Synthesis of Reduced Order Multivariable Feedback Tracking Controllers Using the Q-Parameterization

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Abstract

In this paper we consider the problem of multivariable tracking for stable plants. Specifically, a new method is proposed for constructing a feedback controller which, for certain cases, is of lower order than the controller constructed via the classical robust servomechanism method. Viewed from a tracking standpoint, the servomechanism method results in what may be considered an overdesign. For example, in a two input, two output system, where the first reference input is a step and the second reference input is a sinusoid of a known frequency, ω, the controller obtained via the classical robust servomechanism method allows each output to track a reference input which is of the form $A + Bsin(\omega t)$. For many applications, this may not be necessary. As will be shown, the reduced order controller, which is obtained by carefully designing the Q-parameter, does not exhibit this behavior. An example of a two input, two output system is given to illustrate the method.

1 Introduction

In this paper we consider the problem of multivariable tracking for stable plants. Specifically, a new method is proposed for constructing an output feedback controller which, for certain cases, is of lower order than the controller constructed via the classical robust servomechanism method which is developed in [1]. In addition, this controller does not suffer from the above mentioned tracking overdesign. As will be shown, this lower order controller is obtained by carefully designing the Q-parameter. An example of a two input, two output system is given to illustrate the method. Though only stable plants are considered, some statements are made concerning unstable plants.

This paper is organized as follows. A brief description of the classical multivariable servomechanism will be given, along with a discussion of some of its drawbacks. The new method will then be presented. An example demonstrating the method's effectiveness will be given. Finally, some conclusions will be drawn, and areas for future work will be identified.

2 Classical multivariable servomechanism

The classical output feedback servomechanism consists of three parts: an internal model, a stabilizing compensator, and a state observer. A diagram of this configuration is shown in Figure 1.

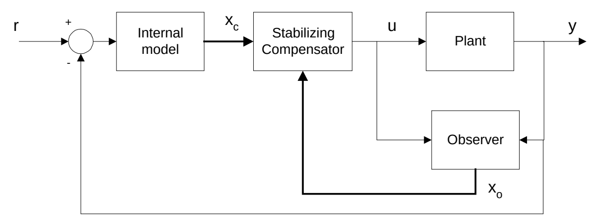

Figure 1: Classical multivariable servomechanism configuration

Let the strictly proper plant be given by

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Then the observer has the familiar state space structure

$$\dot{x}_o = (A - LC)x_o + Bu + Ly$$

while the internal model is given by

$$\dot{x}_c = A_c x_c + B_c e\ .$$

Finally, the output of the stabilizing compensator is given by

$$u = K_1 x_c + K_2 x_o\ .$$

Let β_i denote the characteristic polynomial for the i^{th} reference input. Let $\alpha(s)$ denote the least common multiple of all the β_i's. Next, denote Γ as the matrix in controller canonical form whose characteristic polynomial is $\alpha(s)$.

$$\Gamma = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \\ \vdots & & & \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_p \end{bmatrix}, \gamma = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Then, A_c and B_c are given by $A_c = diag(\Gamma)$, $B_c = diag(\gamma)$. That is, A_c and B_c are block diagonal matrices with Γ and γ as the blocks and with as many blocks as there are inputs (assuming a square system).

Though effective, the classical robust servomechanism has a few drawbacks. First, it can be seen by inspection of the zeros of the error transfer function matrix that every output has the capability of tracking any of the reference inputs. For example, in a two input, two output system, where the first reference input is a step and the second reference input is a sinusoid of a known frequency, ω, it is seen that the controller obtained via the classical robust servomechanism method allows each output to track a reference input which is of the form $A + Bsin(\omega t)$. While there do exist applications where this is desirable, there are also many instances where this is not necessary.

Another drawback is the size of the internal model. Given its structure, it is not difficult to see that for a system with several inputs of several different classes, e.g. steps, exponentials, sinusoids of several different frequencies, the internal model portion of the controller can have a very large number of states. For example, suppose the first output of a two input, two output system is to track a reference input of the form $Asin(\omega_1 t) + Bsin(\omega_2 t)$, while the second output is to track a reference input of the form $Csin(\omega_3 t) + Dsin(\omega_4 t)$. The internal model would then be 16^{th} order.

With these perceived disadvantages in mind, we now consider the following alternate method.

3 Main Result

The primary goal in the tracking problem is to achieve zero steady state error. In many instances it is desirable to obtain a tracking controller which yields a system whose sensitivity transfer function contains a reduced zero structure, thereby avoiding the unneeded, unused tracking capability problem. The problem to be considered is that of continuous tracking of a vector of reference signals, where each signal is assumed to belong to a class that can be specified by a generator such as $\{s^{n_1} \prod_{i=1}^{n_1}(s-\alpha_i) \prod_{j=1}^{n_2}(s^2+\omega_j^2)\}$. This form of generator can include any reference that can be written as a linear combination of polynomials, exponentials and sinusoids of known frequencies.

Now, for a stable plant $P(s)$, the error function is given by $e(s) = (I - P(s)Q(s))r(s)$, where $Q(s)$ is the familiar Q-parameter. Thus, if we can assign the zeros of $I - P(s)Q(s)$, then we can ensure tracking. Specifically, we require that the characteristic polynomial for the i^{th} reference input be present in each row of the i^{th} column of the error transfer function matrix. Then it can easily be seen that achieving asymptotic tracking is equivalent to solving $I - P(s)Q(s) = Y(s)M(s)$ or $P(s)Q(s) + Y(s)M(s) = I$ for some $Q(s)$ and $Y(s)$. It can also be easily seen that each output will be capable of tracking only its associated reference input.

This is perhaps best illustrated through a simple example. Assume the first output of a two input, two output system is to track $r_1 = 2 + sin(t)$, while the second output is to track $r_2 = e^{0.1t} + sin(2t)$. Then

$$M(s) = \frac{\begin{bmatrix} s(s^2+1) & (s-0.1)(s^2+4) \\ s(s^2+1) & (s-0.1)(s^2+4) \end{bmatrix}}{d(s)}.$$

Clearly, regardless of what $Y(s)$ is, $I - P(s)Q(s)$ will have the required zero structure, without unnecessary tracking capability.

To ensure that $Q(s)$ is stable and proper, we make the following definitions. $P(s)$ can be written as $\hat{N}(s)/d(s)$, where $\hat{N}(s)$ is a polynomial matrix, and $d(s)$ is a polynomial. Define $M(s) = \hat{M}(s)/\prod_{i=1}^{v}(s+p_i)$, where $v = \max\left[deg\{d(s)\}, deg\left\{\hat{M}(s)\right\}\right]$. Similarly, let $Y(s)$ be given by $Y(s) = \hat{Y}(s)/d(s)$, and let $Q(s)$ be given by $Q(s) = \hat{Q}(s)/\prod_{i=1}^{v}(s+p_i)$. Then we solve

$$\hat{N}(s)\hat{Q}(s) + \hat{Y}(s)\hat{M}(s) = d(s)\prod_{i=1}^{v}(s+p_i) \cdot I.$$

The controller $C(s)$ is given by $C(s) = Q(s)[I - P(s)Q(s)]^{-1}$, and has the following block diagram representation. Assume that $P(s)$ and $Q(s)$ have the following state space representations, respectively.

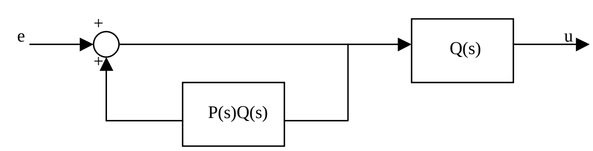

Figure 2: Block diagram representation of $C(s)$

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$\dot{q} = A_q q + B_q v$$
$$y_q = C_q q + D_q v$$

Then the state space representation for the controller is given by

$$\dot{x_c} = A_c x_c + B_c e$$
$$u = C_c x_c + D_c e$$

where, after manipulation and reduction of the above block diagram,

$$x_c = \begin{bmatrix} q_2 \\ x \\ q_1 \end{bmatrix}, \; A_c = \begin{bmatrix} A_q & B_qC & 0 \\ 0 & A + BD_qC & BC_q \\ 0 & B_qC & A_q \end{bmatrix}$$

$$B_c = \begin{bmatrix} B_q \\ BD_q \\ B_q \end{bmatrix}, \; C_c = \begin{bmatrix} C_q & D_qC & 0 \end{bmatrix}, \; D_c = D_q \; .$$

Here, q_1 and q_2 represent the states associated with $Q(s)$ in the feedback and postfilter blocks, respectively. Let $P(s)$ be an n^{th} order, m input, m output system. $Q(s)$ generally will have $m \times v$ states. Thus, this controller would appear to have $2(m \times v) + n$ states. But, as we shall see, $m \times v$ poles (specifically, the poles of $Q(s)$, which are stable by construction) will cancel with $m \times v$ transmission zeros. To see that this is indeed the case, consider the following.

By inspection, we see that the eigenvalues of A_q are also eigenvalues of A_c. The transmission zeros, z, of the controller are given by the solution to

$$\begin{vmatrix} zI - A_c & -B_c \\ C_c & D_c \end{vmatrix} = 0 \; .$$

After carrying out routine row and column operations, we get

$$\begin{vmatrix} zI - A_q & -B_q & 0 & 0 \\ C_q & D_q & 0 & 0 \\ 0 & -BD_q & zI - A & -BC_q \\ 0 & -B_q & 0 & zI - A_q \end{vmatrix} = 0 \; .$$

Clearly, the eigenvalues of A_q are transmission zeros of the controller. Thus, we get $m \times v$ cancellations. We may now state a theorem which summarizes our result.

Theorem 1. *Let β_i denote the characteristic polynomial for the i^{th} reference input, and let $\alpha(s)$ denote the least common multiple of all the β_i's. Let p_r denote the order of $\alpha(s)$. Then, for a stable n^{th} order m input, m output system, if $p_r > v$, then the proposed method will yield a controller which is of lower order than that obtained by the classical robust servomechanism method.*

Proof: The order of the servocompensator is $p_r \times m + n$ (internal model plus observer). The order of the proposed controller is $v \times m + n$. Thus, if $p_r > v$, the servocompensator will be of higher order than the proposed controller. □

Corollary 2. *Assume the system is unstable, but state feedback is available. If $p_r > v + \frac{n}{m}$, the proposed method will yield a controller which is of lower order than that obtained by the classical robust servomechanism method.*

Proof: If state feedback is available, the order of the servocompensator is $p_r \times m$, while the order of the proposed controller is still $v \times m + n$. Thus, if $p_r > v + \frac{n}{m}$, the servocompensator will be of higher order than the proposed controller. □

4 Example

Consider a system of two masses connected with springs and dampers, as shown in Figure 3. It is desired to have

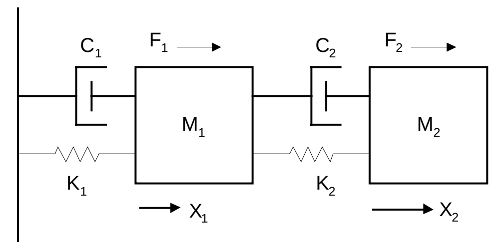

Figure 3: Mass, spring, damper system

x_1 track the reference input $r_1 = sin(1.3t) + sin(2.3t)$. In addition, x_2 is to track the reference input $r_2 = sin(0.5t) + sin(4.3t)$. The inputs which may be used to control the system are the forces F_1 and F_2. The parameter values are $m_1 = 1$, $m_2 = 1.5$, $k_1 = 3$, $k_2 = 5$, $c_1 = 0.75$, $c_2 = 0.5$. $P(s)$ then has the following transfer matrix description:

$$P(s) =$$

$$\frac{\begin{bmatrix} 3.333 + 0.333s + s^2 & 3.333 + 0.333s \\ 3.333 + 0.333s & 5.333 + 0.833s + 0.667s^2 \end{bmatrix}}{s^4 + 1.583s^3 + 11.583s^2 + 3.5s + 10} \; .$$

From this, we get

$$\hat{N}(s) =$$

$$\begin{bmatrix} 3.333 + 0.333s + s^2 & 3.333 + 0.333s \\ 3.333 + 0.333s & 5.333 + 0.833s + 0.667s^2 \end{bmatrix} \; .$$

We construct $\hat{M}(s)$ as follows. The first element is multiplied by 2 to ensure that $\hat{M}(s)$ is non-singular.

$$\hat{M}(s) =$$

$$\begin{bmatrix} 2(s^2 + 1.3^2)(s^2 + 2.3^2) & (s^2 + 0.5^2)(s^2 + 4.3^2) \\ (s^2 + 1.3^2)(s^2 + 2.3^2) & (s^2 + 0.5^2)(s^2 + 4.3^2) \end{bmatrix}$$

By setting $\prod_{i=1}^{v}(s + p_i) = d(s)$ we obtain the following (non-unique) solution:

$$Q(s) = \frac{\begin{bmatrix} \hat{Q}_{11}(s) & \hat{Q}_{12}(s) \\ \hat{Q}_{21}(s) & \hat{Q}_{22}(s) \end{bmatrix}}{s^4 + 1.583s^3 + 11.583s^2 + 3.5s + 10}$$

where

$$\hat{Q}_{11}(s) = 219.5 + 15.2s + 207s^2 + 10.9s^3 + 30.2s^4$$
$$\hat{Q}_{12}(s) = -138.3 - 20.2s - 417.8s^2 - 4.3s^3 - 24.9s^4$$
$$\hat{Q}_{21}(s) = -115.9 - 18s - 111.1s^2 - 10.2s^3 - 13.2s^4$$
$$\hat{Q}_{22}(s) = 80.7 + 9.2s + 192.3s^2 - 34.9s^3 + 1.7s^4 \ .$$

Plots of the response of the system with the controller obtained from this Q-parameter are shown in Figures (4,5). Clearly, tracking was achieved. Note that if we were to use the standard servomechanism method, the resulting controller would have been 20^{th} order (16^{th} order internal model, 4^{th} order observer). However, the controller obtained from the proposed method is 12^{th} order.

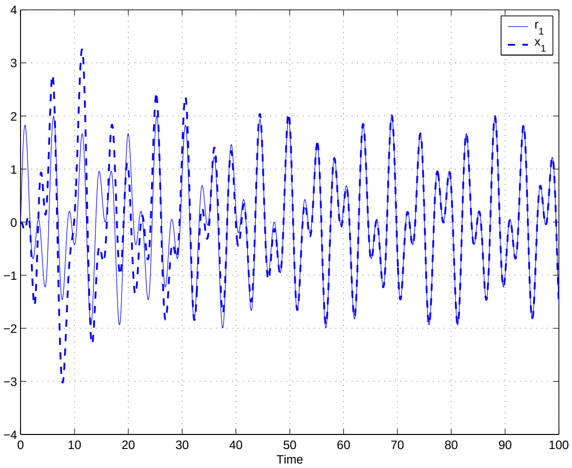

Figure 4: Channel 1 transient response

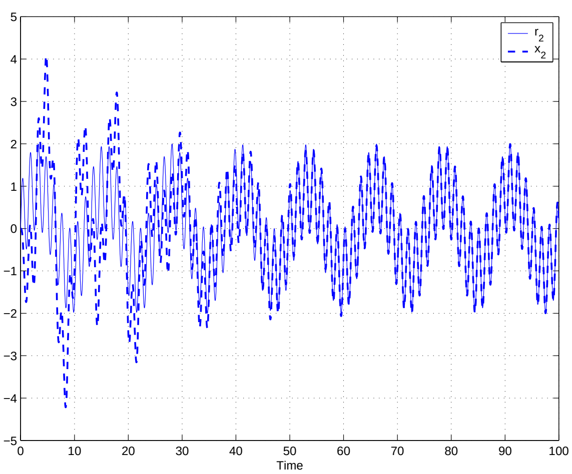

Figure 5: Channel 2 transient response

5 Conclusions

A new method for synthesizing multivariable tracking controllers was presented. In certain cases, this controller was seen to be of lower order than the controller produced by the classical multivariable servomechanism method. In addition, the resulting system does not suffer from the unneeded, unused tracking capability associated with the classical servomechanism. While it has been demonstrated in [2] that the full order internal model is necessary to maintain tracking in the presence of unstructured plant uncertainty, the question of maintaining tracking with a reduced order controller in the presence of structured uncertainty remains open. Thus, future work in this area will involve determining to what extent robustness in the presence of structured uncertainty can be maintained for the technique presented in this paper.

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References

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[2] B.A Francis and W.M. Wonham, "The role of transmission zeros in linear multivariable regulators," *International Journal of Control,* vol. 22, pp. 657-681, 1975.