Practical Output Tracking of Nonlinear Systems with Applications to Underactuated Mechanical Systems *

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Abstract

This paper shows that for a significant class of nonlinear systems with uncontrollable unstable linearization, *global practical output tracking* is achievable by *smooth* feedback, although *asymptotic* output tracking is usually not possible, even locally. Smooth tracking controllers are explicitly constructed via a modified *adding a power integrator* approach. This new design method also leads to solutions to various open control problems, including practical output tracking of an underactuated unstable two degrees of freedom mechanical system.

1 Introduction

One of the most important problems in nonlinear control theory is to design a feedback law having the output of a controlled plant asymptotically track a prescribed smooth reference signal. The problem has a long-standing history and has been thoroughly investigated over the decades. The recent survey [2] and monograph [1] provide a fairly complete review on the major developments and achievements in nonlinear output regulator theory.

When the reference trajectories are generated by an autonomous finite-dimensional dynamical system, known as the exosystem, the problem is commonly called *nonlinear output regulation* or *servomechanism problem,* which has been considered in [5, 7, 8]. The major breakthrough was made in the seminar paper [8], where the output regulation problem was shown to be solvable if and only if a set of partial differential equations, i.e. the regulator equations, permit a solution. The work [8] has not only generated a number of new research issues but also stimulated extensive research; see, for instance, the works [6, 10] on computational approaches to solving the regulator equations, the development of robust regulator theory [1, 11] and inversion-based output tracking scheme [4], and the so-called asymptotically robust perfect tracking approach [14].

Most of the existing solutions to the problem of output regulation are derived based on the assumptions that the Jacobian linearization of nonlinear systems is *stabilizable and detectable.* In fact, stabilizability and detectability, as illustrated in [8, 1, 7, 11], are two crucial conditions for the nonlinear regulator problem to be solvable by either state or error feedback. For a nonlinear system whose linearization is unobservable and/or uncontrollable, little attention has been paid to the output regulation problem in the literature, except the two recent papers [13, 3].

The work [3] studied the *local* output regulation problem for a class of higher-order triangular systems. A local C^0 controller was designed, forcing the tracking error within a prior given bound [3]. In [13] we studied the problem of *global* output tracking for the SISO nonlinear systems

$$\begin{aligned} \dot{x}_1 &= d_1(t,x,u)x_2^{p_1} + \phi_1(t,x,u) \\ \dot{x}_2 &= d_2(t,x,u)x_3^{p_2} + \phi_2(t,x,u) \\ &\cdots \\ \dot{x}_n &= d_n(t,x,u)u^{p_n} + \phi_n(t,x,u) \quad y = x_1 \end{aligned} \tag{1.1}$$

where $p_i \geq 1$ is an *odd* integer, $\phi_i : \mathbb{R}^{n+2} \to \mathbb{R}^1$ and $d_i : \mathbb{R}^{n+2} \to \mathbb{R}^1$ are C^1 functions which may not necessarily be known and may represent an *unknown* time varying function and parameter, respectively. It was proved in [13] that the problem of *global asymptotic* output tracking of a *constant* signal is solvable by smooth feedback, under the following restrictive conditions: $d_i(\cdot) \equiv 1$ and

A1) $p_1 \geq p_2 \geq \cdots \geq p_n \geq 1$ are *odd* integers;

A2) $\phi_1(\cdot) = 0$ and for $i = 2, \cdots, n$, there are smooth functions $\rho_i(x_1, \cdots, x_i) \geq 0$, such that

$$|\phi_i(\cdot)| \leq (|x_2|^{p_i} + \cdots + |x_i|^{p_i})\rho_i(x_1, \cdots, x_i).$$

However, a fundamental question of *whether global output tracking of a time-varying reference signal is solvable via smooth feedback* remains unclear and largely open.

In this paper we concentrate on this challenging problem. In lieu of the assumption that the reference signals are generated by an exosystem, we suppose that the reference signal is a *bounded* and *time-varying* trajectory with bounded first derivative. The control objective is to seek a *smooth* feedback controller such that the output of the system (1.1) globally follows the reference signal. In Section 2, we first present a simple example to illustrate that in general, *asymptotic output tracking* for nonlinear systems (1.1) with *uncontrollable and unobservable linearization is not possible by any smooth feedback,* even locally. Being aware of this negative result, we then formulate the problem of *global practical output tracking (GPOT).* In Section 3 we develop a novel tracking control scheme that is based upon, but modifies the *adding a power integrator* approach proposed in [12, 13]. Using this new design method, a *smooth* controller is explicitly constructed, leading to a solution to the problem of *global practical output tracking* for nonlinear systems (1.1) . As an important consequence, we conclude that the GPOT problem is solvable for a chain of integrators

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perturbed by a triangular vector field, *without imposing any extra growth conditions* such as **A1)-A2).** In Section 4, our output tracking control scheme is applied to an underactuated unstable mechanical system [15], which is exceptionally difficult to control. Although the system is known not to be smoothly stabilizable, it is shown that GPOT is achievable. A smooth tracking controller that solves the problem is explicitly constructed, via the modified adding a power integrator technique. Conclusions are given in Section 5.

2 Problem Formulation

A standard *global asymptotic output tracking* problem is the following one: given a bounded reference signal $y_r(t)$ with bounded derivatives $y_r^{(1)}(t), \cdots, y_r^{(n)}(t)$, find, if possible, a *smooth time-varying state* feedback control law

$$u = \alpha(x, y_r(t), \cdots, y_r^{(n)}(t)), \tag{2.1}$$

such that the system (1.1)–(2.1) satisfies the following: **(i)** all the states of the closed-loop system (1.1)-(2.1) are globally bounded; **(ii)** the output of the closed-loop system (1.1)-(2.1) starting from any initial state $x(0) \in \mathbb{R}^n$ is such that $\lim_{t\to\infty} |y - y_r(t)| = 0$.

In the case of feedback linearizable systems, i.e. $p_i = 1$, $d_i(\cdot) = 1$ and $\phi_i(t, x, u) = \phi_i(x_1, \cdots, x_i)$ with $\phi_i(0) = 0$ in (1.1), (1.1) is feedback equivalent to a chain of *linear* integrators. As a consequence, a solution to the asymptotic tracking problem follows from the global stabilization result for feedback linearizable systems. In fact, with a change of coordinates, it is easy to show that *stabilizability of the feedback linearizable system implies solvability of the problem of asymptotic output tracking.* In the case when $p_i > 1$, (1.1) becomes, however, a highly nonlinear system whose Jacobian linearization may have *uncontrollable unstable modes* associated with eigenvalues on the right-half plane. By the well-known necessary condition, (1.1) is not smoothly stabilizable. Moreover, the system is *not affine in the control input.* All of this makes the problem of *asymptotic output tracking* for (1.1) far more difficult and challenging. Unlike in the feedback linearizable case, *stabilizability of (1.1) does not necessarily imply the existence of a solution to the tracking problem.* To illustrate this subtle point, we examine the *asymptotic output tracking* problem for a planar system.

Example 2.1. Consider the SISO planar system

$$\begin{aligned} \dot{x}_1 &= x_2^p - x_1^4 + x_1^5 \\ \dot{x}_2 &= u^p, \quad y = x_1, \end{aligned} \tag{2.2}$$

where $p = 1$ or 3. The problem of interest is to investigate a possibility of having the system output track $y_r(t) = 1$ asymptotically.

It must be noted that in the case of $p = 1$ or 3, the global stabilization problem is solvable by smooth state feedback, using the adding a power integrator method [12]. However, the asymptotic output tracking problem is radically different in the two cases, as illustrated below.

Let $e = x_1 - y_r(t) = x_1 - 1$. The system (2.2) can be expressed, in the new coordinates, as

$$\dot{e} = x_2^p + (1+e)^4 e, \quad \dot{x}_2 = u^p. \tag{2.3}$$

When $p = 1$, the Jacobian linearization of (2.2) or (2.3) is controllable and the system (2.3) is *feedback linearizable.* As a result, the *asymptotic output tracking* problem of the system (2.2) is solvable and its solution can be simply derived from the stabilization solution of the system (2.3).

When $p = 3$, the linearized system of (2.3) is uncontrollable. Moreover, the uncontrollable mode has a positive eigenvalue, and therefore the system (2.3) cannot be stabilized, even locally, by any smooth (or C^r, $r \geq 1$) state feedback control laws. In conclusion, it is not possible to solve, using *smooth feedback,* the problem of *global asymptotic output tracking* for the system (2.2) with $p = 3$. ∎

Being aware of this negative result, we pursue in this paper a less ambitious goal and concentrate on the problem of global *practical* (instead of asymptotic) output tracking. More precisely, we are interesting in the following problem.

Global Practical Output Tracking (GPOT): Let $y_r(t)$ be a bounded C^1 reference signal with bounded $\dot{y}_r(t)$. For any $\varepsilon > 0$, find, if possible, a smooth controller

$$u = \alpha(x, y_r(t)), \tag{2.4}$$

such that

(a) the state of the closed-loop system (1.1)-(2.4) is well-defined on $[0, +\infty)$ and globally bounded;

(b) for every $x(0) \in \mathbb{R}^n$, there is a finite time $T_{(\varepsilon, x(0))} > 0$ such that the output of the system (1.1)-(2.4) satisfies

$$|y(t) - y_r(t)| < \varepsilon, \qquad \forall\ t \geq T > 0. \tag{2.5}$$

It is worth noticing that in the setting of *practical output tracking,* only boundedness of $y_r(t)$ and $\dot{y}_r(t)$ is required and *no boundedness conditions are imposed on the higher-order derivatives of* $y_r(t)$*.* This is one of the major differences between *practical* and *asymptotic* tracking. Another key difference is that implementing the *practical* tracking controller (2.4) needs only the reference signal $y_r(t)$ but its derivatives $\dot{y}_r, \cdots, y_r^{(n)}$, which makes the controller (2.4) more feasible than the *asymptotic* tracking controller (2.1).

We end this section with two lemmas that will be very useful in deriving solutions to the GPOT problem. They can be easily proved by Young and Gronwall inequalities.

Lemma 2.2 For any real numbers $a \geq 0$, $b > 0$ and $m \geq 1$, the following inequality holds:

$$a \leq b + \left[\frac{a}{m}\right]^m \left[\frac{m-1}{b}\right]^{m-1}. \tag{2.6}$$

Lemma 2.3 Let p_i, $i = 1, 2, \cdots, n$, be *odd* positive integers and $p = \max\{p_i,\ i = 1, \cdots, n\}$. Suppose the Lyapunov function $V = \sum_{i=1}^n \frac{\xi_i^{p-p_i+2}}{p-p_i+2}$, which is positive definite and proper, satisfies

$$\dot{V} \leq -(\xi_1^{p+1} + \cdots + \xi_n^{p+1}) + n\delta, \tag{2.7}$$

where $1 > \delta > 0$ is a real constant. Then, there is a finite time $T > 0$, such that

$$V(\xi(t)) \leq \frac{2}{p+1} \left(\sum_{i=1}^n \frac{p_i - 1}{p - p_i + 2} + n \right) \delta^{\frac{2}{p+1}}, \quad \forall t \geq T. \ ∎$$

3 Main Results

In this section, we present an iterative design method that is based upon, but modifies the technique of *adding a power integrator* [12, 13]. The new feedback design approach leads to an explicit construction of smooth controllers that solve the GPOT problem, for a class of uncertain nonlinear systems (1.1) that are characterized by the following conditions:

A3.1 For $i = 1, \cdots, n$, there are real numbers c_i and $\bar{c}_i$ such that $0 < c_i \le d_i(t, x, u) \le \bar{c}_i$.

A3.2 For $i = 1, \cdots, n$, there exist smooth functions $b_{i,j}(x_1, \cdots, x_i) \ge 0$, such that

$$|\phi_i(t, x, u)| \le \sum_{j=0}^{p_i - 1} |x_{i+1}|^j b_{i,j}(x_1, \cdots, x_i), \qquad x_{n+1} = u.$$

Remark 3.3 **A3.2** requires that the power of x_{i+1} in the function $\phi_i(\cdot)$ be strictly less than p_i. This condition is somewhat necessary, as shown by the example

$$\dot{x}_1 = x_2^3 + x_2^p x_1 + x_1 + 1, \quad \dot{x}_2 = u, \quad y = x_1. \tag{3.1}$$

If $p \ge 3$, **A3.2** fails to be satisfied. In this case, we claim that it is impossible to achieve practical output tracking. To see this, consider $p = 3$ and choose $x_1(0) < -1$. Observe that the vertical line $x_1 = -1$ in the plane is an invariant manifold of the system (3.1), then $x_1(t, x_1(0)) = y \le -1$, $\forall t \ge 0$ and $\forall u$. Thus, no matter how to design the feedback control law u, it is impossible to have the output $y = x_1$ of (3.1) follow any smooth $y_r(t)$ with $|y_r(t)| < 1$. ∎

Under the hypotheses **A3.1-A3.2,** it is possible to prove the following result on global practical output tracking.

Theorem 3.4 For the system (1.1) satisfying **A3.1-A3.2,** there is a smooth controller of the form (2.4) which solves the problem of global practical output tracking.

Proof. We first use a *modified adding a power integrator* technique to construct a smooth controller and a Lyapunov function such that the closed-loop system satisfies a Lyapunov-like inequality (2.7).

Step 1. Introduce $p = \max_{i=1,\cdots,n}\{p_i\}$. Let $\xi_1 = x_1 - y_r$ be the tracking error. Clearly,

$$\dot{\xi}_1 = d_1(t, x, u) x_2^{p_1} + \phi_1(t, x, u) - \dot{y}_r(t). \tag{3.2}$$

By **A3.2** and Young's inequality, there is a smooth function $\gamma_1(x_1) \ge 0$, such that

$$\begin{aligned} |\phi_1(t, x, u)| &\le \sum_{i=0}^{p_1-1} \left[\frac{c_1 |x_2^{p_1}|}{2p_1} + \frac{p_1 - j}{p_1} \left(\frac{2j}{c_1}\right)^{\frac{j}{p_1 - j}} b_{1,j}^{\frac{p_1}{p_1 - j}}(x_1) \right] \\ &\le \frac{c_1 |x_2^{p_1}|}{2} + \gamma_1(x_1). \end{aligned} \tag{3.3}$$

This, together with boundedness of $\dot{y}_r$, implies

$$|\phi_1(t, x, u) - \dot{y}_r(t)| \le \frac{c_1 |x_2^{p_1}|}{2} + \tilde{\gamma}_1(\xi_1) \tag{3.4}$$

where $\tilde{\gamma}_1(\xi_1) \ge 0$ is a smooth function.

Now, consider the Lyapunov function $V_1(\xi_1) = \frac{\xi_1^{p - p_1 + 2}}{p - p_1 + 2}$. Using (3.4), a simple calculation yields

$$\dot{V}_1 \le d_1(t, x, u) \xi_1^{p - p_1 + 1} x_2^{p_1} + |\xi_1^{p - p_1 + 1}| \left[\frac{c_1 |x_2^{p_1}|}{2} + \tilde{\gamma}_1(\xi_1) \right] \tag{3.5}$$

For any positive real number δ, by Lemma 2.2 there is a smooth function $\rho_1(\xi_1) \ge 0$, satisfying

$$|\xi_1^{p - p_1 + 1}| \tilde{\gamma}_1(\xi_1) \le \xi_1^{p+1} \rho_1(\xi_1) + \delta. \tag{3.6}$$

Putting (3.5) and (3.6) together, we have

$$\dot{V}_1 \le d_1(t, x, u) \xi_1^{p - p_1 + 1} x_2^{p_1} + \frac{c_1}{2} |\xi_1^{p - p_1 + 1} x_2^{p_1}| + \xi_1^{p+1} \rho_1(\xi_1) + \delta.$$

Note that the virtual smooth controller

$$x_2^* = -\xi_1 \left[\frac{2n + 2\rho_1(\xi_1)}{c_1} \right]^{\frac{1}{p_1}} := -\xi_1 \beta_1(\xi_1), \quad \beta_1(\xi_1) > 0,$$

renders

$$\begin{aligned} \dot{V}_1 &\le -n\xi_1^{p+1} + \delta + d_1(t, x, u) \xi_1^{p - p_1 + 1} x_2^{p_1} \\ &\quad - \frac{c_1}{2} \xi_1^{p - p_1 + 1} x_2^{*p_1} + \frac{c_1}{2} |\xi_1^{p - p_1 + 1} x_2^{p_1}|. \end{aligned}$$

Using **A3.1** and the fact that $-\xi_1^{p - p_1 + 1} x_2^{*p_1} \ge 0$, we have

$$-\frac{c_1}{2} \xi_1^{p - p_1 + 1} x_2^{*p_1} \le -d_1(t, x, u) \xi_1^{p - p_1 + 1} x_2^{*p_1} - \frac{c_1}{2} |\xi_1^{p - p_1 + 1} x_2^{*p_1}|.$$

Hence,

$$\dot{V}_1 \le -n\xi_1^{p+1} + \delta + (\bar{c}_1 + \frac{c_1}{2}) |\xi_1^{p - p_1 + 1}| \, |x_2^{p_1} - x_2^{*p_1}|. \tag{3.7}$$

Inductive Step. Suppose at step $k-1$, there are a set of smooth virtual controllers $x_1^*, \cdots, x_k^*$, defined by

$$\begin{aligned} x_1^* &= y_r & \xi_1 &= x_1 - x_1^* \\ x_2^* &= -\xi_1 \beta_1(\xi_1) & \xi_2 &= x_2 - x_2^* \\ &\vdots & &\vdots \\ x_k^* &= -\xi_{k-1} \beta_{k-1}(\xi_1, \cdots, \xi_{k-1}) & \xi_k &= x_k - x_k^*, \end{aligned} \tag{3.8}$$

with $\beta_1(\xi_1) > 0, \cdots, \beta_{k-1}(\xi_1, \cdots, \xi_{k-1}) > 0$, being *smooth,* and a Lyapunov function $V_{k-1}(\xi_1, \cdots, \xi_{k-1}) = \sum_{j=1}^{k-1} \frac{\xi_j^{p - p_j + 2}}{p - p_j + 2}$, such that

$$\begin{aligned} \dot{V}_{k-1} &\le -(n - k + 2) \left(\xi_1^{p+1} + \cdots + \xi_{k-1}^{p+1}\right) + (k-1)\delta \\ &\quad + (\bar{c}_{k-1} + \frac{c_{k-1}}{2}) |\xi_{k-1}^{p - p_{k-1} + 1}| \left| x_k^{p_{k-1}} - x_k^{*p_{k-1}} \right|. \end{aligned} \tag{3.9}$$

We claim that (3.9) also holds at step k. To prove this claim, consider the Lyapunov function

$$V_k(\xi_1, \cdots, \xi_k) = V_{k-1}(\xi_1, \cdots, \xi_{k-1}) + \frac{\xi_k^{p - p_k + 2}}{p - p_k + 2}. \tag{3.10}$$

Using (3.9), one has

$$\begin{aligned} \dot{V}_k &\le -(n - k + 2) \left(\xi_1^{p+1} + \cdots + \xi_{k-1}^{p+1}\right) + (k-1)\delta \\ &\quad + \xi_k^{p - p_k + 1} \left[d_k(\cdot) x_{k+1}^{p_k} + \phi_k(\cdot) - \sum_{j=1}^{k-1} \frac{\partial x_k^*}{\partial x_j} \dot{x}_j - \frac{\partial x_k^*}{\partial y_r} \dot{y}_r \right] \\ &\quad + (\bar{c}_{k-1} + \frac{c_{k-1}}{2}) |\xi_{k-1}^{p - p_{k-1} + 1}| \left| x_k^{p_{k-1}} - x_k^{*p_{k-1}} \right|. \end{aligned} \tag{3.11}$$

Similar to (3.3)-(3.6) in Step 1, it is not difficult to prove there is a smooth function $\rho_k(\xi_1, \cdots, \xi_k) \ge 0$, such that

$$\begin{aligned} &\left| \xi_k^{p - p_k + 1} \left(\phi_k(\cdot) - \sum_{j=1}^{k-1} \frac{\partial x_k^*}{\partial x_j} [d_j(\cdot) x_{j+1}^{p_j} + \phi_j(\cdot)] - \frac{\partial x_k^*}{\partial y_r} \dot{y}_r \right) \right| \\ &\quad \le \delta + \frac{c_k}{2} |\xi_k^{p - p_k + 1} x_{k+1}^{p_k}| + \xi_k^{p+1} \rho_k(\xi_1, \cdots, \xi_k). \end{aligned}$$

Using Young's inequality, we have

$$
\begin{aligned}
&(\bar{c}_{k-1} + \frac{c_{k-1}}{2})|\xi_{k-1}^{p-p_{k-1}+1}| \left| x_k^{p_{k-1}} - x_k^{*p_{k-1}} \right| \\
&\leq \xi_1^{p+1} + \cdots + \xi_{k-1}^{p+1} + \xi_k^{p+1} \bar{\rho}_k(\xi_1, \cdots, \xi_k),
\end{aligned}
$$

where $\bar{\rho}_k(\cdot)$ is a smooth non-negative function.

The last two inequalities, together with (3.11), yield

$$
\begin{aligned}
\dot{V}_k \leq& -(n-k+1)\sum_{j=1}^{k-1} \xi_j^{p+1} + k\delta + \xi_k^{p-p_k+1} d_k(\cdot) x_{k+1}^{p_k} \\
&+ \frac{c_k}{2}|\xi_k^{p-p_k+1} x_{k+1}^{p_k}| + \xi_k^{p+1}[\rho_k(\cdot) + \bar{\rho}_k(\cdot)]. \qquad (3.12)
\end{aligned}
$$

Clearly, the virtual smooth controller

$$
x_{k+1}^* = -\xi_k \left[2\frac{n-k+1+\rho_k(\cdot)+\bar{\rho}_k(\cdot)}{c_k} \right]^{1/p_k}
$$

renders

$$
\begin{aligned}
\dot{V}_k \leq& -(n-k+1)\left(\xi_1^{p+1} + \cdots + \xi_k^{p+1}\right) + k\delta \\
&+ (\bar{c}_k + \frac{c_k}{2})|\xi_k^{p-p_k+1}(x_{k+1}^{p_k} - x_{k+1}^{*p_k})|
\end{aligned}
$$

which leads to the claim.

Using repeatedly the argument above, we concludes that at the n-th step, there are a set of transformations $(\xi_1, \cdots, \xi_n)$ of the form (3.8), a smooth Lyapunov function

$$
V_n(\xi_1, \cdots, \xi_n) = \sum_{i=1}^{n} \frac{\xi_i^{p-p_i+2}}{p-p_i+2}
$$

and a smooth controller $u(\xi_1, \cdots, \xi_n)$, such that

$$
\dot{V}_n \leq -\left(\xi_1^{p+1} + \cdots + \xi_n^{p+1}\right) + n\delta. \qquad (3.13)
$$

The last inequality, together with Lemma 2.3, implies that all the solutions $\xi_1(t), \cdots, \xi_n(t)$ of the closed-loop system are globally bounded and well-defined over $[0, +\infty)$. This, in turn, leads to the conclusion that the state $(x_1, \cdots, x_n)$ is globally bounded, because of the relation (3.8) and boundedness of y_r.

To achieve practical output tracking, we shall show that by choosing $\delta > 0$ appropriately, the output error $|\xi_1| = |y - y_r|$ can be made arbitrarily small in a finite time. To this end, we recall from Lemma 2.3 that for any $\delta > 0$, there is a finite time $T(\xi(0), \delta) > 0$, such that for all $t \geq T$,

$$
|y - y_r(t)| \leq \left[\frac{2(p-p_1+2)}{p+1} \left(\sum_{i=1}^{n} \frac{p_i - 1}{p-p_i+2} + n \right) \delta^{\frac{2}{p+1}} \right]^{\frac{1}{p-p_1+2}}
$$

Therefore, for any $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ so that

$$
|y - y_r(t)| < \varepsilon \qquad \forall t \geq T > 0. \quad \blacksquare
$$

Using Theorem 3.4, we arrive at the following important conclusion (*without requiring any growth conditions*).

Corollary 3.5 Consider the nonlinear system

$$
\begin{aligned}
\dot{x}_i &= x_{i+1}^{p_i} + \sum_{j=0}^{p_i-1} x_{i+1}^j a_{i,j}(x_1, \cdots, x_i), \quad i = 1, \cdots, n \\
y &= x_1, \qquad x_{n+1} = u, \qquad (3.14)
\end{aligned}
$$

where $a_{i,j}(x_1, \cdots, x_i)$, $i = 1, \cdots, n$, $j = 0, \cdots, p_i - 1$, are of class C^1. Then, the problem of global practical output tracking is solvable by the C^∞ controller (2.4). ∎

From Corollary 3.5 , we deduce an interesting output tracking result for a chain of power integrators perturbed by a triangular vector field, which is a special case of (3.14) (i.e. $a_{i,j}(\cdot) = 0$ for $1 \leq j \leq p_i - 1$, $1 \leq i \leq n$).

Corollary 3.6 The GPOT problem is solvable for a chain of power integrators perturbed by a lower-triangular vector field. ∎

Example 3.7 Consider the SISO planar system

$$
\dot{x}_1 = x_2^3 + x_1(e^{x_1} + x_2), \quad \dot{x}_2 = u, \quad y = x_1, \qquad (3.15)
$$

which is of the form (3.14). The goal is to design a smooth controller of the form (2.4) such that the output of the system (3.15) *practically* follows the reference signal

$$
y_r(t) = \begin{cases} \sin(\pi t) & t \in [6n, 6n+0.5) \\ 1 & t \in [6n+0.5, 6n+2.5) \\ \sin(\pi(t-2)) & t \in [6n+2.5, 6n+3.5) \\ -1 & t \in [6n+3.5, 6n+5.5) \\ \sin(\pi(t-4)) & t \in [6n+5.5, 6n+6). \end{cases} \quad n = 0, 1, \cdots
$$

Apparently, $y_r(t)$ is only a C^1 signal satisfying $|y_r(t)| \leq 1$ and $|\dot{y}_r(t)| \leq \pi$.

Note that the planar system (3.15) has some significant features which make the problem of stabilization or output tracking non-trivial. Firstly, (3.15) is *not in a lower-triangular form* and the growth condition **(A2)** is not satisfied. Hence, the adding a power integrator technique [13, 12] cannot be applied here. Secondly, the uncontrollable mode of the linearized system has a positive eigenvalue. Thus, there does not exist *any smooth* state feedback control law that can locally stabilize the system (3.15). Using the homogeneous approximation approach [9], one can design a *local continuous* stabilizer for (3.15). However, the *global stabilization* problem of (3.15) remains largely open. Needless to say, the problem of *global output tracking* is, to certain extent, more challenging and difficult than the feedback stabilization. Although solving the problem of *asymptotic* output tracking is usually impossible, Corollary 3.5 indicates that the GPOT problem is solvable.

Given any $\delta > 0$, following the design procedure of Theorem 3.4, one can construct a C^∞ controller

$$
\begin{aligned}
u =& -(x_2 - x_2^*) \Bigg[1 + 3 \times 4^{\frac{1}{3}} + 4\left(3\beta_1^2(x_1)\right)^4 + \frac{1}{(2\delta)^{\frac{1}{3}}} \\
&\times \left(\frac{1}{4} + \left[\frac{\partial x_2^*}{\partial x_1}\right]^2 [x_2^3 + x_1(e^{x_1} + x_2)]^2 + \beta_1^2(x_1)\pi^2 \right) \Bigg] \qquad (3.16)
\end{aligned}
$$

with $\beta_1(x_1) = \left[3 + \frac{2x_1^6}{27\delta^3} + 3\frac{(x_1 e^{x_1})^4 + \pi^4}{\delta^3}\right]^{\frac{1}{3}}$ and $x_2^* = -(x_1 - y_r)\beta_1(x_1)$, such that

$$
\dot{V}_2 \leq -(x_1 - y_r)^4 - (x_2 - x_2^*)^4 + 2\delta, \; V_2 = \frac{(x_1 - y_r)^2}{2} + \frac{(x_2 - x_2^*)^4}{4}
$$

The last inequality, together with Lemma 3.3, implies that all the solutions of the closed-loop system (3.15)-(3.16) are globally bounded and well-defined $\forall t \geq 0$. Moreover, for any $\varepsilon > 0$, we can choose δ sufficient small such that $|x_1(t) - y_r(t)| < \varepsilon$, $t \geq T$, where $T > 0$ is a finite time. ∎

Fig. 1 demonstrates that the output of the planar system (3.15), with $x_1(0) = x_2(0) = 1$, converges to the C^1 reference signal in a very short time and the output tracking error becomes very small after a finite time.

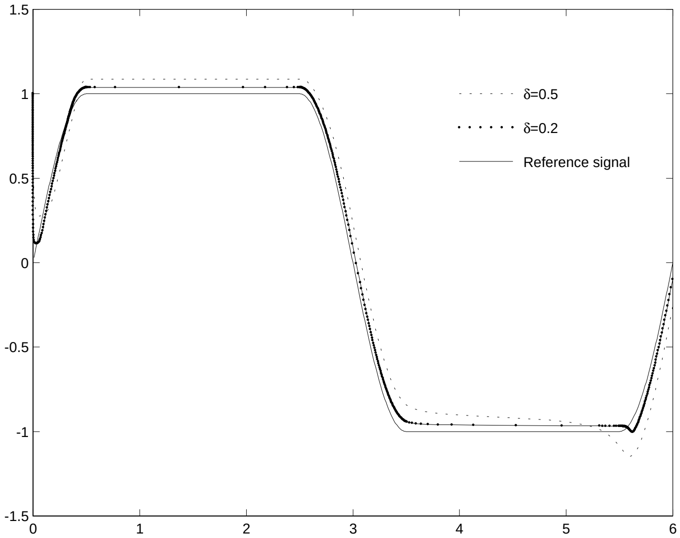

Fig. 1 Output response of the closed-loop system (3.15)-(3.16) and reference trajectory of y_r

4 An Application

The proposed control scheme is now used to study the output tracking problem for an underactuated unstable two degrees of freedom mechanical system considered in [15].

The mechanical system consists of a mass m_1 on a horizontal smooth surface and an inverted pendulum m_2 supported by a massless rod as shown in Fig. 2. The mass is interconnected to the wall by a linear spring and to the inverted pendulum by a nonlinear spring which has cubic force-deformation relation. Let x be the displacement of mass m_1 and let θ be the angle of the pendulum from the vertical such that at $x = 0$ and $\theta = 0$, the springs are unstretched. A control force acts on m_1. The system has two degrees of freedom and is underactuated.

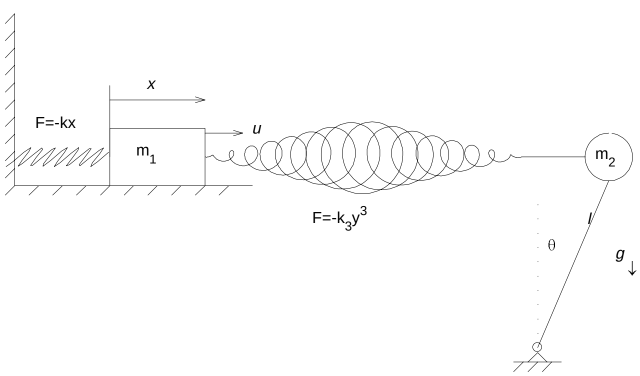

Fig. 2 An underactuated system with weak coupling

The equations of motion for the system are described by

$$\ddot{\theta} \;=\; \frac{g}{l}\sin\theta + \frac{k_s}{m_2 l}(x - l\sin\theta)^3\cos\theta \qquad (4.1)$$

$$\ddot{x} \;=\; -\frac{k}{m_1}x - \frac{k_s}{m_1}(x - l\sin\theta)^3 + \frac{u}{m_1} \qquad (4.2)$$

which can be transformed into (see [15])

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^3 + \frac{g}{l}\sin x_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= v, \qquad y = x_1. \end{aligned} \qquad (4.3)$$

This model represents a class of underactuated mechanical systems that are exceptionally difficult to control. In fact, there does not exist any C^1 state feedback law that stabilizes (4.3), because the linearized system of (4.3) has an uncontrollable mode associated with a positive eigenvalue. For the output tacking problem, when the reference signal is generated by an autonomous system, a *continuous* controller was proposed in [3] recently, achieving *local* practical output tracking for the system (4.3). Note that, however, the output tracking result in [3] is *local* and the designed controller is *continuous.* Since (4.3) a special case of (3.14), the problem of *global practical output tracking* is solvable by *smooth* feedback, according to Corollary 3.5 or 3.6.

Next we show how a *smooth* tracking controller can be constructed for (4.3). For simplicity, we assume that $\frac{g}{l} = 1$ and y_r is a bounded smooth reference signal whose derivatives are available for the feedback design.

Let $\xi_1 = x_1 - y_r$ be the tracking error. Note that $p = \max\{p_i\} = p_2 = 3$. Choose $V_1(\xi_1) = \frac{1}{p-p_1+2}\xi_1^{p-p_1+2} = \frac{\xi_1^4}{4}$. Obviously, the virtual controller $x_2^* = -2\xi_1 + \dot{y}_r$ renders

$$\dot{V}_1 = -2\xi_1^4 + \xi_1^3(x_2 - x_2^*).$$

Now construct the Lyapunov function

$$V_2 = V_1 + \frac{1}{p - p_2 + 2}\xi_1^{p-p_2+2} = \frac{1}{4}\xi_1^4 + \frac{1}{2}\xi_2^2, \qquad \xi_2 = x_2 - x_2^*.$$

The time derivative of V_2 along the solutions of (4.3) is

$$\dot{V}_2 \;\le\; -2\xi_1^4 + \xi_1^3\xi_2 + \xi_2(x_3^3 + \sin x_1 - \dot{x}_2^*). \qquad (4.4)$$

Using Young's inequality and Lemma 2.2, it is easy to show that for any $\delta > 0$, the virtual C^∞ controller

$$x_3^* = -\xi_2\left[1 + \frac{27}{256}\left(1 + \frac{(\sin x_1 - \dot{x}_2^*)^4}{\delta^3}\right)\right]^{\frac{1}{3}} \qquad (4.5)$$

results in

$$\dot{V}_2 \le -\xi_1^4 - \xi_2^4 + \delta + \xi_2(x_3^3 - x_3^{*3}).$$

Since $p_3 = p_4 = 1$, in the last two steps we can use the technique of adding a *linear* integrator to simplify the design. To this end, define $\xi_3 = x_3 - x_3^*$ and consider the Lyapunov function $V_3(\xi_1, \xi_2, \xi_3) = V_2(\xi_1, \xi_2) + \frac{1}{2}\xi_3^2$. Clearly,

$$\dot{V}_3 \;\le\; -\xi_1^4 - \xi_2^4 + \delta + \xi_2\xi_3(x_3^2 + x_3x_3^* + x_3^{*2}) + \xi_3(x_4 - \dot{x}_3^*).$$

An obvious virtual controller would be

$$x_4^* = -\xi_3 + \dot{x}_3^* - \xi_2(x_3^2 + x_3x_3^* + x_3^{*2}),$$

which is smooth and renders

$$\dot{V}_3 \;\le\; -\xi_1^4 - \xi_2^4 - \xi_3^2 + \delta + \xi_3(x_4 - x_4^*).$$

In the last step, choose

$$V_4(\xi_1, \cdots, \xi_4) = V_3(\xi_1, \xi_2, \xi_3) + \frac{1}{2}\xi_4^2, \qquad \xi_4 := x_4 - x_4^*.$$

A direct calculation shows that the C^∞ controller

$$v = -\xi_4 + \dot{x}_4^* - \xi_3 \tag{4.6}$$

is such that

$$\dot{V}_4 \le -\xi_1^4 - \xi_2^4 - \xi_3^2 - \xi_4^2 + \delta \le -2\sqrt{\delta}V_4 + 2\delta, \quad \delta \in (0, 1]. \tag{4.7}$$

Thus, all the solutions $\xi_1(t), \cdots, \xi_4(t)$ of the closed-loop system (4.3)-(4.6) are globally bounded and well-defined over $[0, +\infty)$, so is the state $(x_1, \cdots, x_4)$. Moreover, by choosing $\delta > 0$ appropriately, the output tracking error $|y - y_r|$ can be made arbitrarily small in a finite time.

The simulation result shown in Fig. 3 is based on the following parameters: $\frac{g}{l} = 1$ and $\delta = 0.01$; the desired trajectory to be tracked by the angle of the pendulum θ is $y_r = \sin(0.2t)$. The initial conditions of system (4.3) are chosen as $x_i(0) = 1$, $1 \le i \le 4$. As expected, all the states of the closed-loop system are globally bounded $\forall t \in [0, +\infty)$ and the output tracking error $|y - \sin(0.2t)|$ becomes arbitrarily small after a finite time.

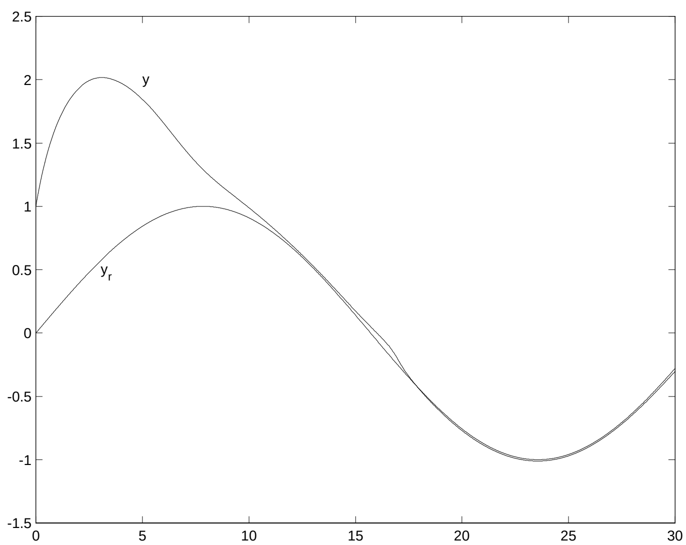

Fig. 3. The reference signal and the output of the closed-loop system (4.3)-(4.6)

5 Conclusions

We have studied the problem of *global practical output tracking (GPOT)* for nonlinear systems with uncontrollable unstable linearization which, as illustrated by Example 2.1, prevents the existence of the solutions to the *asymptotic output tracking* problem. In spite of this difficulty, we have developed a modified *adding a power integrator* technique for the explicit construction of *smooth* controllers that achieve *global practical* output tracking.

For triangular systems, the significance of Corollary 3.6 is that no more growth restrictions are imposed on the chain of power integrators and the perturbed terms. Unlike in the setting of *asymptotic output tracking*, our *GPOT controllers* only use the information of $y_r(t)$ and x. Neither the derivatives of $y_r(t)$ nor the boundedness conditions on the higher-order derivatives of $y_r(t)$ are required. These features certainly enhance the application value of our *practical output tracking* control schemes.

The tracking result for triangular systems has also been generalized to a larger class of nonlinear systems such as (3.14), which go far beyond a triangular structure. The effectiveness of the proposed tracking control schemes has been demonstrated by an underactuated unstable two degree of freedom mechanical system, for which global output tracking has been known a challenging problem.

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