NEW PERTURBATION BOUNDS FOR SYLVESTER EQUATIONS

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Abstract

The sensitivity of Sylvester matrix equations relative to perturbations in the coefficient matrices is studied. New local perturbations bounds are obtained.

Keywords. Perturbation analysis, Sylvester equations.

1 Introduction

In this paper we study the sensitivity of Sylvester matrix equations (SME) arising in linear systems theory. A new local perturbation bound for SME is obtained, which is a non-linear, first order homogeneous and tighter than the local bounds based on condition numbers [1] - [5].

The following notations are used later on: $\mathcal{R}^{m\times n}$ – the space of real $m\times n$ matrices; I_n – the unit $n\times n$ matrix; $A^\top=[a_{ji}]$ – the transpose of the matrix $A=[a_{ij}]$; $\mathrm{vec}(A)\in\mathcal{R}^{mn}$ – the column-wise vector representation of the matrix $A\in\mathcal{R}^{m\times n}$; $A\otimes B=[a_{ij}B]$ – the Kronecker product of the matrices A and B; $\|\cdot\|_2$ – the spectral (or 2-) norm in $\mathcal{R}^{m\times n}$; $\|.\|_{\mathrm{F}}$ – the Frobenius (or F-) norm in $\mathcal{R}^{m\times n}$. The notation ':=' stands for 'equal by definition'.

2 Problem Statement

Consider the SME

$$F(X,P):=AX+XB+Q=0 \tag{1}$$

where $X\in\mathcal{R}^{n\times m}$ is the unknown matrix, $A\in\mathcal{R}^{n\times n}$, $B\in\mathcal{R}^{m\times m}$ and $Q\in\mathcal{R}^{n\times m}$ are given matrices and $P:=(A,B,Q)$. We shall suppose that $0\notin\left\{\lambda_i(A)+\lambda_k(B):i\in\overline{1,n},\ k\in\overline{1,m}\right\}$, where $\lambda_i(A)$ and $\lambda_k(B)$ are the eigenvalues of the matrices A and B, respectively. Under this assumption the partial Fréchet derivative F_X of F in X is invertible and (1) has a unique solution X.

Let the matrices from P be perturbed as

$$A\mapsto A+\Delta A,\ B\mapsto B+\Delta B,\ Q\mapsto Q+\Delta Q$$

and denote by $P+\Delta P$ the perturbed collection P, in which each matrix $Z\in P$ is replaced by $Z+\Delta Z$. Then the perturbed equation is

$$F(Y,P+\Delta P)=0. \tag{2}$$

Since the operator F_X is invertible, the perturbed equation (2) has an unique solution $Y=X+\Delta X$ in the neighborhood of X if the perturbation ΔP is sufficiently small. Denote by

$$\Delta:=[\Delta_A,\Delta_B,\Delta_Q]^\top\in\mathcal{R}^3_+$$

the vector of absolute norm perturbations $\Delta_Z:=\|\Delta Z\|_{\mathrm{F}}$ in the data matrices.

The perturbation problem considered is to find a bound

$$\Delta_X\le f(\Delta)+O(\|\Delta\|^2),\ \Delta\to 0,$$

for the perturbation $\Delta_X:=\|\Delta X\|_{\mathrm{F}}$, where f is a non-linear, first order homogeneous function in Δ.

3 Local perturbation analysis

Since $F(X,P)=0$, the perturbed equation (2) may be written as

$$\begin{aligned}&F(X+\Delta X,P+\Delta P):=\\&F_X(\Delta X)+F_A(\Delta A)+F_B(\Delta B)+F_Q(\Delta Q)\\&+G(\Delta X,\Delta P)=0\end{aligned}$$

where

$$\begin{aligned}&F_X(Z)=AZ+ZB,\ F_A(Z)=ZX,\\&F_B(Z)=XZ,\ F_Q(Z)=Z\end{aligned}$$

are the partial Fréchet derivatives of F in the corresponding matrix arguments, computed at the point (X,P), and $G(\Delta X,\Delta P)$ contains second and higher order terms in ΔX, ΔP.

Since the operator $F_X(.)$ is invertible we get

$$\begin{aligned}\Delta X &= \Phi(\Delta X,\Delta P)\\ &:= -F_X^{-1}\circ F_A(\Delta A) - F_X^{-1}\circ F_B(\Delta B) \quad (3)\\ &\quad - F_X^{-1}(\Delta Q) - F_X^{-1}(G(\Delta X,\Delta P)).\end{aligned}$$

The operator equation (3) may be written in a vector form as

$$\begin{aligned}\mathrm{vec}(\Delta X) &= N_1\mathrm{vec}(\Delta A) + N_2\mathrm{vec}(\Delta B) \quad (4)\\ &\quad + N_3\mathrm{vec}(\Delta Q) - M_X^{-1}\mathrm{vec}(G(\Delta X,\Delta P))\end{aligned}$$

where

$$N_1 := -M_X^{-1}M_A,\ N_2 := -M_X^{-1}M_B,$$
$$N_3 := -M_X^{-1}$$

and

$$M_X = I_m\otimes A + B^\top\otimes I_n,\ \ M_A = X^\top\otimes I_n,$$
$$M_B = I_m\otimes X$$

are the matrix representations of the operators $F_X(\cdot)$, $F_A(\cdot)$ and $F_B(\cdot)$.

The well known, condition number based local perturbation bounds for the Sylvester equation [1] - [4] are a corollary of (4) :

$$\begin{aligned}\Delta_X &= \|\Delta X\|_{\mathrm{F}} = \|\mathrm{vec}(\Delta X)\|_2\\ &\le \mathrm{est}_1(\Delta,N) + O(\|\Delta\|^2)\\ &:= \|N_1\|_2\Delta_A + \|N_2\|_2\Delta_B + \|N_3\|_2\Delta_Q\\ &\quad + O(\|\Delta\|^2)\\ &= K_A\Delta_A + K_B\Delta_B + K_Q\Delta_Q\\ &\quad + O(\|\Delta\|^2), \quad \Delta\to 0\end{aligned}$$

where $N := [N_1,N_2,N_3]$ and

$$K_A = \left\|M_X^{-1}M_A\right\|_2,\ K_B = \left\|M_X^{-1}M_B\right\|_2,$$
$$K_Q = \left\|M_X^{-1}\right\|_2$$

are the absolute condition numbers of (1) relative to the perturbations in A,B and Q, respectively.

However, the local estimates, based on condition numbers, may eventually produce pessimistic results. At the same time it is possible to derive local, first order homogeneous estimates, which are tighter in general.

Relation (4) also gives

$$\begin{aligned}\Delta_X &\le \mathrm{est}_2(\Delta,N) + O(\|\Delta\|^2)\\ &:= \|N\|_2\|\Delta\|_2 + O(\|\Delta\|^2),\ \ \Delta\to 0.\end{aligned}$$

The bounds $\mathrm{est}_1(\Delta,N)$ and $\mathrm{est}_2(\Delta,N)$ are alternative, i.e. which one is less depends on the particular value of Δ.

There is also a third bound, which is always less than or equal to $\mathrm{est}_1(\Delta,N)$. We have

$$\begin{aligned}\Delta_X &\le \mathrm{est}_3(\Delta,N) + O(\|\Delta\|^2)\\ &:= \sqrt{\Delta^\top S(N)\Delta} + O(\|\Delta\|^2),\ \ \Delta\to 0\end{aligned}$$

where $S(N)$ is the 3×3 matrix with elements

$$s_{ij}(N) = \|N_i^\top N_j\|_2.$$

Since

$$\left\|N_i^\top N_j\right\|_2 \le \|N_i\|_2\|N_j\|_2$$

we get

$$\mathrm{est}_3(\Delta,N) \le \mathrm{est}_1(\Delta,N).$$

Hence we have the overall estimate

$$\Delta_X \le \mathrm{est}(\Delta,N) + O(\|\Delta\|^2),\ \ \Delta\to 0 \qquad (5)$$

where

$$\mathrm{est}(\Delta,N) := \min\{\mathrm{est}_2(\Delta,N),\mathrm{est}_3(\Delta,N)\}. \qquad (6)$$

The local bound *est* in (5), (6) is a non-linear, first order homogeneous and piece-wise real analytic function in Δ. It may be shown that *est* is an asymptotically sharp bound.

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