Notions of Observability for Uncertain Linear Systems with Structured Uncertainty[1]

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Abstract

This paper introduces a notion of observability for a class of uncertain linear systems with structured uncertainty described by averaged integral quadratic constraints. The paper presents an algorithm for finding the robust observability function and corresponding unobservable manifold.

1 Introduction

The notion of observability is one of the fundamental properties of a linear system; e.g., see [1]. The aim of this paper is to introduce a notion of observability for uncertain systems which will provide insight into the structure of uncertain systems and the limitations on achievable performance which arise. The notion of observability introduced in this paper involves extending the definition of the observability Gramian to the case of uncertain systems; see also [2].

As in the papers [3, 4], the uncertain systems considered in this paper will use an averaged integral quadratic constraint (IQC) uncertainty description. This uncertainty description allows us to exploit a certain S-procedure theorem in order to calculate the observability function in terms of a certain parameter dependent optimal control problem.

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2 Problem Formulation

We consider the following time-varying uncertain system defined on the finite time interval $[0, T]$:

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t) + \sum_{s=1}^{k} B_s(t)\xi_s(t); \\
y(t) &= C(t)x(t) + \sum_{s=1}^{k} D_s(t)\xi_s(t); \\
z_1(t) &= K_1(t)x(t); \\
&\vdots \\
z_k(t) &= K_k(t)x(t) \qquad (1)
\end{aligned}$$

where $x \in \mathbf{R}^n$ is the *state*, $y \in \mathbf{R}^l$ is the *measured output*, $z_1 \in \mathbf{R}^{h_1}$, $z_2 \in \mathbf{R}^{h_2}, \ldots, z_k \in \mathbf{R}^{h_k}$ are the *uncertainty outputs*, $\xi_1 \in \mathbf{R}^{r_1}$, $\xi_2 \in \mathbf{R}^{r_2}, \ldots, \xi_k \in \mathbf{R}^{r_k}$ are the *uncertainty inputs*, and $A(\cdot)$, $B_1(\cdot)$, ..., $B_k(\cdot)$, $C(\cdot)$, $K_1(\cdot)$, $K_2(\cdot)$, ..., $K_k(\cdot)$ are bounded piecewise continuous matrix functions defined on $[0, T]$.

System Uncertainty The uncertainty inputs and outputs may be collected together into two vectors. That is, we define

$$\xi(t) \triangleq [\xi_1(t)' \; \xi_2(t)' \; \ldots \; \xi_k(t)' \;]'$$

and

$$z(t) \triangleq [z_1(t)' \; z_2(t)' \; \ldots \; z_k(t)' \;]'.$$

The uncertainty is required to satisfy a certain "Averaged Integral Quadratic Constraint".

Averaged Integral Quadratic Constraint Let $d_1 > 0, d_2 > 0, \ldots, d_k > 0$

be given positive constants associated with the system (1). We will consider sequences of uncertainty inputs $\mathcal{S} = \{\xi^1(\cdot), \xi^2(\cdot), \dots \xi^q(\cdot)\}$. The number of elements q in any such sequence is arbitrary. A sequence of uncertainty functions of the form $\mathcal{S} = \{\xi^1(\cdot), \xi^2(\cdot), \dots \xi^q(\cdot)\}$ is an *admissible uncertainty sequence* for the system (1) if the following conditions hold: Given any $\xi^i(\cdot) \in \mathcal{S}$ and any corresponding solution $\{x^i(\cdot), \xi^i(\cdot)\}$ to (1) defined on $[0,T]$, then $\xi^i(\cdot) \in \mathbf{L}_2[0,T]$, and

$$\begin{aligned} \frac{1}{q}\sum_{i=1}^{q}\int_0^T \left(\|\xi_1^i(t)\|^2 - \|z_1^i(t)\|^2\right) dt &\leq d_1; \\ &\vdots \\ \frac{1}{q}\sum_{i=1}^{q}\int_0^T \left(\|\xi_k^i(t)\|^2 - \|z_k^i(t)\|^2\right) dt &\leq d_k. \end{aligned} \tag{2}$$

The class of all such admissible uncertainty sequences is denoted $\mathbf{\Xi}$.

Definition 1 *The* robust observability function *for the uncertain system (1), (2) is defined as*

$$L_o(x_0) \triangleq \inf_{\mathcal{S}\in\Xi} \frac{1}{q}\sum_{i=1}^{q}\int_0^T \|y(t)\|^2 dt \tag{3}$$

where $x(0) = x_0$ in (1).

This definition extends the standard definition of the observability Gramian for linear systems.

Definition 2 *A state $x_0 \in \mathbf{R}^n$ is said to be* unobservable *for the uncertain system (1), (2) if*

$$L_o(x_0) = 0$$

for all constants $d_1 > 0$, $d_2 > 0$, ... $d_k > 0$ in (2). The set of all unobservable states for the uncertain system (1), (2) is referred to as the unobservable manifold *$\mathcal{U}$; i.e.,*

$$\mathcal{U} \triangleq \left\{ \begin{array}{l} x \in \mathbf{R}^n : L_o(x) = 0 \\ \forall d_1 > 0, d_2 > 0, \dots, d_k > 0 \end{array} \right\}.$$

Also, the uncertain system (1), (2) is said to be robustly observable *if $\mathcal{U} = \{0\}$; i.e., the origin is the only unobservable state.*

3 The Main Result

3.1 An unconstrained optimization problem

For the uncertain system (1), (2), we define a function $V_\tau(x_0)$ as follows:

$$V_\tau(x_0) \triangleq \inf_{\xi(\cdot)\in\mathbf{L}_2[0,T]} \int_0^T \left(\begin{array}{l} \|y\|^2 + \\ \sum_{s=1}^k \tau_s \|\xi_s\|^2 - \\ \sum_{s=1}^k \tau_s \|z_s\|^2 \end{array} \right) dt. \tag{4}$$

Here $\tau_1 \geq 0$, $\tau_2 \geq 0$, ... , $\tau_k \geq 0$ are given constants.

In order to calculate $V_\tau(x_0)$, we first introduce some notation. Given $\tau = [\tau_1\ \tau_2\ \dots\ \tau_k]$, let

$$\begin{aligned} B(t) &= [B_1(t)\ B_2(t)\ \dots\ B_k(t)]; \\ D(t) &= [D_1(t)\ D_2(t)\ \dots\ D_k(t)]; \\ K_\tau(t) &= \sum_{s=1}^k \tau_s K_s(t)' K_s(t); \\ \Lambda_\tau &= \left[\begin{array}{ccc} \tau_1 I & & 0 \\ & \ddots & \\ 0 & & \tau_k I \end{array} \right]. \end{aligned}$$

Using this notation, it follows that the system (1) can be re-written as

$$\dot{x}(t) = A(t)x(t) + B(t)\xi(t); \quad x(0) = x_0. \tag{5}$$

Also, the function $V_\tau(x_0)$ can be re-written as

$$V_\tau(x_0) = \inf_{\xi(\cdot)\in\mathbf{L}_2[0,T]} J_\tau(\xi(\cdot)) \tag{6}$$

where

$$J_\tau(\xi(\cdot)) = \int_0^T \left(\begin{array}{l} x'[C'C - K_\tau]x \\ +2x'C'D\xi + \xi'\Lambda_\tau\xi \end{array} \right) dt. \tag{7}$$

If $\tau = [\tau_1\ \tau_2\ \dots\ \tau_k]$ is such that $\tau_1 > 0$, $\tau_2 > 0$, ... , $\tau_k > 0$, then the optimization problem (6) can be solved in terms of the following Riccati differential equation:

$$\begin{aligned} -\dot{P} &= A'P + PA \\ &\quad -(PB + C'D)\Lambda_\tau^{-1}(D'C + B'P) \\ &\quad +C'C - K_\tau; P(T) = 0. \end{aligned} \tag{8}$$

Lemma 1 *Let $\tau = [\tau_1 \ \tau_2 \ \ldots \ \tau_k]$ be given such that $\tau_1 > 0$, $\tau_2 > 0$, ... , $\tau_k > 0$ and consider the corresponding system (5) and cost functional (7). Then*

$$V_\tau(x_0) > -\infty$$

if and only if the Riccati differential equation (8) has a solution $P_\tau(t)$ defined on $[0, T]$. In this case,

$$V_\tau(x_0) = x_0' P_\tau(0) x_0. \qquad (9)$$

Proof This lemma follows directly from a standard result on the linear quadratic regulator problem; e.g., see page 55 of [5]. □

3.2 An S-procedure result

Consider a set of functionals

$$F_0(\xi(\cdot)),\ F_1(\xi(\cdot)), \ldots, F_k(\xi(\cdot))$$

defined for the system (5).

Lemma 2 *Suppose that for any sequence of input functions $\{\xi^1(\cdot),\ \xi^2(\cdot), \ldots, \xi^q(\cdot)\}$ such that $\xi^i(\cdot) \in \mathbf{L}_2[0, T]$ for all i and*

$$\begin{aligned} \sum_{i=1}^{q} F_1(\xi^i(\cdot)) &\geq 0; \\ &\vdots \\ \sum_{i=1}^{q} F_k(\xi^i(\cdot)) &\geq 0; \end{aligned} \qquad (10)$$

we have

$$\sum_{i=1}^{q} F_0(\xi^i(\cdot)) \geq 0. \qquad (11)$$

Then there exist constants $\tau_0 \geq 0$, ... ,$\tau_k \geq 0$ such that $\sum_{i=0}^{k} \tau_i > 0$ and

$$\tau_0 F_0(\xi(\cdot)) \geq \sum_{i=0}^{k} \tau_i F_k(\xi(\cdot)) \qquad (12)$$

for all inputs $\xi(\cdot) \in \mathbf{L}_2[0, T]$.

Proof We first define a subset of $\mathbf{R}^{k+1}$

$$\mathcal{P} \triangleq \left\{ \begin{array}{l} [F_0(\xi(\cdot)), F_1(\xi(\cdot)), \ldots, F_k(\xi(\cdot))]' : \\ \xi(\cdot) \in \mathbf{L}_2[0, T] \end{array} \right\}.$$

Then condition (10), (11) implies that this set satisfies the assumptions of Theorem 3.1 of [6]. From this theorem, (12) follows. □

Observation 1 *If there exists an input $\xi(\cdot) \in \mathbf{L}_2[0, T]$ such that $F_1(\xi(\cdot)) > 0$, $F_2(\xi(\cdot)) > 0$, ..., $F_k(\xi(\cdot)) > 0$ and the assumptions of the above lemma hold, then τ_0 may be chosen as $\tau_0 = 1$ in (12); see Observation 3.1 in [6].*

3.3 A formula for the robust observability function

We first introduce the following notation:

$$\Gamma \triangleq \left\{ \begin{array}{l} \tau = [\tau_1 \ \ldots \ \tau_k] : \tau_1 > 0 \ \ldots \ \tau_k > 0 \\ \text{and } V_\tau(x_0) > -\infty \end{array} \right\}.$$

Also

$$\bar{\Gamma} \triangleq \left\{ \begin{array}{l} \tau = [\tau_1 \ \ldots \ \tau_k] : \tau_1 \geq 0 \ \ldots \ \tau_k \geq 0 \\ \text{and } V_\tau(x_0) > -\infty \end{array} \right\}.$$

Theorem 1 *Consider the uncertain system (1), (2) and corresponding robust observability function (3). Then for any initial condition $x(0) = x_0$,*

$$L_o(x_0) = \max_{\tau \in \bar{\Gamma}} \left\{ V_\tau(x_0) - \sum_{s=1}^{k} \tau_s d_s \right\}. \qquad (13)$$

Proof Given any admissible uncertainty input sequence $\mathcal{S} \in \Xi$ for the uncertain system (1), (2) with initial condition $x(0) = x_0$ and vector $\tau \in \bar{\Gamma}$, we claim

$$\frac{1}{q} \sum_{i=1}^{q} \int_0^T \|y^i(t)\|^2 dt \geq V_\tau(x_0) - \sum_{s=1}^{k} \tau_s d_s. \qquad (14)$$

To establish this claim, we first note that it follows from the definition of $V_\tau(x_0)$ (4) that

$$\int_0^T \left(\|y\|^2 + \sum_{s=1}^{k} \tau_s \|\xi_s\|^2 - \sum_{s=1}^{k} \tau_s \|z_s\|^2 \right) dt \geq V_\tau(x_0)$$

for all $\xi(\cdot) \in \mathbf{L}_2[0, T]$. In particular, this inequality holds for every element in the given sequence $\mathcal{S}$. Hence,

$$\begin{aligned} &\frac{1}{q} \sum_{i=1}^{q} \int_0^T \left(\begin{array}{l} \|y^i\|^2 + \sum_{s=1}^{k} \tau_s \|\xi_s^i\|^2 \\ - \sum_{s=1}^{k} \tau_s \|z_s^i\|^2 \end{array} \right) dt \\ &\geq \frac{1}{q} \sum_{i=1}^{q} V_\tau(x_0) \\ &= V_\tau(x_0). \end{aligned} \qquad (15)$$

However, $\mathcal{S} \in \Xi$ implies that (2) is satisfied and hence from (15), we obtain

$$\frac{1}{q}\sum_{i=1}^{q}\int_0^T \|y^i(t)\|^2 dt + \sum_{s=1}^{k}\tau_s d_s \geq V_\tau(x_0).$$

Thus, (14) holds.

Now since (14) holds for any $\mathcal{S} \in \Xi$, we have

$$\inf_{\mathcal{S}\in\Xi}\frac{1}{q}\sum_{i=1}^{q}\int_0^T \|y^i\|^2 dt \geq V_\tau(x_0) - \sum_{s=1}^{k}\tau_s d_s \tag{16}$$

for all $\tau \in \bar{\Gamma}$. We now claim there exists a $\tau \in \bar{\Gamma}$ such that

$$\inf_{\mathcal{S}\in\Xi}\frac{1}{q}\sum_{i=1}^{q}\int_0^T \|y^i\|^2 dt \geq V_\tau(x_0) - \sum_{s=1}^{k}\tau_s d_s. \tag{17}$$

To establish this claim, we let

$$c \triangleq \inf_{\mathcal{S}\in\Xi}\frac{1}{q}\sum_{i=1}^{q}\int_0^T \|y^i\|^2 dt. \tag{18}$$

Also, we define the functionals in Lemma 1 as follows:

$$\begin{aligned}
F_0(\xi(\cdot)) &\triangleq \int_0^T \|y\|^2 dt - c;\\
F_1(\xi(\cdot)) &\triangleq \int_0^T \left(\|\xi_1\|^2 - \|z_1\|^2\right) dt + d_1;\\
F_2(\xi(\cdot)) &\triangleq \int_0^T \left(\|\xi_2\|^2 - \|z_2\|^2\right) dt + d_2;\\
&\vdots\\
F_k(\xi(\cdot)) &\triangleq \int_0^T \left(\|\xi_k\|^2 - \|z_k\|^2\right) dt + d_k.
\end{aligned}$$

Now for any uncertainty input sequence $\mathcal{S}$ such that

$$\frac{1}{q}\sum_{i=1}^{q}F_1(\xi^i(\cdot)) \geq 0, \ \ldots, \frac{1}{q}\sum_{i=1}^{q}F_k(\xi^i(\cdot)) \geq 0,$$

the averaged IQCs (2) are satisfied and hence $\mathcal{S} \in \Xi$. Then, it follows from (18) that

$$\frac{1}{q}\sum_{i=1}^{q}F_0(\xi^i(\cdot)) \geq 0.$$

Thus, the conditions of the S-procedure result, Lemma 2 are satisfied. Also note that since $d_1 > 0,\ d_2 > 0, \ldots, d_k > 0$, then $F_1(0) > 0,\ F_2(0) > 0, \ldots, F_k(0) > 0$. Thus, it follows from Lemma 2 and Observation 1 that there exist constants $\tau_1 \geq 0$, ... ,$\tau_k \geq 0$ such that

$$F_0(\xi(\cdot)) \geq \sum_{s=1}^{k}\tau_s F_s(\xi(\cdot))$$

for all $\xi(\cdot) \in \mathbf{L}_2[0,T]$. That is

$$\int_0^T \|y\|^2 dt - c \geq \sum_{s=1}^{k}\tau_s\left[\int_0^T \left(\|\xi_s\|^2 - \|z_s\|^2\right) dt + d_s\right]$$

for all $\xi(\cdot) \in \mathbf{L}_2[0,T]$. Hence,

$$\inf_{\xi(\cdot)\in\mathbf{L}_2[0,T]}\int_0^T \begin{pmatrix} \|y\|^2 + \sum_{s=1}^{k}\tau_s\|\xi_s\|^2 \\ -\sum_{s=1}^{k}\tau_s\|z_s\|^2 \end{pmatrix} dt \geq c + \sum_{s=1}^{k}\tau_s d_s.$$

Then using (4) and (18), we have

$$\begin{aligned}
V_\tau(x_0) &\geq \inf_{\mathcal{S}\in\Xi}\frac{1}{q}\sum_{i=1}^{q}\int_0^T \|y\|^2 dt + \sum_{s=1}^{k}\tau_s d_s\\
&\geq 0.
\end{aligned}$$

That is (17) is satisfied. Furthermore, since $V_\tau(x_0) \geq 0$, then $\tau = [\tau_1\ \tau_2\ \ldots\ \tau_k]' \in \bar{\Gamma}$. Combining (16) and (17) now leads to (13). This completes the proof of the theorem. □

Corollary 1 *Consider the uncertain system (1), (2) and corresponding robust observability function (3). Then for any initial condition* $x(0) = x_0$,

$$L_o(x_0) = \sup_{\tau\in\Gamma}\left\{x_0' P_\tau(0)x_0 - \sum_{s=1}^{k}\tau_s d_s\right\}. \tag{19}$$

Proof It is straightforward to verify that

$$\max_{\tau\in\bar{\Gamma}}\left\{V_\tau(x_0) - \sum_{s=1}^{k}\tau_s d_s\right\} = \sup_{\tau\in\Gamma}\left\{V_\tau(x_0) - \sum_{s=1}^{k}\tau_s d_s\right\}.$$

Hence, using Lemma 1, (19) follows. □

Corollary 2 *Consider the uncertain system (1), (2). Then a state $x_0 \in \mathbf{R}^n$ is unobservable if and only if*

$$x_0' P_\tau(0) x_0 \leq 0$$

for all $\tau \in \Gamma$.

Proof If x_0 is unobservable, then $L_o(x_0) = 0$ for all $d_1 > 0,\ d_2 > 0, \dots, d_k$. Hence, using Corollary 1

$$x_0' P_\tau(0) x_0 - \sum_{s=1}^{k} \tau_s d_s \leq 0$$

for all $d_1 > 0,\ d_2 > 0, \dots, d_k > 0$ and all $\tau \in \Gamma$. Thus

$$x_0' P_\tau(0) x_0 \leq 0$$

for all $\tau \in \Gamma$.

Conversely, if

$$x_0' P_\tau(0) x_0 \leq 0$$

for all $\tau \in \Gamma$, then

$$x_0' P_\tau(0) x_0 - \sum_{s=1}^{k} \tau_s d_s \leq 0$$

for all $d_1 > 0,\ d_2 > 0, \dots, d_k > 0$ and all $\tau \in \Gamma$. Thus, using Corollary 1

$$L_o(x_0) = \sup_{\tau \in \Gamma} \left\{ x_0' P_\tau(0) x_0 - \sum_{s=1}^{k} \tau_s d_s \right\} \leq 0$$

for all $d_1 > 0,\ d_2 > 0, \dots, d_k > 0$. However, if follows from the definition of the observability function that $L_o(x_0) \geq 0$. Thus

$$L_o(x_0) = 0$$

for all $d_1 > 0,\ d_2 > 0, \dots, d_k$. That is x_0 is unobservable. □

Observation 2 *From the above corollary, it follows immediately that the unobservable manifold $\mathcal{U}$ can be written in the form:*

$$\mathcal{U} = \{x \in \mathbf{R}^n : x_0' P_\tau(0) x_0 \leq 0 \ \ \forall \tau \in \Gamma\}.$$

Also, it follows that the uncertain system (1), (2) is robustly observable if and only if for all $x_0 \in \mathbf{R}^n : x_0 \neq 0$, there exists a $\tau \in \Gamma$ such that

$$x_0' P_\tau(0) x_0 > 0.$$

4 An Alternative Definition of Robust Observability

In this section, we compare the notion of robust observability defined above with the notion of robust observability considered in the paper [4].

The uncertain system considered in [4] can be considered as a special case of the uncertain system (1), (2) where $k = 1$,

$$\begin{aligned} \xi_1(t) &= \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}; \\ A(t) &= A(t); \quad B_1(t) = [B_1(t)\ 0]; \\ K_1(t) &= K(t); \quad C(t) = C(t); \\ D_1(t) &= [0\ I]. \end{aligned} \tag{20}$$

Also, the uncertainty is assumed to satisfy the IQC

$$\int_0^T \left(\|w(t)\|^2 + \|v(t)\|^2 - \|z(t)\|^2 \right) dt \leq d_1. \tag{21}$$

This IQC can be considered as a special case of the averaged IQCs (2) when we restrict attention to uncertainty input sequences of length one. Also, note that we have assumed that in [4], $R(t) \equiv I$, $Q \equiv I$.

According to the definition given in [4], an uncertain system is robustly observable if, the set of all possible states at time $t = 0$ consistent with the uncertain system model and given output measurements on $[0,\ T]$, is bounded (for all $d_1 > 0$).

Consider the Riccati differential equation:

$$\begin{aligned} -\dot{Y} &= YA + A'Y - YB_1B_1'Y \\ &\quad -K'K + C'C; \quad Y(T) = 0. \end{aligned} \tag{22}$$

It was shown in [4] that the uncertain system under consideration is robustly observable (in the sense of [4]) if and only if (22) has a solution on $[0,\ T]$ and $Y(0) > 0$.

In order to compare our definition of robust observability with the definition given in [4], we first observe that the Riccati differential equation (22) can be given an optimal control interpretation. Indeed, if we consider the system (1), (20) with initial condition $x(0) = x_0$ and the Riccati differential equation (22), then

$$x_0'Y(0)x_0 = \inf_{w(\cdot)\in \mathbf{L}_2[0,T]} \int_0^T \left(\begin{array}{l} \|Cx\|^2 - \|z_1\|^2 \\ +\|w\|^2 \end{array} \right) dt. \tag{23}$$

Now consider the quantity (4) for the uncertain system (1), (20), (21). This quantity can be calculated as follows:

$$\begin{aligned} V_\tau(x_0) &= \inf_{\xi(\cdot)\in \mathbf{L}_2[0,T]} \int_0^T \left(\begin{array}{l} \|y\|^2 - \tau\|z\|^2 \\ +\tau\|w\|^2 \\ +\tau\|v\|^2 \end{array} \right) dt \\ &= \inf_{w(\cdot),v(\cdot)} \int_0^T \left(\begin{array}{l} \|Cx+v\|^2 \\ -\tau\|z\|^2 \\ +\tau\|w\|^2 \\ +\tau\|v\|^2 \end{array} \right) dt. \end{aligned}$$

However,

$$\begin{aligned} &\inf_{v(\cdot)\in \mathbf{L}_2[0,T]} \int_0^T \left(\begin{array}{l} x'C'Cx + 2x'Cv \\ +v'v + \tau\|v\|^2 \end{array} \right) dt = \\ &\inf_{v(\cdot)\in \mathbf{L}_2[0,T]} \int_0^T \left[\begin{array}{l} \left[\begin{array}{c} \frac{1}{\sqrt{1+\tau}}Cx+ \\ \sqrt{1+\tau v} \end{array} \right]' \\ \times \left[\begin{array}{c} \frac{1}{\sqrt{1+\tau}}Cx+ \\ \sqrt{1+\tau v} \end{array} \right] \\ -\frac{1}{1+\tau}x'C'Cx \\ +x'C'Cx \end{array} \right] dt \\ &= \int_0^T \frac{\tau}{1+\tau} x'C'Cx dt. \end{aligned}$$

Thus,

$$V_\tau(x_0) = \tau \inf_{w(\cdot)\in \mathbf{L}_2[0,T]} \int_0^T \left(\begin{array}{l} \frac{1}{1+\tau}x'C'Cx \\ -\|z\|^2 + \|w\|^2 \end{array} \right) dt.$$

Now observe that for any $\tau > 0$, then $V_\tau(x_0) \leq 0$ if and only if $\bar{V}_\tau(x_0) \leq 0$ where

$$\bar{V}_\tau(x_0) = \inf_{w(\cdot)\in \mathbf{L}_2[0,T]} \int_0^T \left(\begin{array}{l} \frac{1}{1+\tau}x'C'Cx \\ -\|z\|^2 + \|w\|^2 \end{array} \right) dt. \tag{24}$$

Also note that it follows from (24) that for any $x_0 \in \mathbf{R}^n$, $\bar{V}_\tau(x_0)$ is monotone increasing as $\tau \to 0$. Hence, it follows from Theorem 1 that a state $x_0 \in \mathbf{R}^n$ is unobservable for the uncertain system (1), (20), (21) if and only if

$$\inf_{w(\cdot)\in \mathbf{L}_2[0,T]} \int_0^T \left(\begin{array}{l} x'C'Cx - \|z\|^2 \\ +\|w\|^2 \end{array} \right) dt \leq 0.$$

Therefore, it follows from (23) that the system has no non-zero unobservable state if and only if

$$Y(0) > 0.$$

That is, in this case, our robust observability condition is equivalent to the robust observability condition of [4] for uncertain systems of the form (1), (20), (21).

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