A new definition of the minimum-phase property for nonlinear systems, with an application to adaptive control*

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Abstract

We introduce a new definition of the minimum-phase property for general smooth nonlinear control systems. The definition does not rely on a particular choice of coordinates in which the system takes a normal form or on the computation of zero dynamics. It requires the state and the input of the system to be bounded by a suitable function of the output and derivatives of the output, modulo a decaying term depending on initial conditions. The class of minimum-phase systems thus defined includes all affine systems in global normal form whose internal dynamics are input-to-state stable and also all left-invertible linear systems whose transmission zeros have negative real parts. We explain how the new concept enables one to develop a natural extension to nonlinear systems of a basic result from linear adaptive control.

1 Introduction

A linear, single-input/single-output (SISO) system is said to be *minimum-phase* if the numerator polynomial of its transfer function has all its zeros in the open left half of the complex plane. This property can be given a simple interpretation which involves the relative degree of the system. Namely, if a linear system of relative degree r is minimum-phase, then the "inverse" system, driven by the r-th derivative of the output of the original system, is stable. For left-invertible, multi-input/multi-output (MIMO) systems, in place of zeros of the numerator one appeals to transmission zeros.

The notion of a minimum-phase system is of great significance in many areas of linear system analysis and design. In particular, it has played an important role in parameter adaptive control. A basic example is provided by the "certainty equivalence output stabilization theorem" [7], which says that when a certainty equivalence, output stabilizing adaptive controller is applied to a minimum-phase linear system, the closed-loop system is detectable through the tuning error. In essence, this result serves as a justification for the certainty equivalence approach to adaptive control of minimum-phase linear systems.

For nonlinear systems that are affine in controls, the minimum-phase property has been defined in [1] in terms of the concept of *zero dynamics*. The zero dynamics are the internal dynamics of the system under the action of the input that holds the output constantly at zero. The system is called minimum-phase if the zero dynamics are (globally) asymptotically stable. In the SISO case, a unique input capable of producing the zero output is guaranteed to exist if the system has a uniform relative degree. Extensions to MIMO systems are discussed in [2, 4].

The need to work with the zero dynamics makes the above definition of a minimum-phase nonlinear system difficult to use, unless one can find a change of coordinates that transforms the system into a certain "normal form". It has also been recognized that just asymptotic stability of the zero dynamics is often inadequate for control design purposes, so that additional requirements need to be placed on the internal dynamics of the system. One such common requirement is that the internal dynamics be *input-to-state stable* with respect to the output and its derivatives up to order $r-1$, where r is the relative degree (see, for instance, [9]).

In this paper we propose a new definition of the minimum-phase property for general smooth nonlinear systems, which does not rely on zero dynamics or normal forms. Loosely speaking, we will call a system minimum-phase if its state and input eventually become small when the output and derivatives of the output are small. It follows from our definition that if a system has a uniform relative degree and is detectable through the output and its derivatives up to some order, uniformly over all inputs that produce a given output, then it is minimum-phase. For SISO systems that are real analytic in controls, the converse is also true, which yields a useful equivalent characterization of the minimum-phase property (Theorem 1). The class of minimum-phase systems as defined here includes all left-invertible linear systems whose transmission zeros have negative real parts (Theorem 2) and all affine systems in global normal form with input-to-state stable internal dynamics. To illustrate the new concept and demonstrate that it is indeed a reasonable extension of the linear notion, we use it to develop a natural nonlinear counterpart of the certainty equivalence output stabilization theorem from

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*Research supported by AFOSR, DARPA, and NSF.

linear adaptive control (Theorem 3).

Due to space constraints, the proofs are omitted, and most of the results are stated for SISO systems. For proofs, extensions, and examples, see [6].

2 Definition and preliminary remarks

We will consider systems of the general form

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{1}$$

where the state x takes values in $\mathbb{R}^n$, the input u takes values in $\mathbb{R}^m$, the output y takes values in $\mathbb{R}^l$ (for some positive integers n, m, and l), and the functions f and h are smooth (C^∞). Admissible input signals are locally essentially bounded, Lebesgue measurable functions $u : [0,\infty) \to \mathbb{R}^m$. Note that whenever the input function u is $k-1$ times continuously differentiable, where k is a positive integer, the derivatives $\dot{y}$, $\ddot{y}$, ..., $y^{(k)}$ are well defined (this issue will be discussed in more detail later).

We will let $\|\cdot\|_{[a,b]}$ denote the essential supremum norm of a signal restricted to an interval $[a,b]$, i.e., $\|z\|_{[a,b]} := \operatorname{ess\,sup}\{|z(s)| : a \le s \le b\}$, where $|\cdot|$ is the standard Euclidean norm. Given an $\mathbb{R}^l$-valued signal z and a nonnegative integer k, we will denote by $\mathbf{z}^k$ the $\mathbb{R}^{l(k+1)}$-valued signal

$$\mathbf{z}^k := (z_1, \dot{z}_1, \dots, z_1^{(k)}, \dots, z_l, \dot{z}_l, \dots, z_l^{(k)})$$

provided that the indicated derivatives exist.

Definition 1. We will call the system (1) *minimum-phase* if there exist a positive integer N, a class $\mathcal{KL}$ function[1] β, and a class $\mathcal{K}_\infty$ function γ such that for every initial state $x(0)$ and every $N-1$ times continuously differentiable input u the inequality

$$\left| \begin{pmatrix} u(t) \\ x(t) \end{pmatrix} \right| \le \beta(|x(0)|, t) + \gamma(\|\mathbf{y}^N\|_{[0,t]}) \tag{2}$$

holds for all t in the domain of the corresponding solution of (1). □

From the bound on the magnitude of the input in (2) we deduce that $y \equiv 0$ implies $u \to 0$. This can be interpreted as saying that the system has a *stable left inverse*, in the input-output sense. However, no explicit construction of such a left inverse is necessary. On the other hand, the bound on the magnitude of the state signifies that the system is *detectable* through the output and its derivatives, uniformly with respect to inputs. Detectability is a state-space concept, whose attractive feature is that it can be characterized by Lyapunov-like dissipation inequalities. These two requirements capture intrinsic properties of the system, which are independent of a particular coordinate representation. They are consistent with the intuition provided by the concept of a minimum-phase linear system.

Example 1. Consider the linear SISO system

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= c^T x \end{aligned} \tag{3}$$

Let r be its relative degree. This means that we have $c^T b = c^T A b = \cdots = c^T A^{r-2} b = 0$ but $c^T A^{r-1} b := g \neq 0$. From the formula $y^{(r)}(t) = c^T A^r x(t) + g u(t)$ we immediately obtain

$$|u(t)| \le \frac{|(A^r)^T c||x(t)| + |y^{(r)}(t)|}{|g|}. \tag{4}$$

Moreover, it is well known that there exists a linear change of coordinates $x \mapsto (\xi, \eta)$, where $\xi \in \mathbb{R}^r$, $\eta \in \mathbb{R}^{n-r}$, and $\xi_1 = y$, which transforms the system (3) into the normal form

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ &\cdots \\ \dot{\xi}_r &= d^T \xi + f^T \eta + gu \\ \dot{\eta} &= P\xi + Q\eta \end{aligned}$$

and (3) is minimum-phase (in the classical sense) if and only if Q is a stable matrix. Stability of Q is equivalent to the existence of positive constants λ and μ such that $|\eta(t)| \le e^{-\lambda t}|\eta(0)| + \mu\|\xi\|_{[0,t]}$ (it is also equivalent to detectability of the transformed system with extended output $\xi = \mathbf{y}^{r-1}$). Combining the last inequality with (4), we arrive at

$$|u(t)| \le \frac{|(A^r)^T c| e^{-\lambda t}}{|g|}|x(0)| + \frac{|(A^r)^T c|(\mu+1)+1}{|g|}\|\mathbf{y}^r\|_{[0,t]}.$$

This yields (2) with $N = r$. On the other hand, if (2) holds, then we know that $y \equiv 0$ implies $x \to 0$, and so (3) must be minimum-phase. Thus we see that for linear SISO systems the above definition reduces to the usual one. Incidentally, note that when $N = r$ in (2), the smoothness of u becomes superfluous, because y is automatically r times (almost everywhere) differentiable for every admissible input u. □

The above remarks suggest that the concepts of relative degree and detectability are related to the minimum-phase property as defined here. In what follows, we develop some machinery which is needed to study this relationship, and explore to what extent the situation described in Example 1 carries over to nonlinear systems.

3 Relative degree

Consider the case when the system (1) is SISO, i.e., when $m = l = 1$. We now give a somewhat non-standard

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[1] Recall that a function $\alpha : [0,\infty) \to [0,\infty)$ is of *class* $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If $\alpha \in \mathcal{K}$ is unbounded, then it is of *class* $\mathcal{K}_\infty$. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \ge 0$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each fixed $s \ge 0$.

definition of relative degree, which is especially suitable for subsequent developments. For each $k = 0, 1, \ldots$ define, recursively, the functions $H_k : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ by the formulas $H_0 := h$ and

$$H_{k+1}(x, u_0, \ldots, u_k) := \frac{\partial H_k}{\partial x} f(x, u_0) + \sum_{j=0}^{k-1} \frac{\partial H_k}{\partial u_j} u_{j+1}$$

where the arguments of H_k are $(x, u_0, \ldots, u_{k-1})$. As an illustration, in the special case of the SISO affine system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{5}$$

we have $H_1(x, u_0) = L_f h + L_g h u_0$ and $H_2(x, u_0, u_1) = L_f^2 h + (L_g L_f h + L_f L_g h)\, u_0 + L_g^2 h u_0^2 + L_g h u_1$ (omitting the argument x in the directional derivatives).

The significance of the functions H_k lies in the fact that if the input $u(\cdot)$ is in C^{k-1} for some positive integer k, then along each solution $x(\cdot)$ of (1) the corresponding output has a continuous k-th derivative satisfying

$$y^{(k)}(t) \;=\; H_k\big(x(t), u(t), \ldots, u^{(k-1)}(t)\big).$$

In particular, suppose that H_k is independent of $u_0, \ldots, u_{k-1}$ for all k less than some positive integer r. Then H_r depends only on x and u_0, as given by

$$H_r(x, u_0) = \frac{\partial H_{r-1}}{\partial x}(x)\, f(x, u_0)\,.$$

We conclude that for every initial condition and every input, $y^{(r-1)}$ exists and is an absolutely continuous function of time, and we have

$$y^{(r)}(t) = H_r(x(t), u(t))$$

for almost all t in the domain of the corresponding solution. The converse is also true, namely, if $y^{(r-1)}$ exists and is absolutely continuous for all initial states and all inputs, then H_k must be independent of $u_0, \ldots, u_{k-1}$ for all $k < r$.

We will say that a positive integer r is the (uniform) *relative degree* of the system (1) if the following two conditions hold:

1. For each $k < r$, the function H_k is independent of $u_0, \ldots, u_{k-1}$.
2. There exist class $\mathcal{K}_\infty$ functions ρ_1 and ρ_2 such that

$$|u_0| \leq \rho_1(|x|) + \rho_2\left(|H_r(x, u_0)|\right) \tag{6}$$

for all $x \in \mathbb{R}^n$ and all $u_0 \in \mathbb{R}$.

It is not hard to see that if there exists such an integer r, then it is unique. In view of the previous remarks, r is the relative degree of (1) if and only if for some functions $\rho_1, \rho_2 \in \mathcal{K}_\infty$, for every initial condition, and every input, $y^{(r-1)}$ exists and is absolutely continuous (hence $y^{(r)}$ exists almost everywhere) and the inequality

$$|u(t)| \leq \rho_1(|x(t)|) + \rho_2(|y^{(r)}(t)|)$$

holds for almost all t.

The above definition reduces to the usual one (as given, e.g., in the book by Isidori [4]) in the case of the SISO affine system (5) with $f(0) = 0$. (According to [4], a positive integer r is the relative degree of (5) if $L_g L_f^k h(x) = 0$ for all x and all integers $k < r - 1$, and $L_g L_f^{r-1} h(x) \neq 0$ for all x.) Note that the definition of relative degree proposed here is not restricted to affine systems. As a simple example, the system $\dot{y} = u^2$ has relative degree 1 according to our definition. This case is also covered by the definition of relative degree for not necessarily affine systems given in [8, p. 417]. However, our definition is more restrictive; for example, the system $\dot{y} = \arctan u$ would have relative degree 1 in the context of [8], but the bound (6) does not hold.

4 Detectability and related notions

Consider a general system of the form

$$\dot{x} = f(x, u).$$

We recall from [10] that this system is called *input-to-state stable* (ISS) if there exist some functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial state $x(0)$ and every input u the corresponding solution satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|_{[0,t]})$$

for all $t \geq 0$.

Given a system with both inputs and outputs

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned} \tag{7}$$

we will say that it is *0-detectable* if there exist some functions $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such for every $x(0)$ and every u the corresponding solution satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_1(\|u\|_{[0,t]}) + \gamma_2(\|y\|_{[0,t]})$$

as long as it exists. In particular, a system without inputs given by

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned}$$

will be called *0-detectable* if there exist some functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial state $x(0)$ the corresponding solution satisfies the following inequality as long as it exists:

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|y\|_{[0,t]}). \tag{8}$$

These concepts were studied in [11] under the names of *input/output-to-state stability* and *output-to-state stability*, respectively.

Let us call the system (7) *uniformly 0-detectable* if there exist some functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial state $x(0)$ and every input u the inequality (8) holds along the corresponding solution. As the name suggests, uniform 0-detectability amounts to 0-detectability that is uniform with respect to inputs. This property was called *uniform output-to-state stability* in [5] and *strong detectability* in [3].

Another definition, which will be needed in Section 7, is the following one (introduced in [12]). The system (7) is said to be *input-to-output stable* if there exist some functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every $x(0)$ and every u the following inequality holds along the corresponding solution:

$$|y(t)| \le \beta(|x(0)|, t) + \gamma(\|u\|_{[0,t]}).$$

5 Minimum-phase systems

We study the SISO case, represented by the system (1) with $m = l = 1$. Take a nonnegative integer k. Restricting the input u to be in C^{k-1}, we can consider the *k-output extension* of the system:

$$\begin{aligned} \dot{x} &= f(x,u) \\ \mathbf{y}^k &= \mathbf{h}_k(x,u,\dots,u^{(k-1)}) \end{aligned} \tag{9}$$

where

$$\mathbf{h}_k(x,u,\dots,u^{(k-1)}) := \big(H_0(x),\dots,H_k(x,u,\dots,u^{(k-1)})\big)$$

is the new output map (here we are using the notation of Section 3). That is, we redefine the output of the system to be $\mathbf{y}^k$. With a slight abuse of terminology, we will apply to such systems the definition of uniform 0-detectability given in the previous section. Of course, for $k = 0$ we recover the original system.

Theorem 1 1. *Suppose that the system* (1) *has a relative degree r and that its k-output extension* (9) *is uniformly 0-detectable for some k. Then* (1) *is minimum-phase in the sense of Definition 1, with $N = \max\{r,k\}$.*

2. *Suppose that the system* (1) *is minimum-phase in the sense of Definition 1, that the function $f(x,\cdot)$ is real analytic in u for each fixed x, and that $f(0,0) = 0$ and $h(0) = 0$. Then* (1) *has a relative degree, and its k-output extension* (9) *is uniformly 0-detectable for $k = N$.*

As an illustration, consider the affine system (5) with $f(0) = 0$. Its right-hand side is obviously real analytic in u, so Theorem 1 applies. The hypothesis $h(0) = 0$ is actually not needed in this case.

We will be especially interested in systems that are covered by part 1 of Theorem 1 with $k = r - 1$. Let us agree to call the system (1) *strongly minimum-phase* if it has a relative degree r and its $(r-1)$-output extension is uniformly 0-detectable. Note that $\mathbf{y}^{r-1}$ is a function of the state x only: $\mathbf{y}^{r-1} = \mathbf{h}_{r-1}(x)$; no differentiability assumptions need to be placed on u in this case.

Example 2. Consider an affine system in global normal form

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ &\dots \\ \dot{\xi}_r &= b(\xi,\eta) + a(\xi,\eta)u \\ \dot{\eta} &= q(\xi,\eta) \\ y &= \xi_1 \end{aligned} \tag{10}$$

where $\xi := (\xi_1,\dots,\xi_r)$ and $a(\xi,\eta) \neq 0\ \forall \xi,\eta$ (so that r is the relative degree). This system is usually called minimum-phase if the *zero dynamics* $\dot{\eta} = q(0,\eta)$ have an asymptotically stable equilibrium at $\eta = 0$ (see [1]). Since $\mathbf{y}^{r-1} = \xi$, the above definition of the strong minimum-phase property in this case demands that the equation for η in (10), which represents the internal dynamics, be input-to-state stable (ISS) with respect to ξ (more precisely, with respect to all possible signals ξ that can be generated by the ξ-subsystem). As we already mentioned, the ISS assumption has been imposed on the internal dynamics of the system in various contexts associated with control design (see, e.g., [9]). ☐

We know from [5, 11] that the $(r-1)$-output extension of (1) is uniformly 0-detectable if there exists a smooth, positive definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ that satisfies

$$\nabla V(x) f(x,u) \le -\alpha(|x|) + \chi(|\mathbf{y}^{r-1}|) \qquad \forall x,u$$

for some functions $\alpha, \chi \in \mathcal{K}_\infty$. This Lyapunov-like dissipation inequality can be used to check the strong minimum-phase property, once the relative degree of the system is known. In fact, it provides a necessary and sufficient condition for uniform 0-detectability if controls take values in a compact set [5].

Consider the affine system (5), and suppose that it is minimum-phase and has a relative degree r. Regardless of whether it has a global normal form, one can apply a state feedback law so that the r-dimensional subsystem that describes the evolution of $\xi = \mathbf{y}^{r-1}$ takes the form $\dot{\xi} = A\xi$, where A is a stable matrix. Then y and all derivatives of y decay to zero exponentially fast. In view of uniform 0-detectability with respect to $\mathbf{y}^N$, the entire system becomes globally asymptotically stable. Note that this is true for *every* feedback law that linearizes and stabilizes the ξ-subsystem, which is not necessarily the case for systems with globally asymptotically stable zero dynamics (see [4, Section 9.2]).

For MIMO systems, the existence of a relative degree is quite a restrictive assumption. For example, linear systems with relative degree form a rather special subclass of those linear systems for which the minimum-phase property (in its classical sense) is well defined.

Fortunately, Definition 1 does not have the shortcoming of applying only to systems with relative degree (as shown by way of examples in [6]). For linear systems, the following result holds.

Theorem 2 *A MIMO left-invertible linear system is minimum-phase in the sense of Definition 1 if and only if all its transmission zeros have negative real parts.*

6 Cascade results

The purpose of this section is to investigate how the minimum-phase property behaves under series connections of several subsystems. To simplify the presentation and to obtain the sharpest results possible, we restrict our attention to strongly minimum-phase systems.

Suppose that we are given two systems:

$$\Sigma_1: \quad \begin{aligned} \dot{x}_1 &= f_1(x_1, u_1) \\ y_1 &= h_1(x_1) \end{aligned}$$

and

$$\Sigma_2: \quad \begin{aligned} \dot{x}_2 &= f_2(x_2, u_2) \\ y_2 &= h_2(x_2) \end{aligned}$$

Upon setting $u_2 = y_1$, we obtain a cascade system, which we denote by Σ_c (see Figure 1).

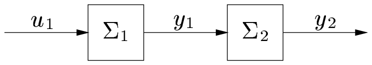

Figure 1: The cascade system

Assume that Σ_2 has a relative degree r. We can then consider the r-output extension $\bar{\Sigma}_c$ of Σ_c, whose input is u_1 and whose output is $\mathbf{y}_2^r$. This corresponds to introducing a new output map of the form

$$\mathbf{h}(x_1, x_2) = \big(h_2(x_2), H_1(x_2), \ldots, H_r(x_2, h_1(x_1))\big)$$

which is defined as explained in Section 3.

Lemma 1 *If Σ_1 is 0-detectable and Σ_2 is strongly minimum-phase, then the system $\bar{\Sigma}_c$ with input u_1 and output $\mathbf{y}_2^r$ is 0-detectable.*

Next, suppose that the system Σ_1 has another output $y_3 = h_3(x_1)$. Letting $u_2 = y_1$ as before, and defining the output $y_4 := y_3 - y_2$, we obtain a cascade-feedforward system Σ_{cf} shown in Figure 2.

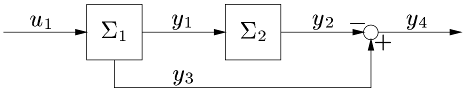

Figure 2: The cascade-feedforward system

Assume that the input u_1 is in C^{r-1}, where r is the relative degree of Σ_2 as before. We can then consider the system $\tilde{\Sigma}_1$ whose input is $\mathbf{u}_1^{r-1}$ and whose output is $\mathbf{y}_3^r$. Indeed, as explained in Section 3, for each $i \in \{1, \ldots, r\}$ the i-th derivative of y_3 exists and can be written as $y^{(i)}(t) = H_i(x(t), u_1(t), \ldots, u_1^{i-1}(t))$ for a suitable function H_i. Moreover, since y_2 is r times differentiable almost everywhere, we can consider the r-output extension $\bar{\Sigma}_{cf}$ of Σ_{cf}, with input $\mathbf{u}_1^{r-1}$ and output $\mathbf{y}_4^r$.

Lemma 2 *Suppose that Σ_1 is 0-detectable (with respect to its input u_1 and both its outputs, y_1 and y_3), Σ_2 is strongly minimum-phase, and the system $\tilde{\Sigma}_1$ with input $\mathbf{u}_1^{r-1}$ and output $\mathbf{y}_3^r$ is input-to-output stable. Then the system $\bar{\Sigma}_{cf}$ with input $\mathbf{u}_1^{r-1}$ and output $\mathbf{y}_4^r$ is 0-detectable.*

7 Adaptive control

Let $\mathbb{P}$ be an unknown process of the form

$$\begin{aligned} \dot{x}_{\mathbb{P}} &= f_{\mathbb{P}}(x_{\mathbb{P}}, u) \\ y &= h_{\mathbb{P}}(x_{\mathbb{P}}) \end{aligned}$$

where $x_{\mathbb{P}} \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, and $y \in \mathbb{R}$ is the measured output. Assume that $\mathbb{P}$ is a member of some family of systems $\bigcup_{p \in \mathcal{P}} \mathcal{F}_p$, where $\mathcal{P}$ is an index set. For each $p \in \mathcal{P}$, the subfamily $\mathcal{F}_p$ can be viewed as consisting of a *nominal process model* $\mathbb{M}_p$ together with a collection of its "perturbed" versions.

Consider the following family of controllers, parameterized by p taking values in $\mathcal{P}$:

$$\begin{aligned} \dot{x}_{\mathbb{C}} &= f_{\mathbb{C}}(x_{\mathbb{C}}, y, p) \\ u_p &= h_{\mathbb{C}}(x_{\mathbb{C}}, p) \end{aligned}$$

For every fixed $p \in \mathcal{P}$, we denote the corresponding controller by $\mathbb{C}_p$. One can think of $\mathbb{C}_p$ as a *candidate controller*, which would be used to control the process $\mathbb{P}$ if this process were known to be a member of $\mathcal{F}_p$.

We assume that on-line controller selection is carried out with the help of some estimation procedure. This is facilitated by a dynamical system $\mathbb{E}$ called the *multi-estimator*, which takes the form

$$\begin{aligned} \dot{x}_{\mathbb{E}} &= f_{\mathbb{E}}(x_{\mathbb{E}}, y, u) \\ y_p &= h_p(x_{\mathbb{E}}), \qquad p \in \mathcal{P} \end{aligned}$$

The signals $y_p, p \in \mathcal{P}$ are used to define the *estimation errors*

$$e_p := y_p - y, \qquad p \in \mathcal{P}.$$

One usually designs the multi-estimator in such a way that e_p converges to zero in the case when the unknown process coincides with the p-th nominal process model $\mathbb{M}_p$ and there are no disturbances or noise.

Take an arbitrary fixed $q \in \mathcal{P}$. The closed-loop system, which results when the q-th candidate controller

$\mathbb{C}_q$ is placed in the feedback loop with the process $\mathbb{P}$ and the multi-estimator $\mathbb{E}$, is described by the equations

$$\begin{aligned} \dot{x}_{\mathbb{P}} &= f_{\mathbb{P}}(x_{\mathbb{P}}, h_{\mathbb{C}}(x_{\mathbb{C}}, q)) \\ \dot{x}_{\mathbb{E}} &= f_{\mathbb{E}}(x_{\mathbb{E}}, h_{\mathbb{P}}(x_{\mathbb{P}}), h_{\mathbb{C}}(x_{\mathbb{C}}, q)) \\ \dot{x}_{\mathbb{C}} &= f_{\mathbb{C}}(x_{\mathbb{C}}, h_{\mathbb{P}}(x_{\mathbb{P}}), q) \end{aligned} \tag{11}$$

(see Figure 3). We will take the output of this system to be the estimation error $e_q = h_q(x_{\mathbb{E}}) - h_{\mathbb{P}}(x_{\mathbb{P}})$. Most of the standard adaptive algorithms are based on varying the index of the candidate controller in the feedback loop according to a tuning/switching law $\sigma : [0, \infty) \to \mathcal{P}$, in such a way that the corresponding estimation error e_σ is maintained small in some sense. The underlying principle behind such a strategy is known as *certainty equivalence*. To justify this paradigm, one must be able to ensure that the smallness of the estimation error implies the smallness of the state of the closed-loop system. Thus it is desirable to design the system (11) so as to make it 0-detectable with respect to e_q.

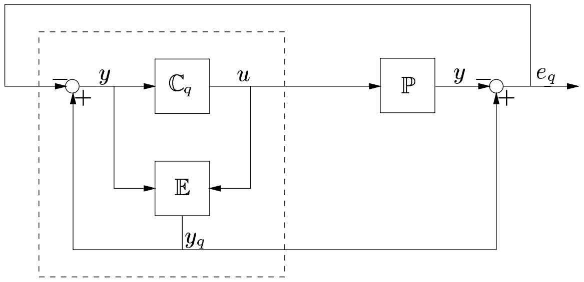

Figure 3: The closed-loop system (11)

Consider the following system, which we call the *injected system* and denote by $\mathbb{EC}_q$:

$$\begin{aligned} \dot{x}_{\mathbb{E}} &= f_{\mathbb{E}}(x_{\mathbb{E}}, h_q(x_{\mathbb{E}}) - e_q, h_{\mathbb{C}}(x_{\mathbb{C}}, q)) \\ \dot{x}_{\mathbb{C}} &= f_{\mathbb{C}}(x_{\mathbb{C}}, h_q(x_{\mathbb{E}}) - e_q, q) \end{aligned}$$

We view it as a system with input e_q, state $(x_{\mathbb{E}}, x_{\mathbb{C}})$, and outputs u and $y_q = h_q(x_{\mathbb{E}})$. It realizes the interconnection of the q-th candidate controller $\mathbb{C}_q$ with the multi-estimator $\mathbb{E}$. This is the system enclosed in the dashed box in Figure 3. It was shown in [3] that if the injected system $\mathbb{EC}_q$ is input-to-state stable (ISS) with respect to e_q and the process $\mathbb{P}$ is 0-detectable, then the closed-loop system (11) is 0-detectable with respect to e_q. This provided a natural nonlinear extension of the Certainty Equivalence Stabilization Theorem proved for linear systems in [7]. Another relevant result from [7] is the so-called Certainty Equivalence *Output* Stabilization Theorem, which we mentioned in the Introduction. It suggests that the desired 0-detectability of the system (11) through e_q should be preserved if one weakens the assumptions on the injected system $\mathbb{EC}_q$ by only requiring input-to-output stability from e_q to y_q instead of input-to-state stability, but demands that the process be minimum-phase rather than 0-detectable. In what follows, we demonstrate that a result along these lines indeed holds for nonlinear systems.

Assume that $\mathbb{P}$ has a known relative degree r. Let us redefine the input and the output of the system $\mathbb{EC}_q$ to be $\mathbf{e}_q^{r-1}$ and $\mathbf{y}_q^r$, respectively. We denote the resulting system by $\widetilde{\mathbb{EC}}_q$; its output map is obtained as explained in the previous sections. We also redefine the output of the closed-loop system (11) to be $\mathbf{e}_q^r$; i.e., we consider the r-output extension of (11), which we denote by $\bar{\Sigma}_{cl}$. We now make the following assumptions:

1. The process $\mathbb{P}$ is strongly minimum-phase.
2. The system $\widetilde{\mathbb{EC}}_q$ is input-to-output stable.
3. The controller $\mathbb{C}_q$ is 0-detectable.
4. The multi-estimator $\mathbb{E}$ is 0-detectable.

The result stated below is a direct consequence of Lemma 2: one needs to apply that lemma with $\Sigma_1 = \mathbb{EC}_q$ (which is a 0-detectable system) and $\Sigma_2 = \mathbb{P}$.

Theorem 3 *Under assumptions 1–4, the closed-loop system* $\bar{\Sigma}_{cl}$ *with output* $\mathbf{e}_q^r$ *is 0-detectable.*

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