A New Version of the Strong Law of Large Numbers For Dependent Vector Processes With Decreasing Correlation

Alex S. Poznyak
CINVESTAV-IPN, Deptartment of Automatic Control
A. P. 14-740, CP-07300, Mexico D.F.,Mexico
e-mail: apoznyak@ctrl.cinvestav.mx

Abstract

The new form of the strong law of large numbers for dependent vector sequences using the "double averaged" correlation function is presented. The suggested theorem generalizes well-known Cramer-Lidbetter's theorem and give more general conditions for fulfilling the strong law of large numbers within the class of vector random processes generated by a non stationary stable forming filters with an absolutely integrable impulse function.

Keywords: Law of large numbers , correlation function, forming filter, dependent processes.

1 Introduction. The strong law of large numbers tackles the problem of the convergence to zero with probability one the time averaged process S_n, that is,

$$S_n := n^{-1} \sum_{t=1}^{n} \xi_t \to 0 \ (P - a.s.)$$

Here $\xi_n \in R^n$ is a discrete-time centered $(E\{\xi_n\} = 0)$ quadratically integrable $\left(E\{\xi_n \xi_n^\intercal\} = \Xi_n,\ \sigma_n^2 := tr\Xi_n < \infty\right)$random vector process. This problem plays a key role in the analysis of the asymptotic behavior of recurrent algorithms analyzed in the identification and the stochastic adaptive control theories. A lot of fundamental results have been obtained for the independent random processes $\{\xi_n\}$ (see, for example, [4] and [6]). Later on, several elegant constructions, generalizing the strong law of large number to the class of dependent (martingales [5], weak and strong mixing [3] and mixingales [2]) processes have been propose. Unfortunately, the most of the characteristics (such as the mixing coefficients), participating in these constructions, are extremely complex for the direct calculation and turn out to be unapplicable in engineering practice. More clear and practically more useful results on the strong law of large numbers for dependent processes are contained in the publications operating with the correlation function as a main characteristic of a statistic dependence. The most advanced results in this direction have been obtained in [9] and [1] where the special decreasing conditions for the corresponding correlation function were introduced to guarantee the fulfilling of the strong law of large number. This note presents a new form of the strong law of large numbers using a special characteristic (**the "double averaged" correlation function**) of dependence which can be easily constructed based on correlation coefficients. This paper generalizes the earlier author's results [7] obtained for the scalar case.

2 Main Result. Let all random sequences considered below be defined on the probability space (Ω, F, P). For the given centered quadratic-integrable R^n-valued random process $\{\xi_n\}$ introduce the special characteristic, so-called, *the "double averaged" correlation function* R_n defined by

$$R_n := n^{-2} \sum_{t=1}^{n} \sum_{s=1}^{n} \rho_{t,s} = E\{S_n^\intercal S_n\} \tag{1}$$

where $\rho_{t,s} := E\{\xi_t^\intercal \xi_s\}$ is the corresponding correlation function.

Theorem 1 (The strong law of large numbers) *If*

$$\sum_{n\ \in N^+} \left(\frac{\sigma_n}{n} \sqrt{R_{n-1}} + \frac{1}{n^2} \sigma_n^2 \right) < \infty \tag{2}$$

*then **"the strong law of large numbers"** holds, that is,* $S_n \overset{a.s.}{\to} 0$.

Remark 2 *If the given process $\{\xi_t\}$ has a bounded variance, that is, $\sigma_n^2 \le \bar{\sigma}^2 < \infty$ and a "double averaged" correlation function R_n, decreasing as $R_n = O(n^{-\varepsilon})$ $(\varepsilon > 0)$, then the conditions of this theorem are fulfilled automatically.*

Proof: Since for any $n = 2, 3, \ldots$

$$\|S_n\|^2 = \left(1 - \frac{1}{n}\right)^2 \|S_{n-1}\|^2 + v_n \le \left(1 - \frac{1}{n}\right) S_{n-1}^2 + v_n$$

where $v_n := 2\frac{1}{n}\left(1 - \frac{1}{n}\right) S_{n-1}^\intercal \xi_n + \frac{1}{n^2}\|\xi_n\|^2$ then the back iterations imply

$$S_n^2 \leq \pi_n S_1^2 + \pi_n \sum_{t=2}^{n} \pi_t^{-1} v_t,\ \pi_n := \prod_{t=2}^{n} \left(1 - \frac{1}{t}\right)$$

By the Kronecker's lemma (see, for example, Appendix in [8]) S_n tends to zero if with probability 1 the following sequence $r_n^{(1)} := \sum_{t=1}^{n} v_t$ converges. To fulfill this, it is sufficient to show that under the conditions of this theorem the series $r_n^{(2)} := \sum_{t=1}^{n} t^{-2} \|\xi_t\|^2$, $r_n^{(3)} := \sum_{t=1}^{n} t^{-1} |S_{t-1}^{\intercal} \xi_t|$ converge with probability one, that is true if $\sum_{t=1}^{\infty} t^{-2} \sigma_n^2 < \infty$, $\sum_{t=1}^{\infty} t^{-1} \{|S_{t-1}^{\intercal} \xi_t|\} < \infty$. By the Cauchy-Bouniakovski - Shwartz inequality, it follows

$$\sum_{t=1}^{\infty} \frac{1}{t} E\{|S_{t-1}^{\intercal} \xi_t|\} \leq \sum_{t=1}^{\infty} \frac{1}{t} \sqrt{E\{\|S_{t-1}\|^2\} \sigma_n^2}$$

that together with the identity $E\{\|S_{t-1}\|^2\} = R_{t-1}$ directly leads to the result of this theorem. ■

Corollary 3 (Cramer-Lidbetter's Condition) *Assume that the correlation coefficients $\rho_{t,s}$ of the given random process satisfy the **Cramer-Lidbetter's condition** [1], that is,*

$$|\rho_{t,s}| \leq K \frac{t^{\alpha} + s^{\alpha}}{1 + |t - s|^{\beta}}$$

where K, α, β - nonnegative constants verifying $2\alpha < \min\{1, \beta\}$. Then the strong law of large numbers holds, that is, $S_n \overset{a.s.}{\to} 0$.

Proof: Since $E\{\|\xi_t\|^2\} = \sigma_n^2 = \rho_{t,t} \leq 2Kt^{\alpha}$ then

$$R_n \leq \frac{K}{n^2} \sum_{t=1}^{n} \sum_{s=1}^{n} \frac{t^{\alpha}+s^{\alpha}}{1+|t-s|^{\beta}} \leq \frac{2K}{n^{1-\alpha}} + 2K\, I_n$$
$$I_n := \frac{1}{n^2} \sum_{t=1}^{n} \sum_{s<t} \frac{t^{\alpha}+s^{\alpha}}{1+(t-s)^{\beta}} = I_n' + I_n''$$

$$I_n' := \sum_{t=1}^{n} \frac{t^{\alpha}}{n^2} \sum_{s<t} \frac{1}{1+(t-s)^{\beta}} \leq \int_0^n \frac{t^{\alpha}}{n^2} \left\{ \begin{array}{l} \frac{t^{\beta-1}-1}{1-\beta}, \beta \neq 1 \\ \ln t, \beta = 1 \end{array} \right\} dt$$
$$\leq Const \left\{ \begin{array}{ll} n^{\alpha-\beta} & , \beta < 1 \\ n^{\alpha+\varepsilon-1} & , \beta = 1 \\ n^{\alpha-1} & , \beta > 1 \end{array} \right\}, \varepsilon > 0$$

$$I_n'' := \frac{1}{n^2} \sum_{t=1}^{n} \sum_{s<t} \frac{s^{\alpha}}{1+(t-s)^{\beta}} \leq I_n'$$

So, finally, the following upper estimate for the "double averaged" correlation function R_n holds:

$$R_n \leq Const \left\{ \begin{array}{ll} n^{\max\{\alpha-1,\alpha-\beta\}} & , \beta < 1 \\ n^{\alpha+\varepsilon-1} & , \beta = 1 \\ n^{\alpha-1} & , \beta > 1 \end{array} \right\}, \varepsilon > 0$$

The substitution of the right-hand side of the last inequality in (2) implies the result of this corollary. ■

Corollary 4 (Output of Stable Forming Filters) *Consider centered random independent vectors ξ_n with finite variances σ_n^2 satisfying*

$$\sum_{n \in N^+} \frac{1}{n(n-1)} \sigma_n \sqrt{\sum_{r=0}^{n-1} \sigma_r^2} < \infty$$

and generating the random vector sequence ζ_n according to the following expression $\zeta_n = \sum_{t=0}^{n} h_{n,t} \xi_t$ where the impulse response matrix function $h_{n,t}$ for any $t \leq n$ satisfies

$$\|h_{n,t}\| \leq \hat{h}(n-t), H := \sum_{\tau=0}^{\infty} \hat{h}(\tau) < \infty$$

(such impulse functions correspond to stable, may be, nonstationary forming filters). Then for the random sequence $\{\zeta_n\}$ the strong law of large numbers holds, that is, $n^{-1} \sum_{t=1}^{n} \zeta_t \overset{a.s.}{\to} 0$.

Proof: The inequality $R_n \leq H^2 n^{-2} \sum_{r=0}^{n} \sigma_r^2$ implies

$$\frac{1}{n} \sqrt{R_{n-1} \sigma_n^2} \leq \frac{H}{n(n-1)} \sigma_n \sqrt{\sum_{r=0}^{n-1} \sigma_r^2}$$

that, proves this corollary. ■

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