

# A characterization of the Lie Algebra Rank Condition by transverse periodic functions

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## Abstract

The Lie Algebra Rank Condition plays a central role in nonlinear systems control theory. We show that the satisfaction of this condition by a set of smooth control vector fields is equivalent to the existence of smooth transverse periodic functions. The proof here outlined—details can be found in [4]—is constructive and provides a method for the determination of such functions. This is illustrated by an example.

## 1 Introduction

The main result of this paper is a theorem which basically states that smooth vector fields  $X_1, \dots, X_m$  on a finite-dimensional manifold  $M$  satisfy the classical Lie Algebra Rank Condition at a point  $p \in M$  ( $LARC(p)$ ) if and only if there exist an integer  $\bar{n} (> m)$  and, for any neighborhood  $U_p$  of  $p$ , a smooth function  $f : \alpha \mapsto f(\alpha)$  from  $\mathbb{R}^{\bar{n}-m}$  to  $U_p$  which, for every  $\alpha$ , is (maximally) “transversal” to the subspace spanned at  $f(\alpha)$  by these vector fields.

The authors believe that the proposed theorem could become instrumental, and a unifying tool, for the development of new solutions to various problems involving nonlinear control systems. Direct application of the theorem concerns, in the first place, “practical” feedback stabilization of either driftless control systems—such as nonholonomic systems—, in relation to time-varying feedback methods, or systems subjected to a nonvanishing drift vector field, in relation to “hybrid” open-loop/feedback control solutions based on the

use of “highly oscillatory” terms and averaging techniques. Other applications are also envisioned in the context of nonholonomic motion planning, again in relation to oscillatory open-loop control techniques which have been proposed to approximate arbitrary trajectories in the state space, and—a more tentative guess—in the domain of state estimation and nonlinear observer design. Results in some of these directions have already been obtained (see [5]).

The following notation is used throughout the paper. For manifolds  $M$  and  $N$ ,  $M_p$  denotes the tangent space of  $M$  at  $p$ , and for  $F \in \mathcal{C}^\infty(M; N)$ ,  $T_p F$  denotes the tangent mapping of  $F$  at  $p$ .  $\mathbb{T}^k$ , with  $k \in \mathbb{N}$ , denotes the  $k$ -dimensional torus.  $B_n(0, \delta)$  denotes the closed ball in  $\mathbb{R}^n$  centered at zero, and of radius  $\delta$ . For  $h \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^m)$ , and  $g \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R})$  with  $g(x) \neq 0$  for  $x \neq 0$ , we write  $h = o(g)$  when  $|h(x)|/|g(x)| \rightarrow 0$  as  $x \rightarrow 0$ . Finally,  $\mathbf{d}$  denotes the exterior derivative.

## 2 Main result

**Theorem 1** *Let  $X_1, \dots, X_m$  denote smooth vector fields on a smooth  $n$ -dimensional manifold  $M$ , such that the accessibility distribution  $\Delta(p) \triangleq \text{Span} \{X(p) : X \in \text{Lie}(X_1, \dots, X_m)\}$  is of constant dimension  $n_0$  in a neighborhood of  $p_0$ . Then, the following properties are equivalent:*

1.  $n_0 = n$ , i.e. the Lie Algebra Rank Condition at  $p_0$ ,  $LARC(p_0)$ , is satisfied for the vector fields  $X_1, \dots, X_m$ .
2. There exist  $\bar{n} \in \mathbb{N}$  and, for any neighborhood  $U$

of  $p_0$ , a function  $F \in C^\infty(\mathbb{T}^{\bar{n}-m}; \mathcal{U})$  such that:

$$\forall \theta \in \mathbb{T}^{\bar{n}-m}, M_{F(\theta)} = \text{Span} \{X_1(F(\theta)), \dots, X_m(F(\theta))\} + T_\theta F(\mathbb{T}^{\bar{n}-m}).$$

**Remark:** For a large class of systems (in particular for free systems defined below),  $\bar{n} = n$  so that  $F$  is an immersion and the sum in the above equality is direct.

We rephrase this theorem by considering a system of local coordinates  $x = (x_1, \dots, x_n)$  on  $M$  which maps  $p_0$  to  $0 \in \mathbb{R}^n$ . We also denote by  $\alpha = (\alpha_{m+1}, \dots, \alpha_{\bar{n}})$  a system of local coordinates on  $\mathbb{T}^{\bar{n}-m}$ .

**Theorem 2** *Let  $X_1, \dots, X_m, F$  in Theorem 1 be given, in local coordinates, by  $g_1, \dots, g_m, f$ . Then, the following properties are equivalent:*

1. *LARC(0): the Lie Algebra Rank Condition at the origin is satisfied for*

$$S: \quad \dot{x} = \sum_{i=1}^m g_i(x) u_i$$

2. *TC(0): there exist  $\bar{n} \in \mathbb{N}$  and a family  $(f_\epsilon)_{\epsilon>0}$  of functions  $f_\epsilon \in C^\infty(\mathbb{T}^{\bar{n}-m}; B_n(0, \epsilon))$  such that, for any  $\epsilon > 0$ , the following Transversality Condition holds:*

$$\forall \theta \in \mathbb{T}^{\bar{n}-m}, \text{Rank} \left( g_1(f_\epsilon(\theta)) \dots g_m(f_\epsilon(\theta)) \frac{\partial f_\epsilon}{\partial \alpha_{m+1}}(\theta) \dots \frac{\partial f_\epsilon}{\partial \alpha_{\bar{n}}}(\theta) \right) = n. \quad (1)$$

In the following sections, we give a sketch of proof of Theorem 2. We refer the reader to [4] for details.

### 3 TC(0) $\implies$ LARC(0)

We assume that LARC(0) is not satisfied, and show that TC(0) cannot be satisfied either. By assumption, the accessibility distribution is of constant dimension  $n_0$  in a neighborhood of the origin. Therefore, if  $n_0 < n$ , the Frobenius theorem guarantees the existence of local coordinates  $\phi(x)$

such that  $\phi_n$  is constant along the trajectories of  $S$ , i.e. for some neighborhood  $\mathcal{U}$  of the origin,

$$\forall i = 1, \dots, m, \forall x \in \mathcal{U}, L_{g_i} \phi_n(x) = 0. \quad (2)$$

Now assume that TC(0) is satisfied, and choose any  $f_\epsilon$  satisfying (1) and such that  $B_n(0, \epsilon) \subset \mathcal{U}$ . By compactness of  $\mathbb{T}^{\bar{n}-m}$ , the smooth function  $\theta \mapsto \phi_n(f_\epsilon(\theta))$  from  $\mathbb{T}^{\bar{n}-m}$  to  $\mathbb{R}$  attains its maximal value for some  $\bar{\theta}$ , i.e.

$$\forall i = m+1, \dots, \bar{n}, \frac{\partial \phi_n}{\partial x}(f_\epsilon(\bar{\theta})) \frac{\partial f_\epsilon}{\partial \alpha_i}(\bar{\theta}) = 0. \quad (3)$$

From (3), and (2) evaluated at  $x = f_\epsilon(\bar{\theta})$ , we obtain

$$\frac{\partial \phi_n}{\partial x}(f_\epsilon(\bar{\theta})) \left( g_1(f_\epsilon(\bar{\theta})) \dots g_m(f_\epsilon(\bar{\theta})) \frac{\partial f_\epsilon}{\partial \alpha_{m+1}}(\bar{\theta}) \dots \frac{\partial f_\epsilon}{\partial \alpha_{\bar{n}}}(\bar{\theta}) \right) = 0,$$

which is in contradiction with TC(0).  $\blacksquare$

## 4 LARC(0) $\implies$ TC(0)

### 4.1 Notation and recalls

**About homogeneity** (see e.g. [1, 2] for details) Given  $\mu > 0$  and a *weight vector*  $r = (r_1, \dots, r_n)$  ( $r_i > 0 \forall i$ ), a *dilation*  $\Delta_\mu^r$  on  $\mathbb{R}^n$  is a map from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  defined by  $\forall z = (z_1, \dots, z_n) \in \mathbb{R}^n, \Delta_\mu^r z \triangleq (\mu^{r_1} z_1, \dots, \mu^{r_n} z_n)$ .

A function  $f \in C^0(\mathbb{R}^n; \mathbb{R})$  is *homogeneous of degree  $l$  with respect to the family of dilations  $(\Delta_\mu^r)_{\mu>0}$* , or more concisely  *$\Delta^r$ -homogeneous of degree  $l$* , if  $\forall \mu > 0, f(\Delta_\mu^r z) = \mu^l f(z)$ .

A  *$\Delta^r$ -homogeneous norm* is defined as a positive definite function on  $\mathbb{R}^n$ ,  $\Delta^r$ -homogeneous of degree one.

A smooth vector field  $X$  on  $\mathbb{R}^n$  is  *$\Delta^r$ -homogeneous of degree  $d$*  if, for all  $i = 1, \dots, n$ , the function  $x \mapsto X_i(x)$  is  $\Delta^r$ -homogeneous of degree  $d + r_i$ .

The system

$$S_{ap}: \quad \dot{z} = \sum_{i=1}^m b_i(z) u_i \quad (4)$$

is a  *$\Delta^r$ -homogeneous approximation* of  $S$ , with  $\min\{r_i, ; i = 1, \dots, n\} = 1$ , if there exists a change of coordinates  $\phi: x \mapsto z$  which transforms  $S$  into

$$\dot{z} = \sum_{i=1}^m (b_i(z) + h_i(z)) u_i, \quad (5)$$

where  $b_i$  is  $\Delta^r$ -homogeneous of degree  $-1$ , and  $h_i$  denotes higher-order terms, i.e. for any  $j$ , the  $j$ -th component  $h_{i,j}$  of  $h_i$  satisfies  $h_{i,j} = o(\rho^{r_j-1})$ , where  $\rho$  is any  $\Delta^r$ -homogeneous norm. The main motivation for introducing such approximations comes from the following result.

**Proposition 1** [1, 6] *For any system  $S$  of smooth v.f. which satisfies  $LARC(0)$ , there exists a  $\Delta^r$ -homogeneous approximation  $S_{ap}$  which also satisfies  $LARC(0)$ .*

Finally, we say that a set  $\{b_1, \dots, b_m\}$  of v.f., or the associated system (4), is *nilpotent of order  $d+1$*  if any Lie bracket of these v.f. of length larger than, or equal to,  $d+1$  is identically zero. It is simple to verify that any set  $\{b_1, \dots, b_m\}$  of smooth v.f. with the  $b_i$ 's  $\Delta^r$ -homogeneous of degree  $-1$ , is nilpotent of order  $1 + \text{Max}\{r_i : i = 1, \dots, m\}$ .

**About free Lie algebras** (see e.g. [3, 7] for details)

Let us consider a finite set of indeterminates  $X_1, \dots, X_m$ , and denote by  $\mathcal{L}(X)$  the free Lie algebra over  $\mathbb{R}$  generated by the  $X_i$ 's. We also denote by  $\mathcal{F}(X)$  the set of formal brackets in the  $X_i$ 's. For any set  $\{b_1, \dots, b_m\}$  of smooth v.f., and any  $B \in \mathcal{F}(X)$ , we denote by  $\text{Ev}_{\{b\}}(B)$  the *evaluation map*, i.e.  $\text{Ev}_{\{b\}}(X_i) = b_i$ , and

$$\text{Ev}_{\{b\}}([B_\lambda, B_\rho]) = [\text{Ev}_{\{b\}}(B_\lambda), \text{Ev}_{\{b\}}(B_\rho)].$$

**Definition 1** *A P. Hall basis  $\mathcal{B}$  of  $\mathcal{L}(X)$  is a totally ordered subset of  $\mathcal{F}(X)$  such that*

- i) Each  $X_i$  belongs to  $\mathcal{B}$ .*
- ii) If  $B = [B_\lambda, B_\rho] \in \mathcal{F}$  with  $B_\lambda, B_\rho \in \mathcal{F}$ , then  $B \in \mathcal{B}$  if and only if  $B_\lambda, B_\rho \in \mathcal{B}$  with  $B_\lambda < B_\rho$ , and either (i)  $B_\rho$  is one of the  $X_i$ 's or (ii)  $B_\rho = [B_{\lambda\rho}, B_{\rho^2}]$  with  $B_{\lambda\rho} \leq B_\lambda$ .*
- iii) If  $B \in \mathcal{B}$  is a bracket of length  $\ell(B) \geq 2$ , i.e.  $B = [B_\lambda, B_\rho]$ , with  $B_\lambda, B_\rho \in \mathcal{B}$ , then  $B_\lambda < B$ .*

In order to simplify the forthcoming analysis we consider a specific P. Hall basis  $\mathcal{B}$  associated with a specific total order. The P. Hall basis so obtained is in fact a Hall basis in the original (narrow) sense.

**Specific order:**

$$\begin{cases} \ell(B) < \ell(B') \implies B < B' \\ X_i < X_j \iff i < j \\ \text{For } \ell(B) = \ell(B') > 1, B < B' \iff \\ (B_\lambda < B'_\lambda, \text{ or } B_\lambda = B'_\lambda \text{ and } B_\rho < B'_\rho) \end{cases} \quad (6)$$

We denote by  $\mathcal{B} = \{B_1, B_2, \dots, B_q, \dots\}$ , with  $B_1 < B_2 < \dots < B_q < \dots$ , the P. Hall basis associated with the total order (6), and also by  $\ell(i)$  the length of any bracket  $B_i$  of this basis. From (6) and the definition of a P. Hall basis,

$$\forall i = 1, \dots, m, \quad B_i = X_i. \quad (7)$$

Note that, for any  $i \geq m+1$ , there exist unique integers  $\lambda(i)$  and  $\rho(i)$  such that

$$B_i = [B_{\lambda(i)}, B_{\rho(i)}]. \quad (8)$$

Let  $0 < d \in \mathbb{N}$ , we denote by  $\mathcal{L}_d(X)$  the subspace of  $\mathcal{L}(X)$  generated by brackets of length at most equal to  $d$ . Then, the subset of  $\mathcal{B}$  composed of all brackets  $B_j$  such that  $\ell(j) \leq d$  is a basis of  $\mathcal{L}_d(X)$  denoted as  $\mathcal{B}_d$ . Let  $n(d)$  denote the dimension of  $\mathcal{L}_d(X)$ , so that

$$\mathcal{B}_d = \{B_1, \dots, B_{n(d)}\} \quad \text{and} \quad \ell(n(d)) = d.$$

One can associate the following *free system* with the basis  $\mathcal{B}_d$ .

$$S(m, d) : \begin{cases} \dot{x}_i = u_i & i = 1, \dots, m \\ \dot{x}_i = x_{\lambda(i)} \dot{x}_{\rho(i)} & i = m+1, \dots, n(d). \end{cases}$$

The following properties of free systems are well known (see [3]).

**Lemma 1** *Let  $b_i$  denote either the control v.f. of  $S(m, d)$  associated with  $u_i$ , if  $i = 1, \dots, m$ , or  $\text{Ev}_{\{b\}}(B_i)$ , if  $i = m+1, \dots, n(d)$ . Then,*

1. *For any  $i = 1, \dots, n(d)$  and any  $x \in \mathbb{R}^{n(d)}$ ,  $b_i(x) = a_i \partial / \partial x_i + \sum_{j>i} b_{i,j}(x) \partial / \partial x_j$  for some non-zero constant  $a_i$ , so that  $S(m, d)$  satisfies  $LARC(x)$  at any  $x \in \mathbb{R}^{n(d)}$ .*

2. *The v.f.  $b_i$  are  $\Delta$ -homogeneous of degree  $-\ell(i)$  with  $\Delta_\mu$  ( $\mu > 0$ ) the dilation defined by*

$$\Delta_\mu x = (\mu^{\ell(1)} x_1, \dots, \mu^{\ell(n(d))} x_{n(d)}), \quad (9)$$

*so that  $S(m, d)$  is nilpotent of order  $d+1$ .*

## 4.2 Main steps of the proof

We can now proceed with the proof of Theorem 2. It is composed of three steps which are summarized in the following three propositions.

**Proposition 2** *For any  $d \in \mathbb{N} - \{0\}$ ,  $TC(0)$  holds for the free system  $S(m, d)$  with  $\bar{n} = n(d)$ .*

**Proposition 3** *Let  $S_{ap}$  denote any driftless system in  $\mathbb{R}^n$ , which satisfies  $LARC(0)$ , and whose control v.f. are  $\Delta$ -homogeneous of degree  $-1$ . Denote  $d$  the order of nilpotency of  $S_{ap}$ . Then, there exists a polynomial mapping  $p : \mathbb{R}^{n(d)} \rightarrow \mathbb{R}^n$  such that, if  $TC(0)$  holds for  $S(m, d)$  with  $\bar{n} = n(d)$  and a family  $(f_\epsilon)_{\epsilon > 0}$ , then  $TC(0)$  holds for  $S_{ap}$  with  $\bar{n} = n(d)$  and the family  $(\tilde{f}_\epsilon)_{\epsilon > 0}$ , where  $\tilde{f}_\epsilon = p(f_{\epsilon'})$  and  $\epsilon' > 0$  is such that  $\|p(f_{\epsilon'})\| \leq \epsilon$ .*

**Proposition 4** *Let  $S_{ap}$  denote any homogeneous approximation of  $S$  which satisfies  $LARC(0)$  — the existence of which is guaranteed by Proposition 1 —, and  $z$  a system of local coordinates for which  $S$  writes as (5). Then, if  $TC(0)$  holds for  $S_{ap}$  with  $\bar{n}$  and a family  $(\tilde{f}_\epsilon)_{\epsilon > 0}$ ,  $TC(0)$  holds for  $S$  with the same  $\bar{n}$ , and with the family  $(\bar{f}_\epsilon)_{\epsilon > 0}$  where  $\bar{f}_\epsilon = \Delta_{\mu(\epsilon)}^r \tilde{f}_1$  and  $\mu(\epsilon)$  such that  $\mu(\epsilon) \leq \mu_0$  and  $\|\Delta_{\mu(\epsilon)}^r \tilde{f}_1\| \leq \epsilon$ , where  $\mu_0$  is some strictly positive constant.*

These three propositions imply that, for a system  $S$  which satisfies  $LARC(0)$ , the problem of finding a family  $(f_\epsilon)_{\epsilon > 0}$  which satisfies the Transversality Condition is basically solved provided the same problem is solved for the class of free systems — i.e. provided Proposition 2 is proved. The rest of this paper focuses on this latter problem.

## 4.3 Proof of Proposition 2

For free systems we can rewrite the transversality condition as follows.

**Lemma 2** *A function  $f \in C^\infty(\mathbb{T}^{n(d)-m}; \mathbb{R}^{n(d)})$  satisfies Condition (1) for the free system  $S(m, d)$  if and only if*

$$\forall \theta \in \mathbb{T}^{n(d)-m}, \quad \text{Det } M(\theta) \neq 0 \quad (10)$$

where  $M(\theta) = (m_{i,j}(\theta))_{i,j=m+1,\dots,n(d)}$ , and

$$m_{i,j}(\theta) \triangleq \frac{\partial f_i}{\partial \alpha_j}(\theta) - f_{\lambda(i)}(\theta) \frac{\partial f_{\rho(i)}}{\partial \alpha_j}(\theta).$$

We rewrite (10) in the formalism of differential forms:

$$\forall \theta \in \mathbb{T}^{n(d)-m}, \quad (\omega_{m+1} \wedge \dots \wedge \omega_{n(d)})(\theta) \neq 0$$

with  $\omega_i$  the one-form on  $\mathbb{T}^{n(d)-m}$  defined by

$$\omega_i = \mathbf{d}f_i - f_{\lambda(i)} \mathbf{d}f_{\rho(i)}. \quad (11)$$

We show below how to find a function  $f$  which satisfies (10). A family  $(f_\epsilon)_{\epsilon > 0}$  for which  $TC(0)$  holds is then given by  $f_\epsilon = \Delta_{\mu(\epsilon)} f$ .

**Design algorithm:** The function  $f$  is defined by

$$f \triangleq f^{n(d)}, \quad (12)$$

and is obtained via a recursive construction which starts with some function  $f^{m+1}$ . For each  $k = m+1, \dots, n(d)$ , the function  $f^k \in C^\infty(\mathbb{T}^{k-m}; \mathbb{R}^{n(d)})$  is required to verify the following property:

$$\forall \theta^k \triangleq (\theta_{m+1}, \dots, \theta_k) \in \mathbb{T}^{k-m}, \quad (\omega_{m+1}^k \wedge \dots \wedge \omega_k^k)(\theta^k) \neq 0, \quad (13)$$

with  $\omega_i^k$  the one-form on  $\mathbb{T}^{k-m}$

$$\omega_i^k \triangleq \mathbf{d}f_i^k - f_{\lambda(i)}^k \mathbf{d}f_{\rho(i)}^k. \quad (14)$$

The functions  $f^{m+1}, \dots, f^{n(d)}$  are defined below.

$\mathbf{f}^{m+1}$  : A possible choice for  $f^{m+1}$  is:

$$f_i^{m+1}(\theta_{m+1}) = \begin{cases} \sin \theta_{m+1} & \text{if } i = \lambda(m+1) \\ \cos \theta_{m+1} & \text{if } i = \rho(m+1) \\ \frac{1}{4} \sin 2\theta_{m+1} & \text{if } i = m+1 \\ 0 & \text{otherwise.} \end{cases} \quad (15)$$

Indeed, it readily follows from (14) that<sup>1</sup>

$$\forall \theta^{m+1} \in \mathbb{T}, \quad \omega_{m+1}^{m+1}(\theta^{m+1}) = 1/2.$$

$\mathbf{f}^{k-1} \longrightarrow \mathbf{f}^k$  : Assume now that, for some  $k-1 \in \{m+1, \dots, n(d)-1\}$ , a function  $f^{k-1} \in C^\infty(\mathbb{T}^{k-1-m}; \mathbb{R}^{n(d)})$  which verifies (13) for  $k-1$  has been obtained. We show below how to construct from this function a new function  $f^k \in C^\infty(\mathbb{T}^{k-m}; \mathbb{R}^{n(d)})$  which verifies (13).

<sup>1</sup>We implicitly identify  $\alpha(\theta) \approx \theta \in \mathbb{R}$ .

Let  $\Delta_\mu^k$  ( $\mu > 0$ ) denote the dilation defined by

$$\Delta_\mu^k(s, c, x) \triangleq (\mu^{\ell(\lambda(k))} s, \mu^{\ell(\rho(k))} c, \Delta_\mu x), \quad (16)$$

with  $\Delta_\mu$  given by (9). Let  $p_i^k$  ( $i = 1, \dots, n(d)$ ) be the functions defined by

$$p_i^k(s, c) = s \delta_i^{\lambda(k)} + c \delta_i^{\rho(k)} + \frac{m_k^k}{2} s c \delta_i^k \quad (17)$$

with  $\delta_i^j$  the Kronecker delta, and  $m_k^k$  given by

$$m_i^k = \begin{cases} 0 & \text{if } \ell(i) \leq \ell(\lambda(k)) \text{ or } \lambda(i) \neq \lambda(k) \\ 1 + m_{\rho(i)}^k & \text{otherwise.} \end{cases} \quad (18)$$

The construction involves solving the following problem:

**P:** For each  $i = 1, \dots, n(d)$ , find —polynomial— functions  $q_{i,j}^k$ , for

$$j = 1, \dots, j_{i,k} \triangleq \max\{j : \ell(i) - j\ell(\lambda(k)) \geq 0\},$$

$\Delta^k$ -homogeneous of degree  $\ell(i) - j\ell(\lambda(k))$  and such that, if  $i \in \{m+1, \dots, k\}$ ,

$$\begin{aligned} \bar{\omega}_i^k &= (\mathbf{d}x_i - x_{\lambda(i)} \mathbf{d}x_{\rho(i)} + \bar{\gamma}_i^k) \\ &+ \sum_{j=m+1}^{i-1} t_{i,j}(s, x) (\mathbf{d}x_j - x_{\lambda(j)} \mathbf{d}x_{\rho(j)} + \bar{\gamma}_j^k) \end{aligned} \quad (19)$$

where

$$\begin{aligned} \bar{\omega}_i^k &\triangleq \mathbf{d}\bar{f}_i^k - \bar{f}_{\lambda(i)}^k \mathbf{d}\bar{f}_{\rho(i)}^k \\ \bar{f}_i^k(s, c, x) &\triangleq x_i + p_i^k(s, c) + \sum_{j=1}^{j_{i,k}} s^j q_{i,j}^k(x), \end{aligned} \quad (20)$$

the  $t_{i,j}$ 's are smooth functions, and  $\bar{\gamma}_i^k$  is a one-form on  $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n(d)}$ :

$$\bar{\gamma}_i^k = \bar{\gamma}_{i,1}^k \mathbf{d}s + \bar{\gamma}_{i,2}^k \mathbf{d}c$$

with  $\bar{\gamma}_{i,1}^k, \bar{\gamma}_{i,2}^k$ ,  $\Delta^k$ -homogeneous of degree  $\ell(i) - \ell(\lambda(k))$  and  $\ell(i) - \ell(\rho(k))$  respectively, and

$$\begin{cases} \bar{\gamma}_{i,1}^k \equiv 0 & \text{if } i < \lambda(k) \\ \bar{\gamma}_{i,1}^k \equiv 1 & \text{if } i = \lambda(k) \\ \bar{\gamma}_{i,1}^k(s, c, 0) = 0 & \text{if } \lambda(k) < i < k \\ \bar{\gamma}_{i,1}^k(s, c, 0) = \frac{m_k^k}{2} c & \text{for } i = k \end{cases} \quad (21)$$

$$\begin{cases} \bar{\gamma}_{i,2}^k \equiv 0 & \text{if } i < \rho(k) \\ \bar{\gamma}_{i,2}^k \equiv 1 & \text{if } i = \rho(k) \\ \bar{\gamma}_{i,2}^k(s, c, 0) = 0 & \text{if } \rho(k) < i < k \\ \bar{\gamma}_{i,2}^k(s, c, 0) = -\frac{m_k^k}{2} s & \text{for } i = k. \end{cases} \quad (22)$$

Once Problem **P** is solved, the functions  $\bar{f}_i^k$  in (20) are known, and we set

$$\begin{aligned} f^k(\theta^k) &= \bar{f}^k \circ \bar{g}_{\eta_k}^k(\theta^k) \quad \text{with} \\ \bar{g}_{\eta_k}^k(\theta^k) &= \begin{pmatrix} \eta_k^{\ell(\lambda(k))} \sin \theta_k, \\ \eta_k^{\ell(\rho(k))} \cos \theta_k, f^{k-1}(\theta^{k-1}) \end{pmatrix} \end{aligned} \quad (23)$$

It is proved in [4] that

1. There exist functions  $q_{i,j}^k$  which satisfy the conditions stated in **P**.
2. There exists  $\bar{\eta}_k > 0$  such that, for  $\eta_k > \bar{\eta}_k$ , the function  $f^k$  defined by (23) satisfies (13).

Furthermore, one can always choose

$$q_{i,j}^k \equiv 0 \quad \text{for} \quad (24)$$

$$\begin{aligned} i &\in \{1, \dots, \text{Max}\{m, \lambda(k)\}\} \cup \{k+1, \dots, n(d)\} \\ \text{and } j &\in \{1, \dots, j_{i,k}\}, \end{aligned}$$

and

$$\begin{cases} q_{i,1}^k \equiv 0 & \text{if } \lambda(i) < \lambda(k) \\ q_{i,1}^k(f) = m_i^k f_{\rho(i)} & \text{if } \lambda(i) = \lambda(k) \\ \text{for } i &\in \{\text{Max}\{m, \lambda(k)\} + 1, \dots, k\}. \end{cases} \quad (25)$$

#### 4.4 An illustrative example

We illustrate the algorithm described previously on the free system  $S(2, 3)$  on  $\mathbb{R}^5$ , associated with the basis  $\mathcal{B}_3 = \{B_1, \dots, B_5\}$  where

$$\begin{aligned} B_1 &\triangleq X_1, B_2 \triangleq X_2, B_3 \triangleq [B_1, B_2] = [X_1, X_2], \\ B_4 &\triangleq [B_1, B_3], B_5 \triangleq [B_2, B_3]. \end{aligned} \quad (26)$$

From (12), we have to compute  $f = f^{n(d)} = f^5$ , starting from  $f^{m+1} = f^3$ . From (8) and (26),  $\lambda(3) = 1, \rho(3) = 2$ . Therefore, in view of (15),

$$f^3(\theta_3) = (\sin \theta_3, \cos \theta_3, \frac{\sin 2\theta_3}{4}, 0, 0)^T. \quad (27)$$

Let us now compute  $f^4$  from  $f^3$ . We first solve Problem **P**. From (8) and (26),  $\lambda(4) = 1, \rho(4) = 3$ . Then, (17), (18), (20), and (24) give

$$\bar{f}^4 = x + \begin{pmatrix} s \\ 0 \\ c \\ s c \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ s q_{3,1}^4 + s^2 q_{3,2}^4 \\ s q_{4,1}^4 + s^2 q_{4,2}^4 + s^3 q_{4,3}^4 \\ 0 \end{pmatrix} \quad (28)$$

where we have omitted the arguments  $(s, c, x)$  for  $\bar{f}^4$ , and  $x$  for the functions  $q_{i,j}^k$ . From (25)

$$\begin{cases} q_{3,1}^4(x) = m_3^4 x_{\rho(3)} = x_{\rho(3)} = x_2 \\ q_{4,1}^4(x) = m_4^4 x_{\rho(4)} = 2x_{\rho(4)} = 2x_3 \end{cases} \quad (29)$$

Now let us calculate  $\bar{\omega}_3^4$ . Since  $q_{4,3}^4$  is by definition homogeneous of degree  $\ell(3) - 2\ell(1) = 0$ , it is a constant function. A direct calculation gives

$$\bar{\omega}_3^4 = \mathbf{d}x_3 - x_1 \mathbf{d}x_2 + (x_2 + 2sq_{3,2}^4) \mathbf{d}s + \mathbf{d}c$$

With the simple choice

$$q_{3,2}^4 \equiv 0, \quad (30)$$

it follows that (19) is verified with  $\bar{\gamma}_3^4 = x_2 \mathbf{d}s + \mathbf{d}c$ , a one-form which satisfies the conditions in  $\mathbf{P}$ . There remains to determine  $q_{4,2}^4$  and  $q_{4,3}^4$ . Again,  $q_{4,3}^4$  is homogeneous of degree zero and thus, it is a constant function. An easy calculation gives

$$\begin{aligned} \bar{\omega}_4^4 = & \mathbf{d}x_4 - x_1 \mathbf{d}x_3 + s(\mathbf{d}x_3 - x_1 \mathbf{d}x_2 + \bar{\gamma}_3^4) \\ & + s^2(\mathbf{d}q_{4,2}^4 - \mathbf{d}x_2) - (x_1 + s) \mathbf{d}c \\ & + (c + 2x_3 + 2sq_{4,2}^4 + 3s^2q_{4,3}^4 \\ & - x_1x_2 - 2sx_2) \mathbf{d}s \end{aligned}$$

Choosing

$$q_{4,2}^4(x) = x_2, \quad q_{4,3}^4 \equiv 0 \quad (31)$$

allows to rewrite  $\bar{\omega}_4^4$  in the form (19), with

$$\bar{\gamma}_4^4 = (c + 2x_3 - x_1x_2) \mathbf{d}s - (x_1 + s) \mathbf{d}c$$

a one-form which satisfies the conditions in  $\mathbf{P}$ . We finally obtain from (28), (29), (30), and (31),

$$\bar{f}^4(s, c, x) = x + \begin{pmatrix} s \\ 0 \\ c + sx_2 \\ sc + 2sx_3 + s^2x_2 \\ 0 \end{pmatrix} \quad (32)$$

Applying (23) yields the expression of  $f^4$ . As for the parameter  $\eta_4$ , it must be chosen large enough so that (13) is satisfied for  $k = 4$ . By inspection the —conservative— condition  $\eta_4 \geq 5/2$  can be obtained.

The computation of  $f^5$  from  $f^4$  is similar. Solving  $\mathbf{P}$ , we get —details are left to the reader—

$$\bar{f}^5(s, c, x) = x + (0, s, c, 0, sc/2 + sx_3)^T \quad (33)$$

Then, (23) gives the expression of  $f = f^5$ . One easily verifies —from (27), (32), and (33)— that

$$\begin{aligned} f_1^5 &= \sin \theta_3 + \eta_4 \sin \theta_4 \\ f_2^5 &= \cos \theta_3 + \eta_5 \sin \theta_5 \\ f_3^5 &= \frac{1}{4} \sin 2\theta_3 + \eta_4^2 \cos \theta_4 + \eta_4 \sin \theta_4 \cos \theta_3 \\ &\quad + \eta_5^2 \cos \theta_5 \\ f_4^5 &= \frac{\eta_4^3}{2} \sin 2\theta_4 + \frac{\eta_4}{2} \sin \theta_4 \sin 2\theta_3 \\ &\quad + \eta_4^2 \sin^2 \theta_4 \cos \theta_3 \\ f_5^5 &= \frac{\eta_5^3}{4} \sin 2\theta_5 + \eta_5 \sin \theta_5 (f_3^5 - \eta_5^2 \cos \theta_5) \end{aligned}$$

There remains to determine specific values for  $\eta_5$ . Here the analysis gets more involved. In order to give an example, let us only mention that for  $\eta_4 = 3$ , the condition  $\eta_5 \geq 7$  —obtained from simulations— seems sufficient to guarantee (13).

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