Lyapunov Stability Analysis for Nonlinear Delay Systems

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Abstract

Sufficient conditions ensuring that a nonlinear system with disturbances having a delay is delay independent globally asymptotically stable are given. The proof carried out relies extensively on a characterization of the stability property in term of Lyapunov function. The result is applied to some biological systems and neural networks. It is also used to construct a stablizing memoryless controller for a second order system with state-delay.

Key words. Lyapunov function, delay system, asymptotic stability.

1 Introduction

One of the most fundamental theoretical results of the nonlinear control theory is the *converse Lyapunov theorem*. It states that for all asymptotically stable system, there exists a Lyapunov function whose derivative along the trajectories of the system is negative definite. Unfortunately, the determination of explicit formulas of Lyapunov functions for nonlinear systems with no specific structure is a problem which cannot always be solved. However, notably during the last decade, many techniques of Lyapunov constructions have been developped for wide classes of systems. In particular for two important families of controlled systems, the systems in feedback form and the systems in feedforward form, the two complementary approaches *backstepping* [14, 18] and *forwarding* [10, 15] allow us to construct *control Lyapunov functions*.

Whereas the knowledge of a Lyapunov function has been extensively exploited to prove robustness results with respect to disturbances without delay, they have not been used to prove robustness properties with respect to perturbations with delay.

In the present work, we exhibit sufficient conditions ensuring that a nonlinear delay system is delay independent globally asymptotically stable. They are simple and in general can be easily checked. Observe that most of the work in the analysis of nonlinear delay systems was concentrated on the use of the Razumikhin-function approach due to its simplicity to handle infinite-dimensional system's stability with finite-dimensional tools. Thus, [7] gives a good overview on this method applied to population dynamics models. Some remarks about control Lyapunov functions based on the Razumikhin approach for asymptotic stability can be found in [6], and ISS small-gain based theorems are proposed in [17]. Note that the Razumikhin approach is not the only Lyapunov function technique used for the asymptotic stability analysis. Thus, Barnea [1] proposes a different analysis method exploiting better the Krasovskii's idea to prove the Razumikhin result [4, 11] (and the references therein). Note also the *vector Lyapunov function* approach combined with *comparison principle* proposed in [13], where the comparison system may be finite-dimensional (see, e.g. [8, 11] and the references therein).

Our approach is quite different and is based on an alternative utilization of the Lyapunov function tool: the proposed technique is intimately related to the 'step-by-step' construction of the solutions of the differential-difference equations (see, e.g. [3] for the method). Indeed, in the differential-difference equation case, the knowledge of the evolution over one delay interval allows us to construct the solution on the next delay interval by a simple integration of a nonlinear ordinary differential equation. Such an idea allows us to derive stability results from an appropriate construction of a Lyapunov function, which can be based in particular on the backstepping and the forwarding approaches. So the main idea of this work opens a new way to the control of nonlinear delay systems (see also the third example in Section 3). Note that the connections with Razumikhin as well as with the comparison principle based techniques are also outlined.

As illustrated in the end of this work, the main result we present can be used in particular to analyze the stability properties of some systems modelling the behaviour of biological quantities, or of artificial Hopfield networks with transmission delays.

The paper is organized as follows: Section 2 contains the problem statement and the main results. Section 3 is devoted to some examples. At last, concluding remarks in Section 4 end the paper.

2 Problem statement

We consider the nonlinear system

$$\dot{x}(t) \;=\; f(x(t)) \;+\; g(x(t-\tau), x(t), t) \qquad (1)$$

where $x \in \Re^{n_x}$, $\tau > 0$, where $f(b)$ is a function of class C^1 and where $g(a, b, t)$ is a function of class C^1 such that for all b and t, $g(0, b, t) \;=\; 0$. The initial conditions are $x(t) = \varphi(t)$ for all $t \in [-\tau, 0]$, where $\varphi(t)$ is a function of class C^1.

Such systems are encountered in Hopfield delayed neural networks [9, 2, 19], under appropriate assumptions on $f(\cdot)$ and $g(\cdot)$: $f(x) = Dx$, where D is a positive definite diagonal matrix, and $g(\cdot)$ is a bounded function globally Lipschitz. Note that (1) is also encountered in population dynamics [7], under some appropriate assumptions on $f(\cdot)$ and $g(\cdot)$ such as for example, $g = g(x_t)$ (where x_t is the translation operator over one delay interval) is a decreasing (or increasing) function, or has a hamp, etc. Furthermore, the well known delay logistic equation:

$$\dot{x}(t) = -rx(t)\left[1 - \frac{x(t-\tau)}{K}\right], \quad r, K, \tau > 0 \qquad (2)$$

belongs to this class of delay systems.

The hardware implementation (VLSI design) of neural network induces delays in the models due to finite switching speeds of amplifiers [5]. Note that such delays are difficult to be computed, and the propagation effect may lead to bad performances when the neural network is used in parallel computation or signal processing, (see, e.g. [19, 16] and the references therein). In such situations, it is better that the stability conditions do *not depend* on the *delay size*, that are *delay-independent stability* conditions. Comments and discussions on the importance of *delay-independent* stability in dynamic population analysis are presented in [7] (see also the references therein).

Let us introduce some assumptions.

Assumption 2.1 *The system $\dot{x}(t) \;=\; f(x(t))$ is globally asymptotically stable. A positive definite radially unbounded function $V(\cdot)$ of class C^1, a positive definite nondecreasing continuous function $\psi(\cdot)$ such that:*

$$\frac{\partial V}{\partial x}(x) f(x) \;\leq\; -2\psi(V(x)) \qquad (3)$$

are known.

Remark 2.2 *It is proved in [12] that, provided that the system $\dot{x}(t) \;=\; f(x(t))$ is globally asymptotically stable, there exists a function $V(x)$ such that (3) is satisfied with $\psi(s) \;=\; \frac{1}{3}s$. However, considering the Lyapunov functions satisfying the requirement (3) with more general functions $\psi(\cdot)$, one can prove stability for delay-systems of a much larger class of systems.*

Assumption 2.3 *For some $\varepsilon \in]0, 1[$,*

$$\left|\frac{\partial V}{\partial x}(a) g(a, b, t)\right| \;\leq\; \psi(V(a)) + \varepsilon\psi(V(b)) \;, \quad \forall\, (a, b) \qquad (4)$$

Remark 2.4 *Assumption 2.3 can be checked even when $g(a, b, t)$ is not exactly known: roughly speaking, it is only required to know what are the local behaviour of $g(a, b, t)$ and its growth properties at the infinity.*

Remark 2.5 *Assumption 2.3 allows us to exploit a decoupling property between the "delayed" and "present" states by computing the derivative of the Lyapunov function. Note that* all *time-domain techniques (Lyapunov or comparison systems) make use of such decoupling property for deriving* delay-independent *stability results.*

We have the following preliminary result:

Lemma 2.6 *Assume that the system (1) satisfies Assumptions 2.1 and 2.3. Then all the trajectories of the system (1) are bounded.*

Proof: First, observe that since $\varphi(t)$ is a continuous function on $[-\tau, 0]$ and $V(\cdot)$ is positive,

$$S_0 \;=\; \sup_{s \in [-\tau, 0]} V(\varphi(s)) \qquad (5)$$

is a finite positive number.

To prove that the solutions are bounded, we proceed by contradiction. Assume that there exist t_1 and t_2 such that $V(x(t)) \leq S_0$ for all $t \leq t_1$ and $V(x(t)) > S_0$ for all $t \in]t_1, t_2[$. Assumptions 2.1 and 2.3 imply that

$$\dot{V}(x(t_1)) \;\leq\; -\psi(S_0) \;+\; \varepsilon\psi(S_0) \;<\; 0 \qquad (6)$$

It follows that it is impossible that $V(x(t)) > S_0$ for all $t \in]t_1, t_2[$. This ends the proof.

Let us state our main result.

Theorem 2.7 *Assume that the system (1) satisfies Assumptions 2.1 and 2.3. Then the system (1) is delay-independent globally asymptotically stable.*

Remark 2.8 *When inequality (4) is satisfied with* $\varepsilon = 1$, *the conclusion of Theorem 2.7 does not hold true any more. For instance, for any real number* ξ, *the one-dimensional system* $\dot{x}(t) = -x(t) + x(t-\tau)$ *admits* $x(t) = \xi$ *as a solution and thereby is not locally asymptotically stable.*

Remark 2.9 *Basically, the Razumikhin idea [4] consists of finding a* Lyapunov function *whose derivative is negative-definite* only *along all the trajectories leaving a ball defined by the system evolution over one delay interval. Further comments and connections with the well-known* $\mathcal{S}$*-procedure can be found in [11].*

A different idea was to use Lyapunov functions combined with some variational problems which involves integration over one (or several) delay interval(s). Further comments, examples, and related optimization problems (by tuning one or several parameters) are given in [1]. The scalar case is also presented in [4].

Our idea is quite different: we use further the boundedness of the solutions, under a 'step-by-step' procedure for proving the asymptotic property by "decoupling" the "past" from the "present".

Proof. First observe that, using the arguments we have invoked to prove Lemma 2.6, i.e. to prove that $V(x(t)) \leq S_0$ for all $t \geq S_0$,where $S_0 = \sup\limits_{s\in[-\tau,0]} V(\varphi(s))$, one can prove that for all integer n the inequality

$$S_{n+1} \leq S_n \tag{7}$$

where

$$S_n = \sup_{s\in[(n-1)\tau,n\tau]} V(x(s)) \tag{8}$$

holds. Since this positive sequence is decreasing, it admits a limit. Let us proceed by contradiction: assume that

$$\lim_n S_n = l > 0 \tag{9}$$

Since, According to Assumption 2.1 the function $\psi(\cdot)$ is continuous, it follows that there exists a sequence $t_n \in [(n-1)\tau, n\tau]$, an integer $N_1 > 0$ and $\delta \in]0,1]$ such that:

1. $S_n = V(x(t_n)) \geq l$ for all n.
2. $S_n \leq l + \delta$ for all $n \geq N_1$.
3. $\psi(l) \geq \frac{1+\varepsilon}{2}\psi(l+\delta)$.

The inequality

$$\dot{V}(x(t)) \leq -\psi(V(x(t))) + \varepsilon\psi(V(x(t-\tau))) \tag{10}$$

is valid for all $t \geq 0$. For $t = t_n$, it allows us to write, since $\psi(\cdot)$ is nondecreasing, for all $n \geq N_1 + 1$

$$\begin{aligned} \dot{V}(x(t_n)) &\leq -\psi(V(x(t_n))) + \varepsilon\psi(V(x(t_n - \tau))) \\ &\leq -\psi(l) + \varepsilon\psi(l+\delta) \\ &\leq -\tfrac{1-\varepsilon}{1+\varepsilon}\psi(l) < 0 \end{aligned} \tag{11}$$

It follows that there exists $\nu_n > 0$ such that

$$V(x(t_n - s)) > V(x(t_n)) > V(x(t_n + s)) \ , \ \forall s \in]0, \nu_n] \tag{12}$$

According to the definition of t_n, it follows that $t_n = n\tau$ for all $n \geq N_1 + 1$.

Next, assume first that there exists T_1 such that for all $t > T_1$, then $\dot{V}(x(t)) \leq 0$. It follows that

$$\lim_{t\to+\infty} V(x(t)) = \lim_{t\to+\infty} V(x(t-\tau)) = l \tag{13}$$

Then, according to (10), there exists $T_2 \geq T_1$ such that for all $t \geq T_2$,

$$\dot{V}(x(t)) \leq -\frac{1-\varepsilon}{1+\varepsilon}\psi(l) \tag{14}$$

Since $V(\cdot)$ is a positive function, this yields a contradiction.

It follows that there exists a strictly increasing sequence of integers $m(n)$, such that there exists $s_{m(n)} \in [m(n)\tau - \tau, m(n)\tau]$ such that $\dot{V}(x(s_{m(n)})) = 0$. Then, according to (11), there exists a strictly increasing sequence of integers $k(n)$ which is a subsequence of $m(n)$, such that there exists a sequence $h_{k(n)} \in [k(n)\tau - \tau, k(n)\tau]$ such that $\dot{V}(x(h_{k(n)})) = 0$ and $V(x(h_{k(n)})) > V(x(k(n)\tau))$.

But, according to (10),

$$\begin{aligned} \dot{V}(x(h_{k(n)})) &\leq -\psi(V(x(h_{k(n)}))) \\ &\qquad +\varepsilon\psi(V(x(h_{k(n)} - \tau))) \\ &\leq -\psi(V(x(k(n)\tau))) \\ &\qquad +\varepsilon\psi(V(x(h_{k(n)} - \tau))) \\ &\leq -\psi(l) + \varepsilon\psi(V(x(h_{k(n)} - \tau))). \end{aligned} \tag{15}$$

According to the definition of l, there exists an integer $N_2 > N_1$ such that, for all $n \geq N_2$, $V(x(h_{k(n)} - \tau)) \leq l + \delta$. It follows

$$\dot{V}(x(h_k)) \leq -\psi(l) + \varepsilon\psi(l+\delta) \leq -\frac{1-\varepsilon}{1+\varepsilon}\psi(l) < 0 \tag{16}$$

for all $n \geq N_2$. This contradicts the definition of $h_{k(n)}$.

This concludes our proof.

Remark 2.10 (Razumikhin interpretation) *Consider a function* $V(x)$ *satisfying the Assumptions*

2.1 and 2.3. Assume now that the system's trajectories will leave at the time t the ball defined by $V(x(t+\theta))$ for all $\theta \in [-\tau, 0]$, that is: $V(x(t)) \geq V(x(t+\theta))$, $\forall \theta \in [-\tau, 0]$. It is clear that $\psi(V(x(t)) \geq \psi(V(x(t+\theta))$, $\forall \theta \in [-\tau, 0]$.

Using the facts mentioned above for the computation of the derivative of $V(\cdot)$ along the system's trajectories, it follows that:

$$\begin{aligned} \dot{V}(x(t)) &\leq -\psi(V(x(t))) + \varepsilon\psi(V(x(t-\tau))) \\ &\leq -(1-\varepsilon)\psi(V(x(t)), \end{aligned} \qquad (17)$$

which leads to the conclusion that $V(x)$ is of Razumikhin *type. However, we may conclude only the* uniform stability *of the original system, which does not depend on the delay size.*

Note that the proofs in Lemma 2.6 and Theorem 2.7 do not make use of any Razumikhin argument, and may be seen as alternative ways *to prove boundedness and asymptotic stability properties. Furthermore, the proposed results are better adapted for the control of delay systems, as seen below (Third example in Section 3).*

Remark 2.11 (Comparison principle) *Consider a function $V(x)$ satisfying the Assumptions 2.1 and 2.3. It seems clear that the constraints imposed in Assumption 2.3 allow us to conclude on the stability of the original delay system from the stability of the system free of delay: $\dot{x} = f(x)$ (see the proof or the Razumikhin interpretation outlined above). In conclusion, the system $\dot{x} = f(x)$ is a* comparison system *[8, 13] for the original system (1).*

Note that some of the constructions proposed in [13] based on vector Lyapunov functions *can be adapted in the new framework.*

3 Examples

3.1 First example

Consider the system (see also [7]):

$$\dot{z}(t) = -F(z(t)) + G(z(t-\tau)), \qquad (18)$$

where $z \in \mathbf{R}$, and $F(\cdot)$, $G(\cdot)$ satisfy the following constraints:

- $F(\theta)$ is of class C^1, $F(0) = 0$, $\lim_{\theta \to +\infty} F(\theta) = +\infty$ and there exists $\chi \in \Re$ such that $F'(x) \geq \chi > 0$;
- $G(\theta)$ is a bounded function, strictly decreasing, $G(0) > 0$, $\lim_{\theta \to +\infty} G(\theta) = 0$.

Remark 3.1 *One of the models used in the literature for describing the blood cell production is:*

$$\dot{z}(t) = \frac{\beta_0 \phi^n}{\phi^n + [z(t-\tau)]^n} - \gamma z(t), \quad \beta_0, \phi, \gamma > 0, \qquad (19)$$

where $z(t)$ denotes the density of mature cells in blood circulation (n is an integer). Further comments are given in [7]. The example is treated below.

Observe that the conditions imposed on $F(\cdot)$ and $G(\cdot)$ imply that a neighborhood of the origin is globally attractive. In order to know more accurately what is the asymptotic behaviour of systems (18), and in particular to determine conditions ensuring that all the trajectories converge to an equilibrium point, it is appealing to use Theorem 2.7. However, Theorem 2.7 cannot be applied straightforwardly to the system (18) since the condition $g(0, b, t) = 0$ for all b and t is not satisfied by $G(\cdot)$. Let us prove that this system can be transformed into a system for which Theorem 2.7 applies.

The properties of $F(\cdot)$ and $G(\cdot)$ guarantee that there exists one and only one $z_c > 0$ such that $F(z_c) = G(z_c)$. Using the coordinate $x = z - z_c$ and the notations $f(x) = F(x(t) + z_c) - F(z_c)$, $g(x(t-\tau)) = G(x(t-\tau) + z_c) - G(z_c)$, the system rewrites

$$\dot{x}(t) = -f(x(t)) + g(x(t-\tau)), \qquad (20)$$

Since $F(z)$ is strictly increasing, the Lyapunov function $V(x) = \frac{1}{2}x^2$ is such that

$$\frac{\partial V}{\partial x}(x) f(x) \ = \ -x^2 \rho(x) \qquad (21)$$

with $\rho(x) = \int_0^1 F'(sx + z_c) ds > \chi$. It follows that

$$\frac{\partial V}{\partial x}(x) f(x) \ \leq \ -2\chi V(x) \qquad (22)$$

On the other hand, $G(\cdot)$ is a bounded function. So we can determine $\Delta_G > 0$ such that the inequalities

$$\begin{aligned} &\frac{\partial V}{\partial x}(x(t)) g(x(t-\tau)) \leq \Delta_G |x(t) x(t-\tau)| \\ &\leq \chi V(x(t)) + \frac{\Delta_G^2}{\chi} V(x(t-\tau)) \end{aligned} \qquad (23)$$

hold. Choosing $\psi(v) = \chi v$, we deduce from Theorem 2.7 that all the solutions of the system (18) converge to z_c when the following condition is satisfied:

$$\Delta_G \ < \ \chi \qquad (24)$$

In conclusion, mention that using the previous result, one can prove that:

Proposition 3.2 *The solutions of the system (19) converge to the solution of the equation*

$$\frac{\beta_0 \phi^n}{\phi^n + z_c^n} = \gamma z_c \tag{25}$$

when β_0, ϕ, n satisfy the inequality:

$$\left| \frac{n\beta_0}{4\phi} \right| \leq \gamma \tag{26}$$

3.2 Second example

The second example is a classical Hopfield delayed neural network [2, 9, 11].

For the sake of simplicity, we assume that the origin is the corresponding equilibrium. The model is:

$$\dot{z}_i(t) = -z_i(t) + \sum_{j=1}^{n} a_{ij} g_j(z_j(t-\tau)), \quad \forall i = \overline{1,n}, \tag{27}$$

where $z \in \Re^n$ is the state, τ is the interconnection delay (supposed to be the same for all the units) from the output of the unit j to the input of the unit i, for all $i, j = \overline{1,n}$. We shall assume that the functions $g_j(\cdot)$, $j = \overline{1,n}$ are bounded, and globally Lipschitz with the Lipschitz constant L_j.

With $V(z) = \frac{1}{2} z^T z$ one can prove using Theorem 2.7 the following result:

Proposition 3.3 *The Hopfield delayed neural network (27) is globally delay-independent asymptotically stable if:*

$$\max_{1 \leq j \leq n} \left(L_j \sum_{i=1}^{n} \mid a_{ij} \mid \right) < 1 \tag{28}$$

If we assume that each unit i is scaled with some positive b_i, and that the interconnection delay from the unit j to the unit i is τ_{ij}, the system (27) becomes:

$$\dot{z}_i(t) = -b_i z_i(t) + \sum_{j=1}^{n} a_{ij} g_j(z_j(t-\tau_{ij})), \quad \forall i = \overline{1,n}, \tag{29}$$

and a result similar to Proposition 3.3 can be proved, by an appropriate *rescaling* of a_{ij} in (29).

Note that the same result was proposed in [19], using a more complicated Lyapunov-Krasovskii (functional) based argument.

3.3 Third example

In this section, we illustrate on a very simple example how Theorem 2.7 can be used to solve stabilization problems for controlled systems with delay dependent terms. This example gives an insight into the vast stabilization results which can be proved for families of controlled systems such as the systems in feedback or in feedforward form when terms with a delay are present.

Consider the two dimensional system

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) & = & x_2(t) + x_1(t-\tau)^2 \\ \dot{x}_2(t) & = & u \end{array} \right. \tag{30}$$

where $u \in \Re$ is the input.

Proposition 3.4 *The system (30) is globally delay-independent asymptotically stabilized by the memoryless control law*

$$u(x_1, x_2) = -\frac{1}{2}X_2 - 12X_2^3 - 6x_1^2 X_2 - x_1 - \left[1 + 3x_1^2\right] x_2 \tag{31}$$

with

$$X_2 = x_2 + x_1 + 3x_1^3 \tag{32}$$

Proof. We apply the backstepping approach to construct for the delay independent two dimensional chain of integrators a Lyapunov function and a control law with the specificity of allowing us to apply Theorem 2.7 to the closed-loop system.

The variable X_2 defined in (32) transforms (30) into

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) & = & -x_1(t) - 3x_1(t)^3 + X_2(t) + x_1(t-\tau)^2 \\ \dot{X}_2(t) & = & u + \left[1 + 3x_1(t)^2\right] \left[x_2(t) + x_1(t-\tau)^2\right] \end{array} \right. \tag{33}$$

Let $V(x_1, X_2) = \frac{1}{2}\left(x_1^2 + X_2^2\right)$. The time derivative of $V(x_1, X_2)$ along the trajectories of (33) satisfies the inequalities

$$\begin{array}{rcl} \dot{V} & \leq & -x_1(t)^2 - 3x_1(t)^4 + x_1(t)X_2(t) \\ & & + x_1(t)x_1(t-\tau)^2 + X_2(t)u \\ & & + X_2(t)\left[1 + 3x_1(t)^2\right]\left[x_2(t) + x_1(t-\tau)^2\right] \\ & \leq & -\frac{1}{2}x_1(t)^2 - 3x_1(t)^4 + x_1(t-\tau)^4 \\ & & + x_1(t)X_2(t) + X_2(t)\left[1 + 3x_1(t)^2\right]x_2(t) \\ & & + \frac{1}{2}X_2(t)^2\left[1 + 3x_1(t)^2\right]^2 + X_2(t)u \end{array} \tag{34}$$

The control law (31) yields

$$\begin{array}{rcl} \dot{V} & \leq & -\frac{1}{2}(x_1(t)^2 + X_2(t)^2) \\ & & -48\left[\frac{1}{2}\left(x_1(t)^2 + X_2(t)^2\right)\right]^2 \\ & & + x_1(t-\tau)^4. \end{array} \tag{35}$$

To conclude, we apply Theorem 2.7 with

$$\psi(v) = \frac{1}{2}v + 24v^2. \tag{36}$$

Remark 3.5 *The proposed control law is memoryless and reflects the decoupling properties cited in Remark 2.9. Not suprisingly, the family of control laws which stabilize the system (30) when* $\tau \neq 0$ *is smaller than the one of the stabilizing control laws for the system (30) when* $\tau = 0$.

4 Concluding remarks

The result of delay independent stability for systems having a globally asymptotically stable delay independent part we have proposed relies heavily on the knowledge of a Lyapunov function. Extentions of our result to the case where several delays are present can be obtained straightforwardly.

As suggested by the last example studied in this work, when controlled systems with delay dependent disturbances are considered, a new problem arises: how to choose a stabilizing control which yields a closed loop system for which a Lyapunov function satisfing Assumptions 2.1 and 2.3 exists ? Further research will be devoted to the investigation of this question for systems in feedback and in feedforward form.

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