A Robustness Result for Stochastic Control

Gino Favero, Wolfgang J. Runggaldier
Università degli Studi di Padova
Dipartimento di Matematica Pura ed Applicata
via Belzoni 7, 35131 Padova – Italy
favero@math.unipd.it, runggal@math.unipd.it

Abstract

The solution of a stochastic control problem depends on the underlying model, i.e., on the probability measure induced by the model. The real world model may not be known precisely, and so one solves the problem for a hypothetical model that induces a measure generally different but close to the real one.

We investigate two ways to derive a bound on the suboptimality of the hypothetical optimal control when it is used in the real problem. Both bounds are in terms of the Radon-Nikodym derivative of the real world measure with respect to the hypothetical one.

Introduction

In various applications the mathematical model leads to a discrete time, finite horizon stochastic control problem of the form

$$(P) \qquad \begin{cases} x_{t+1} = F(x_t, \pi_t, \xi_t^P), \qquad t = 0, 1, \ldots, T \\ \inf_\pi \Phi^P(x, \pi) \end{cases}$$

where x_t is the "state" process, ξ_t^P are the "disturbances" (usually a sequence of i.i.d. random variables independent of the initial state x_0), π_t is the control process (adapted to the σ-algebra generated by the disturbances ξ_s^P, $s \leqslant t$), and

$$\Phi^P(x, \pi) := \mathrm{E}_{x,\pi}^P\{C(\pi)\}$$

where, typically, $C(\pi) = \sum_{t=0}^{T-1} c_t(x_t, \pi_t) + b(x_T)$ for given $c_t(\cdot, \cdot)$ and $b(\cdot)$.

The solution to the problem (P) is a pair $(\pi_x^*, \Phi^*(x))$ (optimal control and optimal value) such that

$$\Phi^*(x) = \Phi^P(x, \pi_x^*) = \inf_\pi \Phi^P(x, \pi).$$

The index P in ξ_t^P and $\mathrm{E}_{x,\pi}^P$ stresses the fact that the model, and therefore its solution, depends on the choice of the underlying probability measure P, which has to correspond to the true "real world" probability measure. A question, that arises naturally, deals with the sensitivity of the optimal solution to the choice of the underlying model, more specifically, to the choice of P.

Suppose that P is not known and that, instead of (P), we solve the hypothetical problem

$$(Q) \qquad \begin{cases} x_{t+1}^Q = F(x_t^Q, \pi_t, \xi_t^Q) \\ \inf_\pi \Phi^Q(x, \pi) \end{cases}$$

for a suitable $\Phi^Q(x, \pi)$ (see descriptions below) where Q is a probability measure different from P. We thus obtain an optimal control π_x^Q, in general different from π_x^*, and an optimal value $\Phi^Q(x, \pi^Q)$.

Our goal here is to give a bound on

$$\Delta_x(P, Q) := |\Phi^P(x, \pi^Q) - \Phi^*(x)|$$

in terms of a suitable measure of the difference between P and Q (equivalently, between $\{\xi_t^P\}$ and $\{\xi_t^Q\}$); this measure will be in terms of the Radon-Nikodym derivative of one measure with respect to the other. Dually, given a tolerance for $\Delta_x(P, Q)$, we are interested in determining which measures Q can be tolerated as "working" measures.

Note that in problem (Q) the functional $\Phi^Q(x, \pi)$ need not be the analogue of $\Phi^P(x, \pi)$ i.e., it is not necessarily defined as $\Phi^Q(x, \pi) = \mathrm{E}_{x,\pi}^Q\{C(\pi)\}$ for the same $C(\pi)$. Actually it may be any suitable value functional, provided that the corresponding solution π^Q of (Q) leads to a small value for $\Delta_x(P, Q)$.

In section 1 we derive an upper approximating problem based on a partition of the range of the target functional and actually obtain a functional in the measure Q whose form is different from the original one. The Radon-Nikodym derivative is handled with an exponential technique analogously to Dai Pra et al. (see [1]). In section 2 we derive the bound in the case when the target functional in the hypothetical problem is the analogue in the new measure of that of the real problem by operating a straightforward measure change. Section 3 is dedicated to comments on the two approaches.

1 An exponential approach

Given the initial state $x = x_0$, suppose that there are $\widetilde{M} > \widetilde{m}$ such that

$$\mathrm{e}^{\widetilde{m}} \leqslant C(\pi) \leqslant \mathrm{e}^{\widetilde{M}}$$

for every control π. Furthermore, suppose that Q is such that:

- there is a $\overline{\gamma} > 0$ for which $\left|\log(\frac{\mathrm{d}P}{\mathrm{d}Q})\right| \leqslant \overline{\gamma}$,
- the r.v. $\log C(\pi)$ has a probability density function $p_{C(\pi)}$ under the measure Q such that for $x = x_0$, $p_{C(\pi)} \leqslant K$ for all π.

Now write $m := \widetilde{m} - \overline{\gamma}$, $M := \widetilde{M} + \overline{\gamma}$ and consider a partition $m = m_1 < M_1 = m_2 < \cdots < M_{n-1} = m_n < M_n = M$ of $[m, M]$, with

$$\mathrm{e}^{M_i} - \mathrm{e}^{m_i} = \sqrt{2(\mathrm{e}^{2M} - \mathrm{e}^{2m})K\overline{\gamma}} \qquad \text{for every } i.$$

Instead of minimizing $\Phi^P(x,\pi) = \mathrm{E}^P_{x,\pi}\{C(\pi)\}$, minimize

$$\Phi^Q(x,\pi) = \sum_{i=1}^{n} Q_i(x,\pi)\mathrm{e}^{M_i}$$

where $Q_i(x,\pi) := Q\{m_i - \overline{\gamma} \leqslant \log(C(\pi)) \leqslant M_i + \overline{\gamma}\}$. Note that $\Phi^Q(x,\pi)$ entirely depends upon the measure Q, and not also on the unknown "real" measure P.

1.1. Theorem. *In the given setting, for every control π we have*

$$0 \leqslant \Phi^Q(x,\pi) - \Phi^P(x,\pi) \leqslant \overline{\varepsilon}(\overline{\gamma})$$

where

$$\overline{\varepsilon}(\overline{\gamma}) := 2\sqrt{2(\mathrm{e}^{2M} - \mathrm{e}^{2m})K\overline{\gamma}}.$$

Consequently,

$$\left|\Phi^P(x,\pi^Q) - \Phi^*(x)\right| \leqslant 2\overline{\varepsilon}(\overline{\gamma}).$$

1.2. Remark. Notice that the number of points in the partition turns out to be greater than one if and only if $\frac{\mathrm{e}^M + \mathrm{e}^m}{\mathrm{e}^M - \mathrm{e}^m} K\overline{\gamma} < \frac{1}{2}$.

2 A direct approach

Suppose now that $C(\pi) > 0$ and that $\left|1 - \frac{\mathrm{d}P}{\mathrm{d}Q}\right| \leqslant \widetilde{\gamma}$ for some $\widetilde{\gamma} > 0$. Define $\Phi^Q(x,\pi) := \mathrm{E}^Q_{x,\pi}\{C(\pi)\}$ and let $M_x := \sup_\pi \Phi^Q(x,\pi)$.

2.1. Theorem. *In the given setting, for every control π and every x we have*

$$\left|\Phi^Q(x,\pi) - \Phi^P(x,\pi)\right| \leqslant \widetilde{\gamma}M_x$$

so that, for every x,

$$\left|\Phi^P(x,\pi^Q) - \Phi^*(x)\right| \leqslant 2\widetilde{\gamma}M_x.$$

Moreover, if $\widetilde{\gamma} < 1$, it is possible to restrict the class Π of the admissible controls to a subclass Π^*, containing both π^Q and π^*, such that for every $\pi \in \Pi^*$

$$\left|\Phi^Q(x,\pi) - \Phi^P(x,\pi)\right| \leqslant \frac{\widetilde{\gamma}(1+\widetilde{\gamma})}{1-\widetilde{\gamma}}\Phi^Q(x,\pi^Q).$$

3 Final Remarks

Explicit calculations of the two bounds give strong evidence that the second bound is always significantly smaller than the first one. Moreover, the "direct" approach (unlike the "exponential" one) can be applied to a wider class of models because $\widetilde{\gamma}$ is finite every time that P is absolutely continuous w.r.t. Q, while for $\overline{\gamma}$ to be finite we need P and Q to be mutually absolutely continuous w.r.t. each other. Finally, the direct approach does not need hypotheses on the probability density function of the r.v. $C(\pi)$, so it can be applied also to atomic measures or discrete models, unlike the exponential one.

The exponential approach, on the other hand, features a cost functional whose form differs from the original one. This may have considerable advantages when $C(\pi)$ has a complicated structure, so that the approach may also be seen as a general approximation approach.

For further details, as well as for the proofs of the theorems, we refer to [2]; an application to a concrete model can be found in [3].

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References

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