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### ANTICIPATION OF LINEAR SYSTEMS WITH STATE-FEEDBACKS

**Abstract.** The concepts of anticipatory discrete-time and continuous-time systems for singular linear systems are introduced. Necessary and sufficient conditions for anticipation of the linear systems are established. It is shown that the anticipation of singular systems is not invariant under state - feedbacks.

#### 1. Anticipatory discrete-time systems.

Let  $R^{p \times n}$  be the set of real  $p \times n$  matrices and  $R^p := R^{p \times 1}$ . Consider the discrete-time linear system

$$Ex_{i+1} = Fx_i + Gu_i, \quad y_i = Cx_i \quad (1)$$

where  $x_i \in R^n$ ,  $u_i \in R^m$ ,  $y_i \in R^p$  are the state vector, input vector and output vector at the point  $i \in Z_+ := \{0, 1, 2, \dots\}$ , respectively and  $E, F \in R^{p \times n}$ ,  $G \in R^{n \times m}$ ,  $C \in R^{p \times n}$ . If  $\det E \neq 0$  then the system (1) is called standard and if  $\det E = 0$  then the system is called singular. It is assumed that the pencil  $(E, F)$  is regular, i.e.

$$\det[Ez - F] \neq 0 \text{ for some } z \in \mathbf{C} \quad (2)$$

where  $\mathbf{C}$  is the field of complex numbers  
If the condition (2) is satisfied then

$$[Ez - F]^{-1} = \sum_{i=-\mu}^{\infty} \Phi_i z^{-(i+1)} \quad (3)$$

where  $\mu$  is the nilpotence index and  $\Phi_i$  is the fundamental matrix defined by

$$E\Phi_i - F\Phi_{i-1} = \Phi_i E - \Phi_{i-1} F = \begin{cases} I_n & \text{for } i=0 \\ 0 & \text{for } i \neq 0 \end{cases} \quad (4)$$

$I_n$  is the  $n \times n$  identity matrix.

From the solution of (1a)

$$x_i = \Phi_i Ex_0 + \sum_{k=0}^{i+\mu-1} \Phi_{i-k-1} Gu_k \quad (5)$$

it follows that if  $\mu > 1$  then  $x_i$  depends on the future values of inputs  $u_k$  for  $k > i$ .

**Definition 1.** The system (1) is called anticipatory if the state vector  $x_i$  and output vector  $y_i$  at the point  $i$  depends on the future values of  $u_k$  for  $k > i$ .

**Theorem 1.** The singular system (1) is anticipatory if and only if

$$\text{rank} E > \text{deg. det}[Ez - F] \quad (6)$$

where deg denotes the degree.

#### 2. Anticipatory continuous - time systems

Consider the continuous-time linear system

$$E\dot{x} = Ax + Bu, \quad y = Cx, \quad x(0) = x_0 \quad (7)$$

where  $\dot{x} = \frac{dx}{dt}$ ,  $x = x(t) \in R^n$ ,  $u = u(t) \in R^m$ ,  $y = y(t) \in R^p$  are the state vector, input vector and output vector, respectively and  $E, A \in R^{n \times n}$ ,  $B \in R^{n \times m}$ ,  $C \in R^{p \times n}$ .

If  $\det E \neq 0$  the system (7) is called standard and if  $\det E = 0$  the system is called singular.

It is assumed that the pencil  $(E, A)$  is regular, i.e.

$$\det[Es - A] \neq 0 \text{ for some } s \in \mathbf{C} \quad (8)$$

If the condition (8) is satisfied then

$$[Es - A]^{-1} = \sum_{i=-\mu}^{\infty} \Phi_i s^{-(i+1)} \quad (9)$$

where  $\mu$  is the nilpotence index and  $\Phi_i$  is the fundamental matrix defined by (4)

The solution  $x(t)$  of the equation (7a) has the form

$$x(t) = e^{\Phi_0 A t} \Phi_0 Ex_0 + \int_0^t e^{\Phi_0 A(t-\tau)} \Phi_0 Bu(\tau) d\tau + \sum_{j=1}^{\mu} \Phi_{-j} (Bu^{(j-1)} + Ex_0 \delta^{(j-1)}) \quad (10)$$

where  $u^{(j)} = \frac{d^j u}{dt^j}$ ,  $\delta^{(j)}$  denotes the derivative of the  $j$ -th order of the Dirac impulse  $\delta(t)$ .

From (10) it follows that if  $\mu > 1$  then the solution  $x(t)$  depends on the derivatives of  $u(t)$ .

**Definition 2.** The system (7) is called anticipatory if the state vector  $x$  and the output vector  $y$  depends on the derivatives of  $u$ .

**Theorem 2.** The singular system (7) is anticipatory if and only if

$$\text{rank} E > \text{deg. det}[Es - A] \quad (11)$$

**Definition 3.** The matrices  $E, A \in R^{n \times n}$ ,  $B \in R^{n \times m}$ ,  $C \in R^{p \times n}$  have the first canonical form if

$$\begin{aligned} E &= \text{diag}[E_1 \ E_2 \ \dots \ E_m] \in R^{n \times n}, \\ E_i &= \begin{bmatrix} I_{q_i} & 0 \\ 0 & 0 \end{bmatrix} \in R^{(q_i+1) \times (q_i+1)} \quad n := m + \sum_{i=1}^m q_i \\ A &= \text{diag}[A_1 \ A_2 \ \dots \ A_m] \in R^{n \times n}, \quad B_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^{q_i+1} \\ A_i &= \begin{bmatrix} 0 & I_{q_i} \\ a_i \end{bmatrix} \in R^{(q_i+1) \times (q_i+1)} \\ a_i &= [a_0^i \ \dots \ a_{r-1}^i \ 1 \ 0 \ \dots \ 0] \quad B = \text{diag}[B_1 \ B_2 \ \dots \ B_m] \in R^{n \times m} \\ C &= \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \dots & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pm} \end{bmatrix} \in R^{p \times n}, \\ c_{ij} &= [b_{ij}^0 \ b_{ij}^1 \ \dots \ b_{ij}^{q_i}] \in R^{1 \times (q_i+1)}, \quad i = 1, \dots, p \\ & \quad j = 1, \dots, m \end{aligned} \quad (12)$$

Let the matrices of the single-input single-output system (1) have the canonical form

$$\begin{aligned} E &= \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \in R^{n \times n}, \quad F = \begin{bmatrix} 0 & I_{n-1} \\ a \end{bmatrix} \in R^{n \times n} \\ a &= [a_0, a_1, \dots, a_{n-2} \ a_{n-1}] \\ G &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^n, \quad C = [b_0 \ b_1 \ \dots \ b_{n-1}] \in R^{1 \times n} \end{aligned} \quad (13)$$

Using the expansion of  $\det[Es - A]$  it is easy to show the lemma

**Lemma.** The system (1) with (13) is anticipatory if and only if the entry  $a_{n-1}$  of  $a$  is zero, i.e.  $a_{n-1} = 0$

### 3. Anticipatory systems with state - feedbacks

Consider the system (1) with the state-feedback

$$u_i = v_i - Kx_i \quad i \in Z_+ \quad (14)$$

where  $K \in R^{1 \times n}$  is a feedback gain matrix and  $v_i$  is the new input.

Substitution of (14) into (1) yields

$$Ex_{i+1} = F_c x_i + Gv_i \quad i \in Z_+ \quad (15)$$

where  $F_c = F - GK$  is the closed-loop system matrix

Let the system (1) with (13) be anticipatory,  $a_{n-1} = 0$  and

$$K = [0 \ \dots \ 0 \ -a_{n-1}] \quad a_{n-1} \neq 0 \quad (16)$$

Using (13) and (16) we obtain

$$F_c = F - GK = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \end{bmatrix}$$

Hence by Lemma the closed-loop system (15) is not anticipatory.

**Theorem 3.** The closed-loop system (15) is not anticipatory if the gain matrix  $K$  has the form (16).

Now let the system (1) with (13) be not anticipatory and  $a_{n-1} \neq 0$ .

Let  $K = [0 \ \dots \ 0 \ a_{n-1}]$ ,  $a_{n-1} \neq 0$ . Then we obtain

$$F_c = F - GK = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-2} & 0 \end{bmatrix}$$

By Lemma the closed-loop system (15) with (16) is anticipatory.

**Theorem 4.** The closed-loop system (32) is anticipatory if  $K = [0 \ \dots \ 0 \ a_{n-1}]$ .

The above considerations are also valid for continuous-time linear systems (7).

### 4. Conclusions.

The concepts of anticipatory discrete - time and continuous - time systems for singular linear systems have been introduced. Necessary and sufficient conditions for anticipation of the singular systems have been established. It has been shown that the anticipation of singular systems is not invariant under state - feedbacks. The considerations can be extended for two - dimensional linear systems.