

ROBUST REFLECTION COEFFICIENTS PLACEMENT BY QUADRATIC PROGRAMMING

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Abstract

A new version of pole placement controller design, called robust reflection coefficients placement, is proposed for discrete-time systems. Instead of a single stable point a stable simplex must be preselected in the closed loop characteristic polynomials coefficients space. A constructive procedure for generating simplex inside the "nice stability region" is given starting from the unit hypercube of reflection coefficients of monic polynomials. This procedure is quite straightforward because, first, an appropriate stable point has to be chosen and then the edges of the stable simplex will be generated by a linear Schur invariant transformation. The procedure of robust controller design by quadratic programming makes use of a stability measure defined as the minimal distance between a preselected stable simplex and vertices of an uncertain interval plant.

Keywords: robust stability, discrete-time systems, pole placement, quadratic programming.

1 Introduction

The pole placement method is a common approach for designing closed-loop controllers in order to meet desired control specifications. The objective of assigning closed loop poles is often replaced by that of assigning a characteristic polynomial because 1) this polynomial plays a central role in the stability analysis of linear control systems and 2) polynomial coefficients are simply (affinely) related to controller and plant parameters.

Another practical issue is that of model uncertainty. If the model uncertainty is relatively small, then it is possible to use sensitivity-based methods. If the model uncertainty is large some robust formulation of the problem is needed, such as multimodel [1,2] or polytopic plant model approach [3-6].

One of the main problems concerned with robust control of plants with polytopic uncertainties is: how to find a reasonable convex approximation of the stability region in the closed-loop characteristic polynomial coefficients space. Unfortunately the discrete analog of the Kharitonov theorem [7] is considerably weaker than the continuous one. However, the crucial drawback of the Kharitonov-like approach is the lack of any measure for stability robustness. That is why algorithmic techniques with linear matrix inequalities [3,4] and convex programming [5,6] play a significant role in robust controller design practice. Some rather conservative approximations of the stability region by simplex are known [2,8]. The following scheme explains the relations in robust controller design tasks (Fig.1).

Ackermann's procedure [1] follows the path $\mathbf{A} \rightarrow \mathbf{P} \rightarrow \mathbf{K}$, i.e. from roots of closed-loop characteristic polynomial through polynomial coefficients space to state feedback controller. Kraus and Kučera [5] as well as Keel and Bhattacharayya [6] followed the path $\mathbf{A} \rightarrow \mathbf{P} \rightarrow \mathbf{C}$. Box theorem [9] formulates the relations immediately between polynomial coefficients space \mathbf{P} and output feedback controller \mathbf{C} , i.e. $\mathbf{P} \rightarrow \mathbf{C}$. Robust controller design for polytopic plants via linear matrix inequalities [3,4] follows the path $\overline{\mathbf{P}} \rightarrow \mathbf{C}$.

In [8] a new concept for robust controller design has been introduced: $\Xi \rightarrow \overline{\mathbf{P}} \rightarrow \mathbf{C}$, i.e. starting from reflection coefficients of Schur polynomials we can find a convex approximation of the stability region in the closed-loop characteristic polynomial coefficients space and via a preselected stable simplex we can obtain an output feedback controller. This approach is called robust reflection coefficients placement.

In this paper a similar idea has been used. However, the main interest is concerned, first, with Schur stable convex subsets building in polynomial coefficients space and, second, with robust controller design by quadratic programming.

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These relations define a multilinear mapping from re-

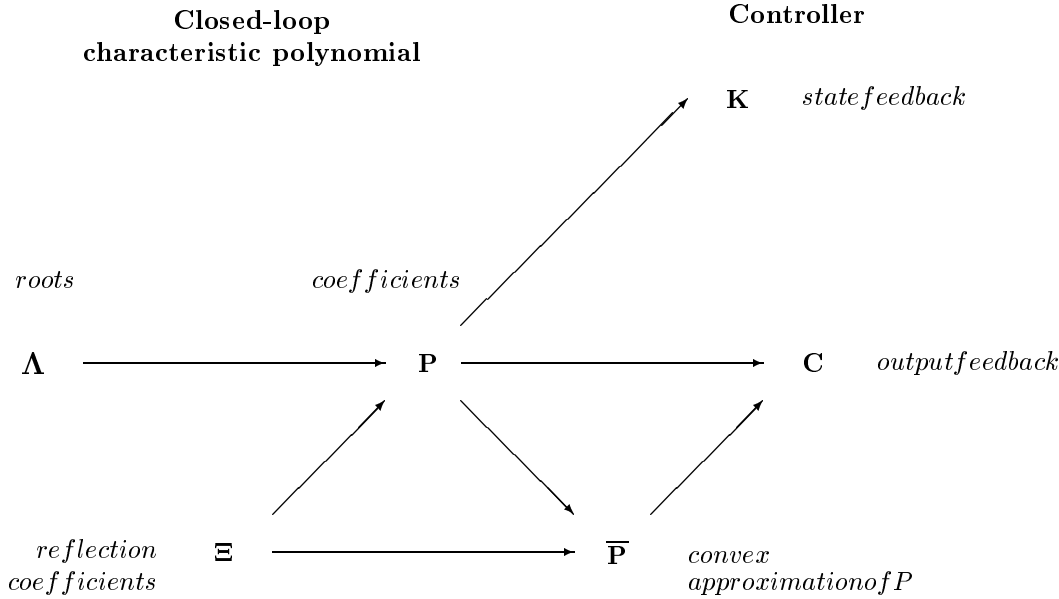


Figure 1. Robust controller design by pole or reflection coefficient placement

The following problems will be considered. First, we recall reflection coefficients of polynomials. Then some convex stable subsets in polynomial coefficients space are defined. The main result of the paper is a constructive procedure for stable polytope (simplex) building starting from a given Schur stable polynomial. The n vertices of a nicely stable simplex will be generated by varying reflection coefficients of a properly chosen point. At last, a procedure is proposed for designing a robust controller for polytopic plants via preselection of an appropriate Schur stable simplex and quadratic programming.

2 Reflection coefficients of Schur polynomials

Consider the polynomials with real coefficients $a_i \in R$

$$a(z) = \sum_{i=0}^n a_i z^i.$$

A polynomial $a(z)$ is said to be Schur stable if all its roots lie inside the unit circle.

The recursive definition of reflection coefficients $k_i \in R$ of a polynomial $a(z)$ is as follows [10]:

$$\begin{aligned} \bar{a}_i^{(n)} &= \frac{a_{n-i}}{a_n}, & i &= 1, \dots, n; \\ \bar{a}_j^{(i-1)} &= \frac{\bar{a}_j^{(i)} + k_i \bar{a}_{i-j}^{(i)}}{1 - k_i^2}, & j &= 1, \dots, i-1; \\ k_i &= -\bar{a}_i^{(i)}. \end{aligned}$$

reflection coefficients space into monic polynomial coefficients space

$$\begin{aligned} a_{i-1} &= a_{n-i}^{(n)}, \\ a_i^{(i)} &= -k_i, \\ a_j^{(i)} &= a_j^{(i-1)} - k_i a_{i-j}^{(i-1)}, \\ i &= 1, \dots, n; j = 1, \dots, i-1. \end{aligned} \quad (1)$$

A necessary and sufficient condition for all the roots of $a(z)$ to be inside the unit circle is [10]

$$|k_i| < 1, \quad i = 1, \dots, n.$$

Reflection coefficients k_i are widely used in digital filtering and signal processing where they are called also k -parameters and *PARCOR*-coefficients.

3 Convex stable subsets in the polynomial coefficients space

Let us introduce a constructive approach for n -dimensional Schur stable convex subset building in polynomial coefficients space. We start with 2 lemmas.

Lemma 1. Through an arbitrary stable point $a = [a_0, a_1, \dots, a_{n-1}]$ with reflection coefficients $k_i^a \in$

$(-1, 1)$, $i = 1, \dots, n$ you can put n stable line segments

$$s_i = \text{conv}\{a|k_i^a = \pm 1\}$$

where $\text{conv}\{a|k_i^a = \pm 1\}$ denotes the convex hull obtained by varying the reflection coefficient k_i^a between -1 and 1 .

Proof follows immediately from the linearity of relations (1). The line segments s_i are called Schur invariant line segments of the point a .

Lemma 2. The polytope S^0 generated by Schur invariant line segments s_i , $i = 1, \dots, n$ through the origin $a = 0$

$$S^0 = \text{conv}\{0|k_i = \pm(1-\delta), 0 < \delta < 1, i = 1, \dots, n\} \quad (2)$$

is stable.

Obviously the Cohn stability criterion [11]

$$\sum_{i=0}^{n-1} |a_i| < 1$$

holds for S^0 .

Lemma 2 (or Cohn stability condition) is quite conservative. The question is: is it possible to relax the initial condition of Lemma 2 in some neighbourhood of the origin? The answer is given by the following proposition.

Theorem 1. Let $k_1^a \in (-1, 1)$ and $k_2^a = \dots = k_n^a = 0$. Then the polytope S^a generated by the Schur invariant line segments

$$S^a = \text{conv}\{a|k_i^a = \pm(1-\delta), 0 < \delta < 1, i = 1, \dots, n\} \quad (3)$$

is Schur stable.

Proof: Let $k_1^a = k_1^*$. Then by (1) we find the vertices

of S^a

$$\begin{aligned} \bar{a}^1 &= a[k_1 = \bar{k}_1 = \pm(1-\delta)] = \\ &= [0, \dots, 0, 0, 0, \bar{k}_1] \\ \bar{a}^2 &= a[k_2 = \bar{k}_2 = \pm(1-\delta)] = \\ &= [0, \dots, 0, 0, 0, -\bar{k}_2, -k_1^*(1-\bar{k}_2)] \\ \bar{a}^3 &= a[k_3 = \bar{k}_3 = \pm(1-\delta)] = \\ &= [0, \dots, 0, 0, -\bar{k}_3, k_1^*\bar{k}_3, -k_1^*] \\ \bar{a}^4 &= a[k_4 = \bar{k}_4 = \pm(1-\delta)] = \\ &= [0, \dots, 0, -\bar{k}_4, k_1^*\bar{k}_4, 0, -k_1^*] \\ &\dots \\ \bar{a}^n &= a[k_n = \bar{k}_n = \pm(1-\delta)] = \\ &= [-k_n, k_1^*\bar{k}_n, 0, \dots, 0, 0, -k_1^*] \end{aligned}$$

Let now for some n the polytope $S^a(n)$ is stable. We have to prove that the polytope $S^a(n+1)$ will be stable.

Obviously,

$$\bar{a}^i(n+1) = [0, \bar{a}^i(n)], \quad i = 1, \dots, n$$

and

$$\bar{a}^{n+1}(n+1) = [-\bar{k}_{n+1}, k_1^*\bar{k}_{n+1}, 0, \dots, 0, -k_1^*].$$

The polynomials

$$\bar{a}^i(z, n+1) = z\bar{a}^i(z, n)$$

will be stable if only the polynomials $\bar{a}^i(z, n)$ are stable (they have an extra root in the origin). So we have to prove stability of edges $\text{conv}\{\bar{a}^{n+1}(n+1, \delta), \bar{a}^i(n+1, \delta)\}$, $i = 1, \dots, n$ for $0 < \delta < 1$. Taking into account the multilinearity of transformation (1) we obtain $S^a(\delta_1) \subset S^a(\delta_2)$ if $\delta_2 < \delta_1$. It means, we have to check the stability of edges $\text{conv}\{\bar{a}^{n+1}(n+1, \delta), \bar{a}^i(n+1, \delta)\}$, $i = 1, \dots, n$ for $\delta \rightarrow 0$. It can be easily done by stability conditions given in [12]. \square

Remark. Condition (3) is less conservative than (2) because for S^a we have $\sum_{i=0}^{n-1} |a_i| < 3$.

If the only requirement for a controller is Schur stability of closed-loop systems then the region of suitable reflection coefficients is the unit hypercube $k_i \in (-1, 1)$, $i = 1, \dots, n$. If there are some stronger requirements for dynamics of the system (rise time, overshoot, settling time etc.) then we are looking for a subset Γ_k in the unit hypercube of reflection coefficients. Numerical experiments indicated that a reasonably damped system has reflection coefficients of the characteristic polynomial with following properties:

- k_1 is nonnegative, $k_1 \in (0, 1)$;
- the sign of successive reflection coefficients is alternating $\text{sign}(k_i) = \text{sign}(-1)^{i+1}$, $i = 1, \dots, n$;

- c) the absolute values of successive reflection coefficients are decreasing $|k_i| \geq |k_{i+1}|$, $i = 1, \dots, n$.

The following assertion holds.

Theorem 2. Let $k_1^a \in (-1, 1)$, $k_2^a \in (-1, 1)$ and $k_3^a = \dots = k_n^a = 0$. Then the simplex \tilde{S}^a generated by the Schur invariant line segments

$$\tilde{S}^a = \text{conv}\{a | k_i^a = [-(1-\delta)]^{i-1}, 0 < \delta < 1, i = 1, \dots, n\} \quad (4)$$

is Schur stable.

Proof is similar to that of Theorem 1.

Theorems 1 and 2 give us a constructive tool for Schur stable polytope (simplex) building in polynomial coefficients space. You need only to choose a convenient starting (generating) point a . Directions of the primary edges of the simplex \tilde{S}^a (or the polytope S^a) are defined by Schur invariant line segments corresponding to reflection coefficients k_i^a , $i = 1, \dots, n$ of the point a . The length of a primary edge is defined by the variation of the corresponding reflection coefficient $k_i \in (-1 + \delta, 1 - \delta)$, $0 < \delta < 1$, $i = 1, \dots, n$.

Corollary 2.1. For $n = 3$ the simplex \tilde{S}^a generated by Schur invariant line segments of an arbitrary stable point a is Schur stable.

4 Robust controller design by reflection coefficients placement and quadratic programming

Consider a discrete-time linear SISO system. Let the plant transfer function $G(z)$ of dynamic order m and the controller transfer function $C(z)$ of dynamic order r be given respectively by

$$G(z) = \frac{b(z)}{a(z)} = \frac{b_{m-1}z^{m-1} + \dots + b_1z + b_0}{a_mz^m + \dots + a_1z + a_0}$$

and

$$C(z) = \frac{q(z)}{r(z)} = \frac{q_rz^r + \dots + q_1z + q_0}{r_rz^r + \dots + r_1z + r_0}$$

It means that the closed loop characteristic polynomial

$$f(z) = a(z)r(z) + b(z)q(z)$$

is of degree $m + r$.

It is known in the literature that when $r = m - 1$ the above equation admit a solution for the controller coefficients for arbitrary $f(z)$ whenever the plant has no common pole-zero pairs. In general for $r < m - 1$ exact attainment of the desired polynomial is impossible. Here we suggest the following approach.

Let us relax the requirement of attaining the desired polynomial $f(z)$ exactly and enlarge the target to a simplex S in coefficient space containing the point representing the desired characteristic polynomial. Without any restrictions we can assume that $a_m = r_r = 1$ and deal with monic polynomials.

Let us now introduce a stability measure ρ in accordance with the simplex S

$$\rho = c^T c$$

where

$$c = S^{-1} f$$

and S is the $(m+r+1) \times (m+r+1)$ matrix of vertices of the target simplex. Obviously, for monic polynomials

$$\sum_{i=1}^{n+1} c_i = 1$$

where $n = m + r$. If all coefficients $c_i > 0$, $i = 1, \dots, n + 1$ then the point f is placed inside the simplex S .

It is easy to see that the minimum of ρ is obtained by

$$c_1 = c_2 = \dots = c_{n+1} = \frac{1}{n+1}$$

Then the point f is placed in the center of the simplex S .

Now we can formulate the following problem of controller design: find a controller $C(z)$ such that the stability measure ρ is minimal. In other words, we are looking for a controller which places the closed loop characteristic polynomial $f(z)$ as close as possible to the center of the target simplex S .

In matrix form we have

$$f = Gx \quad (5)$$

where G is the plant Sylvester matrix

$$G = \begin{bmatrix} a_0 & 0 & \dots & 0 & b_0 & 0 & \dots & 0 \\ a_1 & a_0 & \dots & 0 & b_1 & b_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & a_{n-2} & \dots & a_0 & b_{n-1} & b_{n-2} & \dots & b_0 \\ 0 & a_{n-1} & \dots & a_1 & 0 & b_{n-1} & \dots & b_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n-1} & 0 & 0 & \dots & b_{n-1} \end{bmatrix}$$

of dimensions $(m + r + 1) \times (2r + 1)$ and x is the $(2r + 1)$ -vector of controller parameters $x = [q_0, \dots, q_{r-1}, r_0, \dots, r_r]^T$.

The above controller design problem is equivalent to the quadratic programming problem: find x such that the minimum

$$\min_x x^T G^T (SS^T)^{-1} Gx$$

is obtained by the linear restriction

$$1^T S^{-1} Gx = 1$$

where $1^T = [1 \dots 1]$ is an n vector.

If we have the optimal solution x^* with all positive elements $x_i^* > 0$, $i = 1, \dots, n+1$ then the characteristic polynomial $f(z)$ is placed inside the simplex S , i.e. the above problem of controller design has a solution and x^* is the best one.

If some $x_i^* \leq 0$ then the controller design problem has not any solution.

Let us now consider the case where the plant is subject to parameter uncertainty. We represent this by supposing that the given plant transfer function coefficients a_0, \dots, a_{m-1} and b_0, \dots, b_{m-1} are placed in a polytope P with vertices p^1, \dots, p^M

$$P = \text{conv}\{p^j, j = 1, \dots, M\}.$$

Because the relation (5) is linear in plant parameters we can claim that for an arbitrary fixed controller x the vector f of closed loop characteristic polynomial coefficients is placed in a polytope F with vertices f^1, \dots, f^M

$$F = \text{conv}\{f^j, j = 1, \dots, M\}$$

where

$$f^j = P^j x$$

and P^j is a $2m \times 2m$ matrix composed by the vertex plant $p^j = [a_0^j, \dots, a_{m-1}^j, b_0^j, \dots, b_{m-1}^j]$.

The problem of robust controller design can be formulated as follows: find a controller x such that all vertices f^j , $j = 1, \dots, M$ are placed inside the simplex S .

This problem can be solved by quadratic programming task: find x which minimizes

$$\min_x x^T \tilde{P}^T (I \otimes ((S^T)^{-1}) (I \otimes S^{-1}) \tilde{P} x$$

by linear restrictions

$$1^T S^{-1} P^j x = 1, \quad j = 1, \dots, M.$$

Here I is the unit matrix, \otimes denotes the Kronecker product and $\tilde{P}^T = [P_1^T, \dots, P_M^T]$.

The problem has a solution if for the optimal x^* we have $x_i^* > 0$, $i = 1, \dots, n+1$.

Example. Let us consider an uncertain second order interval plant

$$G(z) = \frac{b_0}{z^2 + a_1 z + a_0}$$

with parameters in the intervals $1.85 \leq b_0 \leq 1.95$, $-1.525 \leq a_1 \leq -1.475$, $a_0 = 0.55$ and we are looking for a first order robust controller.

Let the nominal closed loop characteristic polynomial be

$$f^0 = z^3 - 0.25z^2 + 0.03z - 0.001.$$

Then by pole placement algorithm we can easily find the controller

$$C_0(z) = \frac{0.7132z - 0.3624}{z + 1.25}$$

for the nominal plant

$$G_0(z) = \frac{1.9}{z^2 - 1.5z + 0.55}.$$

The simplex S will be chosen according to considerations of section 3. Starting from the origin $a = 0$ we, first, decrease the reflection coefficient k_1^a , $k_1^a \in (-1, 0)$ to find

$$\rho^0(k_1^*) = \min_{k_1} \rho^0$$

where ρ^0 is the stability measure for the nominal closed loop characteristic polynomial f^0 with respect to simplex $\tilde{S}(k_1)$. Then we increase the reflection coefficient k_2^a , $k_2^a \in (0, 1)$ to find

$$\rho^0(k_1^*, k_2^*) = \min_{k_2} \rho^0(k_1^*).$$

For the above example we obtain $k_1^* = -0.5$, $k_2^* = 0.2$ and

$$S = \tilde{S}(k_1^*, k_2^*) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ -0.2 & -0.2 & 1 & -0.6 \\ 0.4 & -0.8 & 1 & 0.6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

By above quadratic programming procedure we have find a robust controller

$$C(z) = \frac{0.9053z - 0.4428}{z + 1.4539}.$$

5 Conclusions

To find a robust controller by quadratic programming a stable simplex must be preselected in the closed loop characteristic polynomials coefficients space. A constructive procedure for generating simplex inside the "nice stability region" of polynomial coefficients space is given. This procedure of stable simplex (or polytope) building is quite straightforward because you need to choose only one stable point with some restrictions for reflection coefficients of it. Then all the vertices of the simplex will be generated by a linear Schur invariant transformation.

This approach for robust controller design is called robust reflection coefficients placement since it starts from a preselected hyperrectangle of reflection coefficients of closed loop characteristic polynomials.

The procedure of controller design by quadratic programming is based on a stability measure ρ which indicates the placement of a (vertex) point against the preselected stable simplex.

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