Dead-beat control laws for impacting systems in presence of uncertainties.

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1 Control of the nominal system

Consider a 1 d.o.f. rigid mechanical system, whose configuration at time t is described by the Lagrangian coordinate $x(t) \in \mathbb{R}$, and let its dynamics be

$$\ddot{x}(t) = u(t), \quad t \geq 0, \tag{1a}$$

$$x(t) \geq 0, \quad t \geq 0. \tag{1b}$$

Define the set of admissible initial conditions

$$\hat{\mathcal{A}} = \{(x,\, \dot{x}) \in \mathbb{R}^2 \,:\; x \geq 0 \text{ and } \dot{x} \geq 0 \text{ if } x = 0\}.$$

The rigidity assumption implies that the system is subject to *non-smooth* impacts with the surface (see [1, 2]); we also assume that there is no energy dissipation due to the impacts. An impact occurs at time $t_I \geq 0$ if $x(t_I) = 0$ and $\dot{x}(t_I^-) < 0$ (here, and in the following, $g(t^-) := \lim_{\tau \to t^-} g(\tau)$, $g(t^+) := \lim_{\tau \to t^+} g(\tau)$, for any $g(\cdot)$). The motion of the system after the impact can be described by letting $\dot{x}(t_I^+) = -\dot{x}(t_I^-)$.

We assume that $x(t)$ and $\dot{x}(t)$ are measured. To obtain the dead-beat stability of the origin $[x\ \dot{x}]^T = \mathbf{0}$, a piece-wise constant control law is proposed:

$$u(t) = \mathbf{K}_D \begin{bmatrix} x(\lfloor t/T \rfloor T) \\ \dot{x}(\lfloor t/T \rfloor T) \end{bmatrix}, \tag{2}$$

where $T > 0$ and, for any $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the greatest integer smaller than or equal to r. The feedback gain matrix $\mathbf{K}_D$ can be computed easily:

$$\mathbf{K}_D = \begin{bmatrix} -1/T^2 & -3/(2\,T) \end{bmatrix}. \tag{3}$$

Theorem 1 *Consider the closed-loop system described by* (1) *and* (2)*. For any $T > 0$, for any initial condition $(x(0),\, \dot{x}(0)) \in \hat{\mathcal{A}}$, there exists $t_F \in \mathbb{R}$, $t_F \geq 0$, such that $x(t) = 0$ and $\dot{x}(t) = 0$, for any $t \geq t_F$. In particular:*

$$t_F = \begin{cases} t_F \leq 2\,T & \text{if} \quad \dot{x}(0)\,T + 2\,x(0) \geq 0, \\ t_F = 3\,T & \text{if} \quad \dot{x}(0)\,T + 2\,x(0) < 0. \end{cases}$$

2 Control of the perturbed system

We study the system described above assuming that we don't know exactly the position of the surface, *i.e.*, equation (1b) is substituted by

$$x(t) \geq \varepsilon, \tag{4}$$

where ε is a small real number which is not known *a priori*. We assume that the initial configuration of the system satisfies $(x(0),\ \dot{x}(0)) \in \hat{\mathcal{A}}_\varepsilon$, where

$$\hat{\mathcal{A}}_\varepsilon = \{(x,\, \dot{x}) \in \mathbb{R}^2 \,:\; x \geq \varepsilon \text{ and } \dot{x} \geq 0 \text{ if } x = \varepsilon\}.$$

First, we analyze the behaviour of the control law (2); then, we show how to solve the problems deriving from the uncertainty in case of $\varepsilon > 0$.

By some computations, it is easy to see that, if $\varepsilon < 0$, the closed loop system described by (1a), (4) and (2) will in any case reach the equilibrium $[x\ \dot{x}]^T = \mathbf{0}$ in a finite time $t_F \leq 3\,T$ (the set of initial conditions such that $t_F \leq 2\,T$ is actually larger than the one given in Theorem 1 for the nominal system). However, such an equilibrium does not constitute a condition of contact. On the contrary, if $\varepsilon > 0$, the point $x = 0$ does not belong to the admissible region of the perturbed system. In this case, it can be seen that, for any $\varepsilon > 0$, there exists a periodic trajectory of the closed-loop perturbed system, of period $2\,T$, which is locally asymptotically stable (*i.e.*, a limit cycle).

Theorem 2 *Consider the closed-loop system described by* (1a), (4) *and* (2)*. For any $T > 0$, for any $\varepsilon > 0$, let $x_r(t)$ be the trajectory having as initial conditions $x_r(0) = 4\,\varepsilon$ and $\dot{x}_r(0) = 8\,\varepsilon/T$. Then, $x_r(\cdot)$ is periodic with period $2\,T$,* i.e.*, for any $t \geq 0$ and for any $k \in \mathbb{Z}^+$, $x_r(t + 2\,k\,T) = x_r(t)$. Moreover, such a trajectory is locally asymptotically stable.*

To solve the problems described above, related to the uncertainty $\varepsilon > 0$ in the position of the surface, we

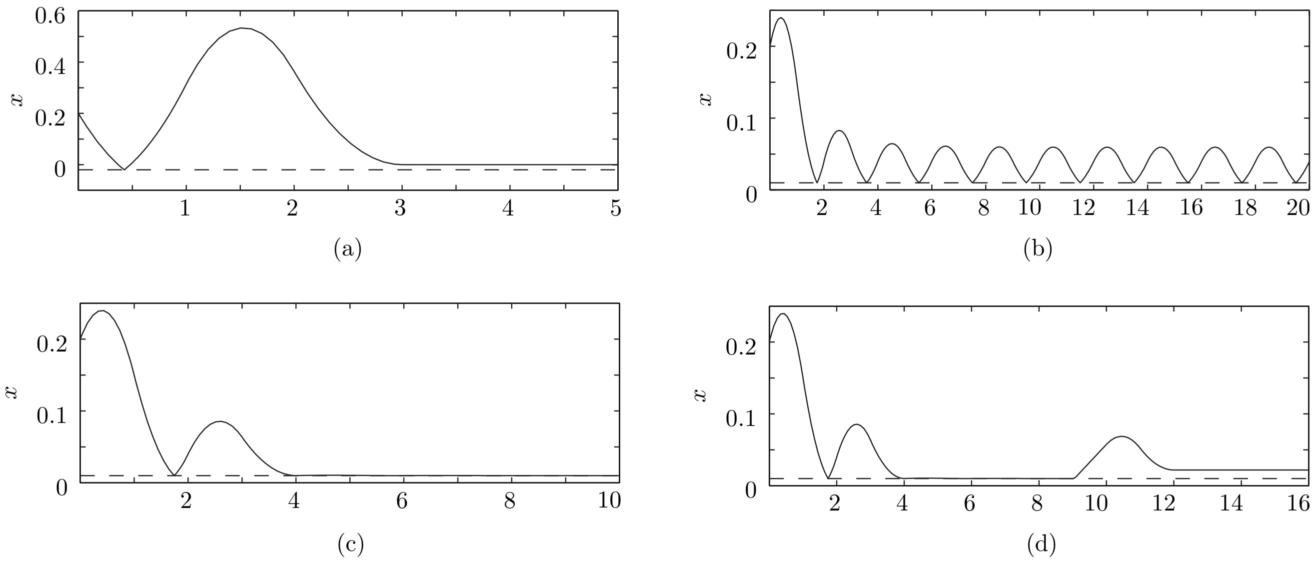

Figure 1: Time behaviour of the perturbed system with different control laws (the dashed line denotes the admissible region). In (a) and (b) the control law (2) is used, with $\varepsilon < 0$ and $\varepsilon > 0$, respectively. In (c) and (d) the control law (5)-(6) is used with $\varepsilon > 0$. It can be seen in (c) that the contact is obtained in spite of the perturbation, whereas in (d) it is shown that external disturbances can cause the loss of contact.

now propose the following control law:

$$u(t) \;=\; \mathbf{K}_D \begin{bmatrix} x(\lfloor t/T \rfloor T) - \hat{\varepsilon}(\lfloor t/T \rfloor T) \\ \dot{x}(\lfloor t/T \rfloor T) \end{bmatrix}, \quad (5)$$

where $\mathbf{K}_D$ is still given by (3) and, for each $k \in \mathbb{Z}^+$, $\hat{\varepsilon}(k)$ is an estimate of the position ε of the surface, obtained with the following updating mechanism:

$$\hat{\varepsilon}(2\,k+1) \;=\; \hat{\varepsilon}(2\,k), \quad (6a)$$

$$\hat{\varepsilon}(2\,k+2) \;=\; \frac{x_{D,1}(2\,k+2) + 3\,\hat{\varepsilon}(2\,k+1)}{4}. \quad (6b)$$

The control law constituted by (5) and (6) obtains dead-beat stability of the equilibrium $x = \varepsilon$, $\dot{x} = 0$ (which is the best approximation of $x = 0$, $\dot{x} = 0$ which is admissible for the perturbed system), for a certain set of initial conditions, as stated in the following theorem.

Theorem 3 *Consider the closed-loop system described by* (1a), (4), (5) *and* (6)*. For any* $T > 0$*, for any initial condition* $(x(0),\ \dot{x}(0),\ \hat{\varepsilon}(0)) \in \hat{\mathcal{A}}_\varepsilon \times \mathbb{R}$ *such that* $\hat{\varepsilon}(0) \le \varepsilon$ *and* $\dot{x}(0) \ge v_{min}(x(0),\ \hat{\varepsilon}(0))$*, where, for each* $(x,\ \dot{x}) \in \hat{\mathcal{A}}_\varepsilon$*:*

$$v_{min}(x,\, \hat{\varepsilon}) := \begin{cases} \dfrac{-x+\varepsilon}{T} & \text{if} \quad \varepsilon \le x \le 3\,\varepsilon - 2\,\hat{\varepsilon}, \\[2ex] \dfrac{-2\,x + 4\,\varepsilon - 2\,\hat{\varepsilon}}{T} & \text{if} \quad x > 3\,\varepsilon - 2\,\hat{\varepsilon}, \end{cases}$$

we have $x(t) = \varepsilon$ *and* $\dot{x}(t) = 0$*, for any* $t \ge 4\,T$*.*

The main disadvantage of the control law (5)-(6) is that its efficacy is limited to the case $\hat{\varepsilon}(0) \le \varepsilon$, *i.e.*, assuming $\hat{\varepsilon}(0) = 0$, it eliminates the limit cycle if $\varepsilon \ge 0$, but does not obtain the permanent contact if $\varepsilon < 0$. The problem can be easily solved in practice, by making sure that the initial estimate of the position of the surface is smaller than the actual position, but, as it will be clarified by a subsequent simulation, this difficulty also arises in presence of disturbances, which have been not considered until now.

We report in Fig. 1 the results of four simulations, obtained for $T = 1$. Figs. 1 (a) and (b) are related to the control law (2), whereas Figs. 1 (c) and (d) are related to (5). In Fig. 1 (a), $\varepsilon = -0.02$, $x(0) = 0.2$ and $\dot{x}(0) = -0.7$, whereas in Figs. 1 (b),(c) and (d), $\varepsilon = 0.01$, $x(0) = 0.2$ and $\dot{x}(0) = 0.2$. In Fig. 1 (d) a disturbance $d(t) = 0.5$ is added to the control input for times $t \in [9 \quad 9.1]$. The effect of the disturbance is that $\hat{\varepsilon}(10) > \varepsilon$, so that for time $t \ge 12$, the closed-loop system remains at the equilibrium $\hat{\varepsilon}(\lfloor t \rfloor) = \hat{\varepsilon}(10)$, $x(t) = \hat{\varepsilon}(10)$, $\dot{x}(t) = 0$.

References

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