A Lyapunov approach to incremental stability

David Angeli *
Dip. di Sistemi e Informatica
Università di Firenze
Via di S. Marta 3, 50139 Firenze, Italy
Email: `angeli@dsi.unifi.it`

Abstract. This paper deals with several notions of incremental stability. In other words, we focus on stability of trajectories with respect to one another, rather than with respect some attractor or equilibrium point. The aim is to present a framework for understanding such questions fully compatible with the well-known Input-to-State Stability approach.

1 Introduction

Input-to-State stability (ISS for short) has proven a valid instrument in order to study questions of robust stability for finite-dimensional nonlinear systems. One reason for that is the possibility of dealing at the same time with a body of theory which nicely extends the classic Lyapunov approach to non-autonomous systems, while still allowing for input-output descriptions of the system behavior [15]. In this way tools such as small-gain theorems and Lyapunov dissipation inequalities [1, 13] have come together in a unified framework which bridges the gap between the state-space and input-output approaches.

Stability properties are described through the use of comparison functions, the so called class $\mathcal{K}$ and $\mathcal{KL}$ functions, which can be thought of as nonlinear versions of linear gains and exponentially fading transients. This results in notions which are invariant with respect to nonlinear changes of coordinates and at the same times avoids the use of the $\varepsilon - \delta$ formalism which is usually less intuitive. This same way of thinking was exploited in order to study detectability questions, [14, 7]. The quest for nonlinear analogues of the separation principle already involved input-to-state stability as one of the ingredients, [11, 17]. As a matter of fact, it is especially looking at the issue of state-detection and observer synthesis that it becomes relevant to understand which systems may enjoy incremental stability properties, viz. systems whose trajectories converge to one another, besides being attracted towards some equilibrium position. Works along these lines have recently appeared in the literature, [12, 2], together with some examples of applications, [9, 4]. As a matter of fact, the notion of incremental input-to-state stability that will be introduced, can be thought of also as as "open-loop observability" property, that is as the possibility of designing an observer for the system which only processes past input data. It is well-known that for linear systems such a property is equivalent to asymptotic stability. It is indeed a much stronger property when dealing with nonlinear ones.

As already pointed out, our aim is to present such notions in a framework compatible with the ISS approach. For reasons of space all the results are stated without proof which we defer to a forthcoming paper.

2 Lyapunov characterizations of Incremental Stability

Let us consider dynamical systems of the following form:

$$\dot{x} = f(x, d) \tag{1}$$

with state $x \in \mathbb{R}^n$ and input d, here seen as a disturbance rather than a control, taking values on a closed set $\mathcal{D} \subset \mathbb{R}^m$. By input signal we mean any measurable, locally essentially bounded function of time and we denote the set of such functions by $\mathcal{M}_{\mathcal{D}}$.

We are interested in characterizing in terms of Lyapunov dissipation inequalities the following property of solutions of (1):

Definition 2.1 We say that (1) is Incremental Globally Asymptotically Stable (δGAS for short) if there exists β of class $\mathcal{KL}$ so that for all $d \in \mathcal{M}_{\mathcal{D}}$, all ξ,$\eta \in \mathbb{R}^n$ and all $t \geq 0$ the following holds

$$|x(t,\xi,d) - x(t,\eta,d)| \leq \beta(|\xi - \eta|, t). \qquad (2)$$

□

It is convenient to recast the notion of incremental stability as a standard problem of robust asymptotic stability with respect to sets. The following lemma holds:

Lemma 2.2 System (1) is δGAS if and only if the auxiliary system

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & f(x_1,d) \\ \dot{x}_2 & = & f(x_2,d) \end{array} \right. \qquad (3)$$

with state $\chi \doteq [x_1', x_2']'$ is uniformly GAS (as in [8]) with respect to the diagonal Δ,

$$\Delta = \left\{ \chi \in \mathbb{R}^{2n} : \exists x \in \mathbb{R}^n : \chi = [x', x']' \right\}. \qquad (4)$$

□

Remark 2.3 Notice that, for $\mathcal{D}$ compact, by virtue of Lemma 2.2 and by applying the converse Lyapunov theorem in [8] we obtain a Lyapunov characterization of δGAS . However, the Main Result in [8] does not extend to $\mathcal{D}$ being a generic closed set even for the apparently simpler case of equilibrium points as it is pointed out in Section 8 of [8]. Nevertheless, For the special case of incremental stability, the theorem still holds provided that we allow for *continuous* Lyapunov functions V. □

Theorem 1 *System (1) is δGAS as in definition 2.1 if and only if there exists a continuous function $U(x_1,x_2)$ and $\mathcal{K}_\infty$ functions α_1,α_2 such that*

$$\alpha_1(|x_1 - x_2|) \leq U(x_1,x_2) \leq \alpha_2(|x_1 - x_2|) \quad (5)$$

and along trajectories of (3) satisfies for any ξ_1, ξ_2 in $\mathbb{R}^n$, any $t \geq 0$ and any $d \in \mathcal{M}_{\mathcal{D}}$

$$\begin{array}{rl} & U\big(x(t,\xi_1,d), x(t,\xi_2,d)\big) - U(\xi_1,\xi_2) \\ \leq & -\displaystyle\int_0^t \alpha(|x(\tau,\xi_1,d) - x(\tau,\xi_2,d)|)\, d\tau \end{array} \quad (6)$$

with α positive definite. ■

The sufficiency part is not surprising and follows by a standard comparison principle. The converse implication is more interesting. Existence of a continuous Lyapunov function is proved in two steps, via the following technical lemmas. First we build a proto-Lyapunov function g which is continuous and non-decreasing along trajectories; then we modify it in order to get an estimate as in (6).

Lemma 2.4 Consider the function $g(\cdot) : \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$ defined as

$$g(\chi_0) = \sup_{t \geq 0, d} |\chi(t,\chi_0,d)|_\Delta. \qquad (7)$$

We claim that g is proper, decrescent (with respect to the distance from Δ) and satisfies the following continuity condition for some $\gamma \in \mathcal{K}_\infty$:

$$|g(\chi_1) - g(\chi_2)| \leq \gamma(|\chi_1 - \chi_2|). \qquad (8)$$

□

Lemma 2.5 Let $U(\xi)$ be defined according to

$$U(\xi) = \sup_{t \geq 0, d} g(\chi(t,\xi,d))\; k(t). \qquad (9)$$

where $k(.) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is any smooth, increasing function with $0 < c_1 < k(t) < c_2$ for some constants c_1,c_2 and such that there is a bounded, positive and decreasing function $\tau(\cdot)$, satisfying $k'(t) \geq \tau(t)$ for all $t \geq 0$. Then U is still continuous as is (8) and satisfies (5), (6). □

3 Local and global aspects of incremental stability

In this section we investigate the relationships between local, semiglobal and global versions of the newly introduced stability notions.

Definition 3.1 We say that system (1) is *semiglobally asymptotically stable* with respect to a closed set $\mathcal{A}$, if for any $M > 0$ there exists β_M of class $\mathcal{KL}$ such that for all $t \geq 0$, $d \in \mathcal{M}_{\mathcal{D}}$ and all ξ with $|\xi|_{\mathcal{A}} \leq M$

$$|x(t,\xi,d)|_{\mathcal{A}} \leq \beta_M(|\xi|_{\mathcal{A}}, t). \qquad (10)$$

□

In (10) $|\cdot|_{\mathcal{A}}$ denotes the euclidean point-set distance.

Definition 3.2 We say that system (1) is *semiglobally* δGAS if for any $M > 0$ there exists $\beta_M \in \mathcal{KL}$ such that for all $t \geq 0$, $d \in \mathcal{M}_\mathcal{D}$ and all ξ_1, ξ_2 with $|\xi_i| \leq M, i = 1, 2$

$$|x(t, \xi_1, d) - x(t, \xi_2, d)| \leq \beta_M(|\xi_1 - \xi_2|, t). \quad (11)$$

☐

Notice that semiglobal incremental asymptotic stability is not the same as semiglobal asymptotic stability of (3) with respect to the diagonal Δ.

Definition 3.3 A system (1) is *locally* δGAS if there exists $\varepsilon > 0$ and $\beta \in \mathcal{KL}$ such that for all $t \geq 0$, $d \in \mathcal{M}_\mathcal{D}$, and all ξ_1, ξ_2 with $|\xi_1 - \xi_2| < \varepsilon$

$$|x(t, \xi_1, d) - x(t, \xi_2, d)| \leq \beta(|\xi_1 - \xi_2|, t). \quad (12)$$

☐

By simple considerations on class $\mathcal{KL}$ functions the following proposition holds:

Proposition 3.4 A system is semiglobally asymptotically stable (with respect to a set $\mathcal{A}$) iff it is globally asymptotically stable. ☐

The main results in this section are the following:

Proposition 3.5 Consider a system $\dot{x} = f(x, d)$, with $f(0, d) = 0$ for all $d \in \mathcal{D}$, then global asymptotic stability of the origin is equivalent to semiglobal incremental asymptotic stability. ☐

Proposition 3.6 A system is locally incrementally asymptotically stable iff it is δGAS . ☐

Remark 3.7 As a consequence of Proposition 3.5 and 3.6 it is clearer why we decided to state the notion of semiglobal stability in a somewhat inconsistent way. Also it turns out that incremental asymptotic stability only makes sense when considered globally. ☐

Corollary 3.8 Let system (1) be globally asymptotically stable; then a sufficient condition for incremental GAS is the existence of two strictly positive real M, ε and of a differentiable function $V(x_1, x_2)$, with $\alpha_1(|x_1 - x_2|) \leq V(x_1, x_2) \leq \alpha_2(|x_1 - x_2|)$ for some α_1, α_2 of class $\mathcal{K}_\infty$, such that for all $x_1, x_2 \in \mathbb{R}^n$ with $|x_1 - x_2| \leq \varepsilon$ and $|x_i| \geq M$, $i = 1, 2$, the following dissipation inequality holds

$$D_{x_1} V\, f(x_1, d) + D_{x_2} V\, f(x_2, d) \leq -\rho(|x_1 - x_2|).$$

☐

4 Incremental Input-to-State Stability

Throughout this section we consider systems of the following form

$$\dot{x} = f(x, u) \quad (13)$$

with state $x \in \mathbb{R}^n$ and input $u \subset \mathcal{U}$, with $\mathcal{U}$ a closed and convex set of $\mathbb{R}^m$ containing the origin. The function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz continuous and such that $f(0, 0) = 0$. We are interested in studying systems which tend to forget their past inputs and states as made precise by the following definition.

Definition 4.1 We call a system (13) *incrementally iss*, (δISS for short) if there exists a $\mathcal{KL}$ function β and $\gamma \in \mathcal{K}_\infty$ such that for any $t \geq 0$, any $\xi_1, \xi_2 \in \mathbb{R}^n$ and any couple of input signals u_1, u_2 the following is true

$$\begin{aligned} & |x(t, \xi_1, u_1) - x(t, \xi_2, u_2)| \quad (14) \\ \leq \quad & \beta(|\xi_1 - \xi_2|, t) + \gamma(\|u_1 - u_2\|_\infty). \end{aligned}$$

☐

Since $f(0, 0) = 0$ it is easy to check that δISS implies ISS just comparing an arbitrary trajectory with $x(t) \equiv 0$. A first interesting fact about δISS is stated in the next proposition.

Proposition 4.2 For all constant value of the input $\bar{u} \in \mathbb{R}^m$ there exists a unique, globally asymptotically stable, equilibrium point of (13). ☐

Remark 4.3 It follows from Proposition 4.2 that a necessary condition for δISS is the following:

$$\forall\, u \in \mathcal{U},\ \exists \text{ unique } x_u :\ f(x_u, u) = 0. \quad (15)$$

Furthermore, x_u is continuous in u. ☐

Proposition 4.4 A δISS system, forced with a periodic input u (of period T), has a state response which asymptotically tends to a periodic function of the same period. Analogous results, for slightly different definitions of incremental stability, are proved in [2, 9]. ☐

The Converging-Input Converging-State property has its counter-part in δISS in the fact that $u_1 - u_2 \to 0$ implies $x(t, \cdot, u_1) - x(t, \cdot, u_2) \to 0$.

Proposition 4.5 Let u_1 and u_2 be inputs signals such that

$$\lim_{t\to+\infty} |u_1(t) - u_2(t)| = 0.$$

and ξ_1, ξ_2 be arbitrary in $\mathbb{R}^n$. Then $|x(t,\xi_1,u_1) - x(t,\xi_1,u_2)| \to 0$. □

Incremental GAS is not preserved under general nonlinear changes of coordinates, nevertheless the following proposition holds:

Proposition 4.6 Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a global diffeomorphism such that

$$\begin{array}{rcl} |\phi(x_1) - \phi(x_2)| & \leq & \gamma(|x_1 - x_2|) \\ |\phi^{-1}(x_1) - \phi^{-1}(x_2)| & \leq & \delta(|x_1 - x_2|) \end{array} \tag{16}$$

for some δ of class $\mathcal{K}$. If the system $\dot{x} = f(x,u)$ is δISS , then

$$\dot{z} = \nabla\phi(\phi^{-1}(z)) f(\phi^{-1}(z), u) \tag{17}$$

is also δISS . □

Proposition 4.7 Consider the interconnected system

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x,y,u) \\ \dot{y} & = & g(y,u) \end{array} \right. \tag{18}$$

Let the x-subsystem be δISS with respect to u, y and the y-subsystem with respect to u. Then the overall system (18) is δISS with respect to the input u. □

5 Lyapunov conditions for incremental input-to-state stability

Definition 5.1 smooth function $V(x_1,x_2) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, is called a δISS *-Lyapunov function* if $\alpha_1(|x_1 - x_2|) \leq V(x_1,x_2) \leq \alpha_2(|x_1 - x_2|)$ for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and there exists $\kappa \in \mathcal{K}_\infty$ so that for any $u_1, u_2 \in \mathcal{U}$ and any $x_1, x_2 \in \mathbb{R}^n$

$$\begin{array}{rcl} \kappa(|x_1 - x_2|) \geq |u_1 - u_2| & \Rightarrow & \\ D_{x_1} V\, f(x_1,u_1) & + & D_{x_2} V\, f(x_2,u_2) \\ & \leq & -\rho(|x_1 - x_2|) \end{array} \tag{19}$$

is satisfied with ρ positive definite. □

We are now ready to state the main result of this Section.

Theorem 2 *System (13) is* δISS *provided that it admits a* δISS *-Lyapunov function. Moreover, if* $\mathcal{U}$ *is compact, existence of a* δISS *-Lyapunov function is equivalent to* δISS . ■

Remark 5.2 It is not known if the Lyapunov condition (19) is necessary for δISS even in the case of unbounded $\mathcal{U}$. As a matter of fact, smooth converse Lyapunov results for incremental robust stability with respect to disturbances taking values in arbitrary closed sets are still not available. In Section 2 a continuous Lyapunov function for δGAS is constructed, but, unlike the compact case, there might be a gap between continuity and smoothness. It is not surprising that sufficient conditions for δISS can be stated relaxing smoothness of V to just lower semicontinuity provided that appropriate integral inequalities are considered. □

The main result will be derived as a corollary of the following Proposition 5.3. In order to state the result we need to define the function

$$\mathrm{sat}_{\mathcal{U}}(u) = \left\{ \begin{array}{cl} u & \text{if } u \in \mathcal{U} \\ \arg\min_{\nu\in\mathcal{U}} |\nu - u| & \text{if } u \notin \mathcal{U} \end{array} \right. . \tag{20}$$

Since $\mathcal{U}$ is closed and convex and $|\cdot|$ is a proper, convex function, definition (20) is well-posed. Moreover, by convexity of $\mathcal{U}$ we have

$$|\mathrm{sat}_{\mathcal{U}}(u_1) - \mathrm{sat}_{\mathcal{U}}(u_2)| \leq |u_1 - u_2|. \tag{21}$$

Proposition 5.3 System (13) is δISS if and only if there exists a smooth gain margin ρ of class $\mathcal{K}_\infty$ that makes the auxiliary system

$$\left\{ \begin{array}{l} \dot{x}_1 = f(x_1, \mathrm{sat}_{\mathcal{U}}(d_1 + \rho(|x_1 - x_2|)\, d_2)) \\ \dot{x}_2 = f(x_2, \mathrm{sat}_{\mathcal{U}}(d_1 - \rho(|x_1 - x_2|)\, d_2)) \end{array} \right. \tag{22}$$

with state $\chi = [x_1', x_2']' \in \mathbb{R}^{2n}$ and input $d = [d_1, d_2]$ taking value in $\mathcal{D} \doteq \mathcal{U} \times \mathcal{B}$ uniformly GAS with respect to the diagonal set Δ (where $\mathcal{B}$ denotes the closed unit ball in $\mathbb{R}^m$) . □

Going back to Theorem 2. By Proposition 5.3, incremental ISS is equivalent to uniform global asymptotic stability of (22) with respect to the diagonal set Δ. For compact $\mathcal{U}$, by virtue of the converse Lyapunov theorem in [16], this is the case if and only if there exist a smooth $V(x_1,x_2)$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, with $\alpha_1(|x_1 - x_2|) \leq V(x_1,x_2) \leq \alpha_2(|x_1 - x_2|)$, so that

$$\begin{array}{rl} & D_{x_1} V\, f(x_1, \mathrm{sat}_{\mathcal{U}}(d_1 + \rho(|x_1 - x_2|) d_2)) \\ + & D_{x_2} V\, f(x_2, \mathrm{sat}_{\mathcal{U}}(d_1 - \rho(|x_1 - x_2|) d_2)) \\ \leq & -\alpha(|x_1 - x_2|) \end{array} \tag{23}$$

holds for any $x_1, x_2 \in \mathbb{R}^n$ and any $[d_1, d_2] \in \mathcal{D}$. Then, whenever $\rho^{-1}(|x_1 - x_2|/2) \geq |u_1 - u_2|$, the input disturbances $d_1 = (u_1 + u_2)/2$ and $d_2 = (u_1 - u_2)/2\rho(|x_1 - x_2|)$ are such that $[d_1, d_2]$ belongs to $\mathcal{D} = \mathcal{U} \times \mathcal{B}$. Then (23) implies

$$\begin{aligned}\kappa(|x_1 - x_2|) \geq |u_1 - u_2| \quad &\Rightarrow \\ D_{x_1} V f(x_1, u_1) \quad &+ \quad D_{x_2} V f(x_2, u_2) \\ &\leq \quad -\alpha(|x_1 - x_2|)\end{aligned}$$

with $\kappa(r) \doteq \rho^{-1}(r/2)$. This completes the converse implication for $\mathcal{U}$ compact. The sufficiency part can be proved along the same lines as in [13] even dropping the compactness assumption on $\mathcal{U}$.

6 A few examples

6.1 Chaos synchronization

We look at syncronization of coupled chaotic systems from an incremental ISS perspective. Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1 &= -\beta x_1 + \mathrm{sat}(x_2 x_3) \\ \dot{x}_2 &= -\sigma x_2 + \sigma x_3 \\ \dot{x}_3 &= -x_3 + u \\ y &= \rho x_2 - x_1 x_2\end{aligned} \tag{24}$$

where β, σ, ρ are given constant parameters and $\mathrm{sat}(\cdot)$ is a piecewise linear saturation. It is not difficult to verify, by virtue of Proposition 4.7 on cascaded systems, that (24) is incrementally ISS. Besides we choosed the output signal y in such a way that under unitary feedback $u = y$, the system evolution is governed (in the linear region of the saturation function), by the celebrated Lorenz equations. Since we were able to isolate a δISS subsystem we can easily synchronize two such systems providing that we force them with the same input signal. In particular we consider the interconnection:

$$\begin{aligned}\dot{x}_1 &= -\beta x_1 + \mathrm{sat}(x_2 x_3) \\ \dot{x}_2 &= -\sigma x_2 + \sigma x_3 \\ \dot{x}_3 &= -x_3 + \rho x_2 - x_1 x_2 \\ y &= \rho x_2 - x_1 x_2 \\ \\ \dot{\hat{x}}_1 &= -\beta \hat{x}_1 + \mathrm{sat}(x_2 x_3) \\ \dot{\hat{x}}_2 &= -\sigma \hat{x}_2 + \sigma \hat{x}_3 \\ \dot{\hat{x}}_3 &= -\hat{x}_3 + \rho \hat{x}_2 - \hat{x}_1 \hat{x}_2 + (y - \hat{y}) + d \\ \hat{y} &= \rho \hat{x}_2 - \hat{x}_1 \hat{x}_2\end{aligned} \tag{25}$$

By virtue of incremental ISS of (24) and letting $x = [x_1, x_2, x_3]$ and $\hat{x} = [\hat{x}_1, \hat{x}_2, \hat{x}_3]$ we have the following estimate:

$$|x(t, \xi) - \hat{x}(t, \hat{\xi}, d)| \leq \beta(|\xi - \hat{\xi}|, t) + \gamma(\|d\|_\infty) \tag{26}$$

which in turns guarantees exact synchronization of the two systems in the ideal case of a noise-free channel, and robustness in the presence of persistent channel disturbances.

6.2 Observer design

In a recent paper on detectability notions for nonlinear systems, [14], Prof. Sontag and Prof. Wang give a definition of full-state observer, robust with respect to sensor and input noise, for systems

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x).\end{aligned} \tag{27}$$

It follows from their definition that such an observer must take the "output injection" form

$$\dot{z} = f(z, u + d_u) + L(z, u + d_u, y - h(z) + d_y) \tag{28}$$

where the vector field L satisfies $L(a, b, 0) = 0$ for all a, b. In (28) the vector $z \in \mathbb{R}^n$ is the estimate of $x(t)$, whereas d_u and d_y are input and sensor noises. In order for (28) to be an observer it is required that feeding the output y of (27) to (28) the following estimate holds

$$\begin{aligned}&|x(t, \xi, u) - z(t, \zeta, u + d_u, h(x) + d_y)| \\ \leq \; &\beta(|\xi - \eta|, t) + \gamma_1(\|d_u\|_\infty) + \gamma_2(\|d_y\|_\infty)\end{aligned}$$

for some β of class $\mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$. It is proved in [14] that a necessary condition for the existence of an observer is incremental input-output to state stability of (27). Hereby we present a sufficient condition in terms of incremental ISS of the output-injected system.

Proposition 6.1 System (27) admits an observer if there exists an output injection such that system

$$\dot{z} = f(z, u) + L(z, u, y - h(z)) \tag{29}$$

is δISS with respect to u and y, (viz. u and y are seen here as inputs of (29)). □

Remark 6.2 As a matter of fact condition (29) can be seen as a robustified version of the detectability notion introduced by Prof. Tsinias in [17]. □

7 Conclusions

We introduced and discussed several notions of robust incremental stability together with their characterizations in terms of Lyapunov dissipation inequalities. Such properties extend the input-to-state stability paradigm to study stability of solutions with respect to one another rather than with respect to some equilibrium position. The central idea is that incremental stability notions can be seen as standard asymptotic stability properties of some duplicated auxiliary system with respect to the diagonal set. Hopefully, this paper will serve as a starting point for the extension of more specific analysis techniques to the context of incremental stability and for the developing of synthesis tools, such as control-Lyapunov functions, backstepping, feedforwarding, [6], which are well-known in nonlinear control theory but usually limited to more traditional stability properties.

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