sum_{j=1}^k \xi_j^{1+\frac{1}{p_1}} + \xi_k^{q_k} [x_{k+1}^{p_k} - x_{k+1}^{*p_k}]$$

This completes the inductive proof.

Using repeatedly the above inductive argument, we conclude immediately, at the n -th step, that there exists a *static continuous* state feedback control law

$$u = x_{n+1}^* = -(\xi_n \beta_n(x_1, \dots, x_n))^{\frac{1}{p_1 \dots p_n}} \quad (3.14)$$

with ξ_n being defined by (3.4), and a C^1 positive definite and proper Lyapunov function $V_n(x_1, \dots, x_n)$ of the form (3.6), such that

$$\dot{V}_n(x_1, \dots, x_n) \leq -(\xi_1^{1+\frac{1}{p_1}} + \dots + \xi_n^{1+\frac{1}{p_1}}). \quad (3.15)$$

By a direct calculation, it is easy to verify that the function on the right-hand side of (3.15) is *negative definite*. Using Kurzweil's Theorem (Theorem 2.2), it is concluded from Proposition 2 and (3.15) that the trivial solution $x = 0$ of the closed-loop system (1.1)–(3.14) is *globally strongly stable*. \blacksquare

Remark 3.3 From the constructive proof of Theorem 3.1, it is not difficult to see that the GSS controller we designed has the following structure

$$\begin{aligned} u^{p_1 \dots p_n} &= -\beta_n(\cdot) [x_n^{p_1 \dots p_{n-1}} + \beta_{n-1}(\cdot) (x_{n-1}^{p_1 \dots p_{n-2}} + \dots \\ &\quad + \beta_2(\cdot) (x_2^{p_1} + \beta_1(\cdot) x_1) \dots)]. \end{aligned} \quad (3.16)$$

In other words, $u^{p_1 \dots p_n}$ is *smooth* (i.e. $u^{p_1 \dots p_n} \in C^\infty(\mathbb{R}^n, \mathbb{R})$) although the GSS controller (3.14) is only *continuous* ($u \in C^0(\mathbb{R}^n, \mathbb{R})$). For this reason, we say, with a little abuse of the terminology, that the GSS controller (3.14) is “*almost smooth*” in \mathbb{R}^n . \blacksquare

4 Examples and Discussions

In this section we demonstrate, by means of two examples, how C^0 globally strongly stabilizing (GSS) controllers can be explicitly constructed, using the *continuous type* of adding a power integrator technique developed in the previous section. Simulation results are also presented to illustrate the main features of our continuous feedback control schemes.

Example 4.1 Consider the planar system

$$\begin{aligned} \dot{x}_1 &= x_2^3 + x_1 e^{x_1} \\ \dot{x}_2 &= u \end{aligned} \quad (4.1)$$

whose linearization has the exactly same form as that of the system (1.2), and hence contains an uncontrollable mode associated with a positive eigenvalue. Due to the existence of the severe nonlinearity $x_1 e^{x_1}$ which grows exponentially, the system (4.1) is *not homogeneous*. It *neither belongs* to the class of triangular systems considered in [22, 15]. Using the idea of *homogeneous approximation* [12, 13, 9, 10], it is straightforward to prove that there is a continuous state feedback control law, making (4.1) *locally* asymptotically stable. In fact, a homogeneous approximation of the system (4.1) is given by (1.2) for which a *homogeneous* stabilizing controller was already known; see, for instance, the papers [12, 13, 6]. Using the theorem on robust stability of homogeneous systems [9, 18], it is immediately concluded that the same *homogeneous* controller *locally* asymptotically stabilizes the system (4.1). However, global stabilization of the system (4.1) is, to the best of our knowledge, an open problem. It cannot be solved by any existing methods.

Next we shall show how the global stabilization problem can be solved by the proposed continuous feedback control scheme, without requiring homogeneity or growth conditions of the system.

As proceeded in the proof of Theorem 3.1, we first choose the C^1 Lyapunov function $V_1(x_1) = \frac{3}{4} x_1^{\frac{4}{3}}$, which is positive definite and proper. A simple calculation shows that the virtual controller $x_2^* = -(x_1(2 + e^{x_1}))^{\frac{1}{3}}$ renders

$$\dot{V}_1(x_1) \leq -2x_1^{\frac{4}{3}} + x_1^{\frac{1}{3}} (x_2^3 - x_2^{*3}). \quad (4.2)$$

Denote $\xi_2 = x_2^3 - x_2^{*3}$ and construct a positive definite and proper Lyapunov function

$$V_2(x_1, x_2) = V_1(x_1) + \frac{1}{20} \left[\frac{x_2^4}{4} - x_2 x_2^{*3} + \frac{3x_2^{*4}}{4} \right], \quad (4.3)$$

which is continuously differentiable. Then, the time derivative of $V_2(\cdot)$ along the trajectories of (4.1) is

$$\dot{V}_2 \leq -2x_1^{\frac{4}{3}} + x_1^{\frac{1}{3}}\xi_2 + \frac{\xi_2 u}{20} + \frac{\alpha(x_1)}{20}[x_2 - x_2^*](\xi_2 - 2x_1), \quad (4.4)$$

where

$$\alpha(x_1) = 2 + (1 + x_1)e^{x_1}. \quad (4.5)$$

By Lemma 2.4, we have

$$|x_1^{\frac{1}{3}}\xi_2| \leq \frac{1}{4}x_1^{\frac{4}{3}} + \frac{3}{4}\xi_2^{\frac{4}{3}} \quad (4.6)$$

Using Lemmas 2.3–2.4, it can be shown that

$$\begin{aligned} & \left| \frac{1}{20}(x_2 - x_2^*)\alpha(x_1)(\xi_2 - 2x_1) \right| \\ & \leq \xi_2^{\frac{4}{3}} \frac{1 + \alpha^2(x_1)}{20 \times 2^{\frac{1}{3}}} + \frac{3x_1^{\frac{4}{3}}}{4} + \frac{1}{20} \left[\frac{\alpha(x_1)}{4} \right]^4 \xi_2^{\frac{4}{3}}. \end{aligned} \quad (4.7)$$

Substituting (4.6) and (4.7) into (4.4), we have

$$\dot{V}_2 \leq -x_1^{\frac{4}{3}} + \frac{1}{20}\xi_2 u + \xi_2^{\frac{4}{3}} \left(\frac{3}{4} + \frac{1 + \alpha^2(x_1)}{20 \times 2^{\frac{1}{3}}} + \frac{1}{20} \left[\frac{\alpha(x_1)}{4} \right]^4 \right)$$

Clearly, the continuous state feedback control law

$$u = -\xi_2^{\frac{1}{3}} \left(35 + \left(\frac{\alpha(x_1)}{4} \right)^4 + \frac{1 + \alpha^2(x_1)}{2^{\frac{1}{3}}} \right) \quad (4.8)$$

is such that

$$\dot{V}_2 \leq -x_1^{\frac{4}{3}} - \xi_2^{\frac{4}{3}} = -x_1^{\frac{4}{3}} - [x_2^3 + (x_1(2 + e^{x_1}))]^{\frac{4}{3}}.$$

This, in turn, implies that the trivial solution of the closed-loop system (4.1)–(4.8)–(4.5) is **GSS**.

A simulation result is shown in Fig. 1, illustrating effectiveness of the C^0 controller (4.8)–(4.5) as well as dynamic performance of the closed-loop system. ■

Remark 4.2 As a consequence of Example 4.1, one concludes that the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2^3 + x_1 e^{x_1} \\ \dot{x}_2 &= x_3^5 \\ \dot{x}_3 &= u \end{aligned} \quad (4.9)$$

is also *globally strongly stabilizable* by *continuous* state feedback. A C^0 GSS controller can be explicitly constructed, using the *continuous type of adding a power integrator* technique developed in the last section. Indeed, the controller for the system (4.9) can be easily derived based on the result from Example 4.1. It is interesting to notice that (4.9) satisfies *neither the hypothesis (A1) nor (A2) in [15]*, and cannot be handled by any existing methods. The example (4.9) has clearly illustrated the power of *continuous* feedback design as well as some significant advantages of *continuous* feedback over *smooth* feedback. ■

The next example illustrates how a C^0 GSS controller can be designed in the three-dimensional case.

Example 4.3 Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^3 + \frac{1}{2}(\ln(1 + x_2^2) + \sin x_2) \\ \dot{x}_3 &= u \end{aligned} \quad (4.10)$$

Apparently, the linearized system shares the similar property of the system (4.1), and hence *no smooth* controllers can stabilize the above system. Moreover, (4.1) is *not homogeneous*. Using the homogeneous approximation method, a C^0 stabilizer can be designed for the homogeneous system $\dot{x} = (x_2, x_3^3 + \frac{1}{2}x_2, u)^T$, as done in [13, 3, 6, 22]. By the robust stability of homogeneous systems [9, 18], the controller thus obtained also renders the system (4.10) *locally asymptotically stable*.

To achieve *global stabilization* of (4.10), one may try to apply the existing design methods in [16, 15, 22]. Unfortunately, due to the nonlinear term $(\ln(1 + x_2^2) + \sin x_2)$, global stabilization of the system (4.10) cannot be dealt with by any existing methods including those suggested in [16, 15, 22]. However, according to Theorem 3.1, there does exist a C^0 controller that GSS the system (4.10).

Similar to the design procedure in Example 4.1, one can construct a C^0 state feedback control law

$$u = -245 (x_3^3 + 6(x_2 + x_1))^{\frac{1}{3}} \quad (4.11)$$

which renders the trivial solution $x = 0$ of the system (4.10) **GSS**. The conclusion can actually be verified by a straightforward but tedious calculation, using the C^1 control Lyapunov function (which is positive definite and proper)

$$V(x_1, x_2, x_3) = 2x_1^2 + \frac{(x_1 + x_2)^2}{2} + \frac{1}{76} \int_{x_3^*}^{x_3} (s^3 - x_3^{*3})^{\frac{5}{3}} ds, \quad (4.12)$$

where $x_3^* = -[6(x_2 + x_1)]^{\frac{1}{3}}$.

The response of the closed-loop system (4.10)–(4.11) is shown in Fig. 2, demonstrating that the controller (4.11) globally strongly stabilizes (GSS) the three dimensional system (4.10), with a satisfactory dynamic performance. ■

5 Conclusions

The main contribution of this paper has been the development of a constructive algorithm that shows how to explicitly design a *continuous, globally strongly stabilizing (GSS)* controller, for a chain of odd power integrator perturbed by a C^1 lower-triangular vector field. The systems under consideration in general involve an *uncontrollable unstable* Jacobian linearization which implies the *non-existence of smooth stabilizers*. By combining the idea of the use of homogeneous-like Lyapunov functions [22] and *extending the adding a power integrator technique* [16, 15] to its *continuous counterpart*, we developed a new continuous feedback control scheme that allows one to solve the problem of stabilization in the large, for a class of highly nonlinear systems having a triangular structure, *without imposing any growth conditions* (e.g. [16, 22]) or homogeneous property (e.g. [22]) on the systems.

The machinery developed in the paper, namely a *continuous version of adding a power integrator*, is believed to be very useful in studying a number of robust control problems for uncertain lower-triangular systems. For example, the problems of robust stabilization, adaptive regulation and disturbance attenuation can be naturally considered in the continuous framework. These problems are currently under investigation and the results will be reported elsewhere.

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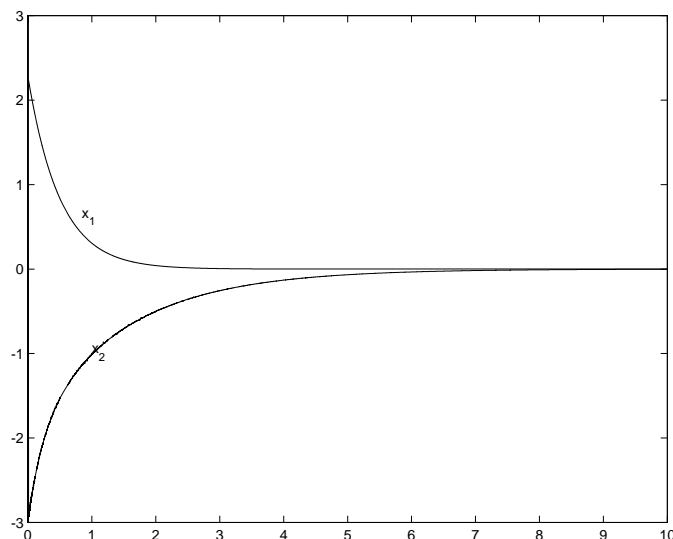


Fig. 1: State trajectories of the closed-loop system (4.1)-(4.8)-(4.5) with $x_1(0) = 2$, $x_2(0) = 3$

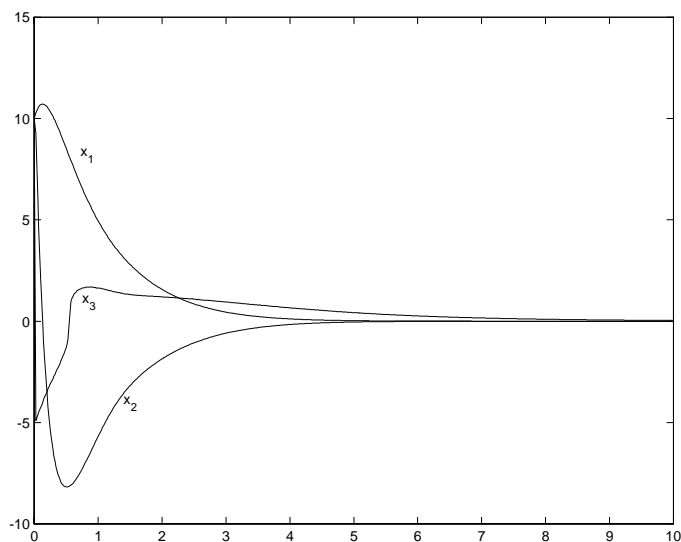


Fig. 2: State trajectories of the system (4.10)–(4.11) with $x_1(0) = x_2(0) = x_3(0) = 10$.