Gramian based interaction measure.

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Abstract

In this paper, stable multivariable systems are considered and a gramian based measure of dynamic channel interaction is proposed. This measure also supports decisions regarding input-output pairing in decentralized control, triangular control and other controller structures.

1 Introduction.

In industrial control, channel interaction is a common feature which generates difficulties to control a process variable without perturbing other variables of interest. On the other hand, the control engineer will normally look, in the first place, for a decentralized control architecture. Decentralized control, although it is a limited flexibility choice, has advantages in different aspects, including control synthesis, tuning, sequencing of loop closing and the possibility to use the knowledge and intuition gained through the single input-single output control design. In this chosen strategy, a key issue is the way in which inputs and outputs are paired. This issue has received a lot of attention over the last four decades. The most significant result is the seminal work of E. Bristol [1], who developed the idea of *Relative Gain Array* (RGA). In the RGA, the channel interaction measure is built upon the d.c. gain of the MIMO process. The RGA has proved to be a useful tool, however it has limitations which has been explored elsewhere, among them it is the inability to cope with certain non minimum phase structures, its insensitivity to delays and the fact that only one point of the process frequency response is considered.

After Bristol's work was published, several researchers have studied the properties and usage of the RGA, see e.g. [7]. Some others have proposed new measures of interaction and criteria to choose a sensible input-output pairing. These include the Niederlinski index [6], [2], the *Relative Interaction Array* [9], the *Relative Dynamic Gains* (RDG), [8], the *Generalized Relative Dynamic Gains* (GRDG) [3], and others.

Control of industrial interacting processes is not only connected to decentralized architectures. There are other control architectures which are simpler than the full MIMO control (where every process input depends on all process outputs), but more versatile than the simplest decentralized (diagonal) controller. These additional architectures include block diagonal, triangular, sparse controllers, etc. Few, if any, of the existing indices are useful to evaluate alternative controller structures other than diagonal controllers.

This paper focuses on interaction quantification and alternative controller structures. We here propose an interaction measure which is based upon a dynamic model of the process. This measure also quantifies interaction as a function of chosen channel bandwidths, gives criteria for input-output pairing and allows to asses alternative controller architectures. The proposed index, which is built on the system gramians, also provides a measure of the relative performance of a given controller architecture with respect to the full MIMO case. We consider stable square MIMO systems in the continuous time and discrete time domains.

2 Gramian fundamentals

Gramians are matrices which describe certain controllability and observability properties of a given stable system. They can be computed for continuous time and discrete time systems. For simplicity, in the sequel we will only refer to the continuous time case.

Assume that a stable MIMO system has a state space representation given by the 4-tuple ($\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$, $\mathbf{0}$), then the controllability gramian, $\mathbf{P} \in \mathbb{R}^{n\times n}$, and the observability gramian, $\mathbf{Q} \in \mathbb{R}^{n\times n}$, are symmetric non negative definite matrices which satisfy the Lyapunov equations (1)

$$\mathbf{AP} + \mathbf{PA}^T + \mathbf{BB}^T = 0 \qquad \mathbf{A}^T\mathbf{Q} + \mathbf{QA} + \mathbf{C}^T\mathbf{C} = 0 \tag{1}$$

Gramians quantify how hard is to control and to observe the system state, and the ranks of $\mathbf{P}$ and $\mathbf{Q}$ are the dimensions of the controllable subspace and observable subspace respectively. However, gramians depend on the state space realization. To extract valuable information, the product $\mathbf{PQ}$ is formed and its eigenvalues, λ_i ($i = 1, 2, \dots, n$), are computed. It can be proved that these eigenvalues (known as the system Hankel singular values, HSV) are non negative and that they do not depend on the particular realization, see e.g. [4].

The HSV properties will be exploited to build an interaction measure in the next section.

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3 Gramians and MIMO interaction

3.1 Elementary system

Consider the (stable) system state space description ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{0}$). We can then associate with this MIMO system, a set of elementary (SISO) systems, each of them having a single input u_i ($i \in \{1,2,\ldots,m\}$) and a single output y_j ($j \in \{1,2,\ldots,m\}$) and a state space model given by ($\mathbf{A}$, $\mathbf{b_i}$, $\mathbf{c_j}^T$, 0) with gramians $\mathbf{P_i}$ and $\mathbf{Q_j}$ satisfying

$$\mathbf{AP_i} + \mathbf{P_i A}^T + \mathbf{b_i b_i}^T = \mathbf{0}; \;\; \mathbf{A}^T\mathbf{Q_j} + \mathbf{Q_j A} + \mathbf{c_j c_j}^T = \mathbf{0} \quad (2)$$

where $\mathbf{b_i}$ is the i^{th} column of $\mathbf{A}$, and $\mathbf{c_j}$ is the j^{th} column of $\mathbf{C}^T$. Then, the HSV associated to the pair ($\mathbf{P_i}$,$\mathbf{Q_j}$) describe the ability (or otherwise) of the input u_i and the output y_j to control and to observe the system state.

3.2 Gramian decomposition.

We next observe that the system gramians can be expressed as functions of the gramians for the elementary systems. This is precisely stated in the following lemma.

Lemma 3.1 (Gramian decomposition). *Let $\mathbf{P_i}$ and $\mathbf{Q_j}$ be the controllability and observability gramians for the elementary system ($\mathbf{A}$, $\mathbf{b_i}$, $\mathbf{c_j}^T$, 0).*

Then, the original system controllability and observability gramians $\mathbf{P}$ and $\mathbf{Q}$ are given by:

$$\mathbf{P} = \sum_{i=1}^{m} \mathbf{P_i} \qquad and \quad \mathbf{Q} = \sum_{j=1}^{m} \mathbf{Q_j} \qquad (3)$$

Proof. *Lyapunov equations in* (2) *are built for $i,j \in \{1,...,m\}$, then the equations for the $\mathbf{P_i}$'s are added and the result is the Lyapunov equation for $\mathbf{P}$. The same procedure is applied for $\mathbf{Q}$. We use the fact that $\mathbf{BB}^T = \sum_{i=1}^{m} \mathbf{b_i b_i}^T$ and $\mathbf{C}^T\mathbf{C} = \sum_{i=1}^{m} \mathbf{c_i c_i}^T$.*

Remark 1 (Gramian decomposition interpretation). *From the gramian decomposition introduced in Lemma 3.1, it can be seen that the product $\mathbf{PQ}$ for the multivariable process is given by* (4)

$$\mathbf{PQ} = \left(\sum_{i=1}^{m} \mathbf{P_i}\right)\left(\sum_{j=1}^{m} \mathbf{Q_j}\right) = \sum_{i,j=1}^{m} \mathbf{P_i Q_j} \qquad (4)$$

Then, the product $\mathbf{PQ}$ can be computed as the sum of the corresponding products $\mathbf{P_i Q_j}$ associated to the m^2 single-input single-output elementary systems.

Also, if in some sense (to be defined later) the products $\mathbf{P_i Q_j}$ and $\mathbf{P_j Q_i}$ are much smaller that $\mathbf{P_i Q_i}$ and $\mathbf{P_j Q_j}$ ($i \neq j$), then channels i and j have little coupling.

It is straightforward to prove that if the system transfer function $\mathbf{G}(s)$ is diagonal, then $\mathbf{P_i Q_j} = \mathbf{0}$ for all $i \neq j$.

When a full MIMO controller architecture is chosen, then all terms of the form $\mathbf{P_i Q_j}$ have to be added to compute the system HSV. However if a restricted complexity controller is chosen, then only a subset of those terms is required.

3.3 Quantification

The above analysis requires, to have practical interest, a way to quantify and to compare.

It turn outs that the trace of the product $\mathbf{P_i Q_j}$ is state realization independent and it is a convenient basis to measure the interaction and the ability of different controller structures to control and to observe the system state. A crucial fact is that the trace of $\mathbf{P_i Q_j}$ is equal to the sum of the HSV for the elementary systems. This choice has some other properties which derive from standard linear algebra results (trace of a sum of matrices having positive eigenvalues).

This measure can be organized in a matrix $\Phi = [\phi_{ij}] \in \mathbb{R}^{m\times m}$, which we will call **the participation matrix**, defined by

$$\phi_{ij} = \frac{trace\,[P_i Q_j]}{trace\,[PQ]} \leq 1 \qquad (5)$$

Note that the trace measure has been normalized by $trace(\mathbf{PQ})$. This implies that the sum of all elements ϕ_{ij} is equal to one.

Thus, the complexity of a controller structure should be traded off against the closeness to one, of the sum of the corresponding ϕ_{ij} elements. For future reference we will denote this sum by Σ.

To appreciate the role of the participation matrix we consider a system with three inputs and three outputs ($m = 3$), and two different controller structures[1].

Case 1: Diagonal (decentralized) controller: In this case we have three elementary subsystems, we thus have that

$$\Sigma = \sum_{i=1}^{3} \phi_{ii} \qquad (6)$$

Case 2: Lower triangular controller: In this case we have three subsystems, S_{t1}, S_{t2} and S_{t3}, with state space models given by $(\mathbf{A}, \mathbf{b_1}, \mathbf{c_1}^T, 0)$, $(\mathbf{A}, \mathbf{b_2}, [\mathbf{c_1}\;\mathbf{c_2}]^T, 0)$ and $(\mathbf{A}, \mathbf{b_3}, [\mathbf{c_1}\;\mathbf{c_2}\;\mathbf{c_3}]^T, 0)$ respectively.

Then the controllability and observability of this architecture can be quantified through

$$\Sigma = \underbrace{\phi_{11}}_{S_{t1}} + \underbrace{\phi_{21} + \phi_{22}}_{S_{t2}} + \underbrace{\phi_{31} + \phi_{32} + \phi_{33}}_{S_{t3}} \qquad (7)$$

The two cases above illustrate how the quantification proposed can be applied to arbitrary controller structures. **The main general aim is to obtain a value of Σ close to one with the minimum controller complexity, hopefully a decentralized controller.**

4 Applications

Input-output pairing for fully decentralized control can be decided on the basis of the largest elements in the participation matrix Φ. This is done by ordering the m^2 elements

[1] We consider the simplest situation when neither column nor row permutations in $\mathbf{G}$ are required

in according to their magnitudes. The pairing is then built trying to include, whenever possible, the m largest elements. To illustrate the idea consider the participation matrix for a 3×3 system given by

$$\Phi = \begin{bmatrix} 0.1030 & 0.1371 & 0.0002 \\ 0.3348 & 0.1371 & 0.0802 \\ 0.0258 & 0.0014 & 0.1804 \end{bmatrix} \tag{8}$$

In this case, the ordering is: ϕ_{21}, ϕ_{33} and then two elements, ϕ_{12} and ϕ_{22} , sharing the third place.

Then a natural pairing is (u_2,y_1), (u_3,y_3) and (u_1,y_2). The sum of the corresponding elements results to be $\Sigma = \phi_{21} + \phi_{33} + \phi_{12} = 0.6523$.

The same ordering of the elements built for diagonal controllers can be used to define a more complex controller structure. This will yield Σ closer to one, i.e. increased controllability and observability. This can be illustrated with the participation matrix (8) used in the diagonal controller case. We observe that, starting from the diagonal choice $(2,1)$, $(3,3)$ and $(1,2)$, the largest element which was left out is ϕ_{22}. If we add this element we obtain $\Sigma = \phi_{21} + \phi_{33} + \phi_{12} + \phi_{22} = 0.7894$. The associated controller has the block diagonal form

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{9}$$

where $E_i = R_i - Y_i$, $\forall i \in \{1,2,3\}$, with R_i being the reference for channel i.

The next step regarding increasing controller complexity would be to add the term ϕ_{11}. This would lead to $\Sigma = \phi_{21} + \phi_{33} + \phi_{12} + \phi_{22} + \phi_{11} = 0.8924$. The controller would still be a block diagonal controller where the upper block would be a full 2×2 MIMO controller. Inspection of the participation matrix also suggests that there is no significant benefit to make u_1 dependent also on y_3 (ϕ_{13} is much smaller than the other elements). Similar comment applies to ϕ_{32}, which connects u_3 and y_2.

It is well known that loop interaction, in general, is frequency dependent. Thus, different closed loop bandwidths may require different control structures and/or different input output pairings. To introduce this factor we recall that the frequency contents in the plant input, $\mathbf{U}$, depends on the control sensitivity $\mathbf{S_{uo}}$ [5] and on the frequency contents of reference, noise and disturbances. Thus, a way to asses how interaction changes with different closed loop bandwidths is to filter the system transfer function $\mathbf{G}(s)$ to obtain a modified plant model $\mathbf{G_f}(s)$ given by $\mathbf{G_f}(s) = \mathbf{F}(s)\mathbf{G}(s)$, where $\mathbf{F}(s) \in \mathbb{C}^{m \times m}$ is a filter which captures the essential frequency domain characteristics of the control sensitivity.

Once the modified plant model is built, the participation matrix can be computed and analyzed to decide on input output pairings and to assess the benefits of different controller structures.

5 Conclusions.

In this paper a new measure of channel interaction in stable MIMO systems has been proposed. This measure is based on the system controllability and observability gramians. It thus makes use of the ability of the gramians to describe the difficulty to observe and to control the system state. The gramian based measure is built upon a dynamic plant model and it has no limitations regarding the number of plant inputs and outputs. This measure has associated rules for input output pairings in decentralized control. Furthermore, this measure allows the designer to asses the benefits of other controller structures (triangular, block diagonal, sparse, etc.)

It has been also sugested how this measure, when applied to a suitable modified plant model, can provide information regarding interaction as function of a projected closed loop bandwidth.

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