Controller Design for the Integral-Input-to-State Stabilization of Switched Nonlinear Systems: a Cycle Analysis Method

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Abstract

This paper considers the integral-input-to-state stabilization of a class of switched nonlinear systems, which consist of integral-input-to-state stable (IISS) subsystems and subsystems whose continuous states may eventually be unbounded even if integrals of the corresponding inputs are small. In general, a switched controller has two parts: local controllers for subsystems and their switching law. Here, a cycle analysis method is used to derive a switched controller for the integral-input-to-state stabilization of the considered system. The results obtained here extend our previous work in that all subsystems are not required to be IISS.

1 Introduction

This paper investigates the issue on the integral-input-to-state stabilization of switched systems modeled by

$$\dot{X}(t) = f_{m(t)}(X(t), u(t)) \tag{1}$$

where $X(t) \in \mathbb{R}^r$ is the continuous state, $u(t) : \mathbb{R}_+ \to \mathbb{R}^p$ is the control input, $m(t) \in \bar{M} = \{1, \cdots, n\}$ is the index for the discrete states, and $\{f_{\bar{M}} : \mathbb{R}^r \times \mathbb{R}^p \to \mathbb{R}^r\}$ is a family of sufficiently regular functions. Each $i \in \bar{M}$ stands for a location where the system dynamics is governed by the corresponding vector field $f_i(X(t), u(t))$ called a subsystem.

For a subsystem in mode $i \in \bar{M}$, let $X_i(t)$ and $u_i(t)$, respectively, denote the continuous state and control input of the subsystem. Following [1], the subsystem is said to be integral-input-to-state stable (IISS) with respect to the input u_i if for some functions $\alpha_i \in \mathcal{K}_\infty$, $\beta_i \in \mathcal{KL}$ and $\gamma_i \in \mathcal{K}$, the following estimate holds for any $t > 0$

$$\alpha_i(|X_i(t)|) \leq \beta_i(|X_i(0)|, t) + \int_0^t \gamma_i(|u_i(s)|)ds, \tag{2}$$

where $|\cdot|$ denotes the Euclidean norm. As established in [2], a necessary and sufficient condition for IISS is the existence of an IISS-Lyapunov function, i.e., a positive definite and radially unbounded smooth function $V_i : \mathbb{R}^r \to \mathbb{R}$ such that for a positive definite function ρ_i and a $\mathcal{K}$ function σ_i,

$$DV_i(X)f_i(X_i, u_i) \leq -\rho_i(|X_i|) + \sigma_i(|u_i|). \tag{3}$$

If condition (3) is changed into the following form

$$DV_i(X)f_i(X_i, u_i) \leq \rho_i(|X_i|) + \sigma_i(|u_i|), \tag{4}$$

then we call such a subsystem as a pseudo-IISS subsystem. Condition (4) implies that continuous states are not necessarily eventually small even if integrals of the corresponding inputs are small.

In [3], some results were derived to integral-input-to-state stabilize switched nonlinear systems under the assumption that each subsystem of the considered switched nonlinear system is IISS. In this paper, we enlarge the class of systems by including pseudo-IISS subsystems in (1). In other words, not all subsystems are required to be IISS. A cycle analysis method proposed in [4] is used to derive proper conditions for the integral-input-to-state stabilization of system (1).

The remainder of the paper is organized as follows. Section 2 establishes a state property of a pseudo-IISS system. Section 3 contains main results. Concluding remarks are given in Section 4.

2 Pseudo-IISS Systems

In what follows, we will explore a state property of a pseudo-IISS system. To this end, we need some supporting results. Firstly, we establish the following comparison lemma.

Lemma 1. *For any (locally) absolutely continuous function $v : \mathbb{R}_+ \to \mathbb{R}_+$, if there exists a continuous and positive definite function ρ such that*

$$\dot{v}(t) \leq \rho(v(t)), \quad \forall t \geq 0, \tag{5}$$

then there exists a continuously increasing function $\Gamma_\rho : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with $\Gamma_\rho(0, \cdot) = 0$ such that

$$v(t) \leq \Gamma_\rho(|v(0)|, t), \quad \forall t \geq 0.$$

Proof. From condition (5), we have

$$\begin{aligned} v(t) &\leq \int_0^t \rho(v(\tau))d\tau + v(0) \\ &\leq \rho'(t)t + v(0) \end{aligned}$$

where $\rho'(t) = \max_{0\leq\tau\leq t}\{\rho(v(\tau))\}$. If $v(0) = 0$, then $v(t) \equiv 0$ for all $t \geq 0$ since $\rho(0) = 0$. Let

$$\Gamma_\rho(s, t) = \begin{cases} 0, & if\ s = 0; \\ \rho'(t)t + s, & if\ s > 0. \end{cases}$$

Then

$$v(t) \leq \Gamma_\rho(|v(0)|, t), \quad \forall t \geq 0.$$

□

Similar to Lemma 4.2 and Corollary 4.3 in [2], the following lemma and corollary are obtained based on Lemma 1.

Lemma 2. *For some $0 < t' \leq \infty$, a continuous function $z : [0, t') \to \mathbb{R}_+$ and a (locally) absolutely continuous function $v : [0, t') \to \mathbb{R}$ with $v(0) \geq 0$ satisfy*

$$\dot{v}(t) \leq \rho(\max\{v(t) + z(t), 0\}) \tag{6}$$

for all $t \in [0, t')$. Here, $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and positive definite function. Then, there exists a continuously increasing function $\Gamma_\rho : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with $\Gamma_\rho(0, \cdot) = 0$ such that the following estimate holds:

$$v(t) \leq \max\{\Gamma_\rho(|v(0)|, t), \|z(t)\|\}, \quad \forall t \in [0, t') \tag{7}$$

where $\|z(t)\| = ess\ sup\{|z(s)|, s \in [0, t]\}$.

Proof. Without loss of any generality, we may assume that ρ is locally Lipschitz. Otherwise, one may replace ρ by any function majorized by ρ.

Consider the function $w : [0, t') \to \mathbb{R}$ such that $\dot{w}(t) = \rho(\max\{w(t) + z(t), 0\})$ with $w(0) = v(0)$.

Let

$$t_0 = \max\{t \geq 0 | w(t) \leq \|z(t)\|\}$$

Then, $w(t) \leq \|z(t)\|$ for any $t \in [0, t_0]$ since $w(t)$ is a nondecreasing function.

For any $t \in [t_0, t')$, we have $w(t) \geq \|z(t)\| \geq z(t) \geq 0$ from the definition of $\|z(t)\|$. That is $0 \leq w(t) + z(t) \leq 2w(t)$ for any $t \in [t_0, t')$. Thus, from the definition of $w(t)$

$$\dot{w}(t) \leq \rho(2w(t))$$

From Lemma 1, there exists a continuously increasing function $\Gamma_\rho : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with $\Gamma_\rho(0, \cdot) = 0$ such that

$$w(t) \leq \Gamma_\rho(|w(t_0)|, t - t_0), \ \forall t \in [t_0, t')$$

Thus, for all $t \in [0, t')$

$$w(t) \leq \max\{\Gamma_\rho(|w(t_0)|, t - t_0), \|z(t)\|\}$$

It follows from the standard comparison principle that $v(t) \leq w(t)$ for all $t \in [0, t')$. Therefore, (7) holds. □

The following is a consequence of Lemma 2

Corollary 1. *For some $0 < t' \leq \infty$, any (locally) absolutely continuous function $v : [0, t') \to \mathbb{R}_+$ and any measurable, locally essentially bounded function $z : [0, t') \to \mathbb{R}_+$ satisfy*

$$\dot{v}(t) \leq \rho(v(t)) + z(t), \quad \forall t \in [0, t')$$

where $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and positive definite function. Then, there exists a continuously increasing function $\Gamma_\rho : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with $\Gamma_\rho(0, \cdot) = 0$ such that

$$v(t) \leq \Gamma_\rho(|v(0)|, t) + \int_0^t 2z(\tau)d\tau, \quad \forall t \in [0, t')$$

Proof. Without loss of any generality, we may assume that ρ is locally Lipschitz. Otherwise, one may replace ρ by any function majorized by ρ.

Consider the function $w : [0, t) \to \mathbb{R}_+$ such that $\dot{w}(t) = \rho(w(t)) + z(t)$ with $w(0) = v(0)$. It follows

from the standard comparison principle that $0 \leq v(t) \leq w(t)$ for all $t \in [0, t')$. Let

$$z_1(t) = \int_0^t z(\tau)d\tau \tag{8}$$

and

$$w_1(t) = w(t) - z_1(t), \tag{9}$$

then for almost all $t \in [0, t')$ we have

$$\begin{aligned} \dot{w}_1(t) &= \rho(w(t)) + z(t) - z(t) = \rho(w(t)) \\ &= \rho(\max\{w_1(t) + z_1(t), 0\}), \end{aligned}$$

where the last equation holds since $w(t) \geq 0$. It follows from Lemma 2 that there exists a continuously increasing function $\Gamma_\rho : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with $\Gamma_\rho(0, \cdot) = 0$ such that

$$w_1(t) \leq \max\{\Gamma_\rho(|w_1(0)|, t), \|z_1(t)\|\}, \quad \forall t \in [0, t').$$

Further, from (8) and (9) we have

$$\begin{aligned} v(t) &\leq w(t) \leq \max\{\Gamma_\rho(|w_1(0)|, t), \|z_1(t)\|\} + z_1(t) \\ &= \max\{\Gamma_\rho(|v(0)|, t), \|z_1(t)\|\} + z_1(t) \\ &\leq \Gamma_\rho(|v(0)|, t) + \|z_1(t)\| + z_1(t) \\ &= \Gamma_\rho(|v(0)|, t) + \int_0^t 2z(\tau)d\tau \end{aligned}$$

for all $t \in [0, t')$. Thus, we complete our proof. □

Now we consider a pseudo-IISS subsystem corresponding to mode $i \in \bar{M}$. For a pseudo-IISS subsystem, there exist a continuously differentiable function $V_i : \mathbb{R}^r \to \mathbb{R}$, a positive definite function ρ_i and some functions $\sigma_{1,i}, \sigma_{2,i} \in \mathcal{K}_\infty$, $\sigma_{3,i} \in \mathcal{K}$ such that

$$\sigma_{1,i}(|X_i(t)|) \leq V_i(X_i(t)) \leq \sigma_{2,i}(|X_i(t)|), \ \forall X_i(t) \in \mathbb{R}^r$$

and $\forall X_i(t) \in \mathbb{R}^r$ and $u_i(t) \in \mathbb{R}^p$,

$$DV_i(X_i(t))f_i(X_i(t), u_i(t)) \leq \rho_i(|X_i(t)|) + \sigma_{3,i}(|u_i(t)|).$$

Then, we have

$$\begin{aligned} \dot{V}_i(X_i(t)) &\leq \rho_i(|X_i(t)|) + \sigma_{3,i}(|u_i(t)|) \\ &\leq \rho_i(\sigma_{1,i}^{-1}(V_i(X_i(t)))) + \sigma_{3,i}(|u_i(t)|) \\ &\triangleq \rho_i'(V_i(X_i(t))) + \sigma_{3,i}(|u_i(t)|). \end{aligned}$$

where the last inequality holds since $|X_i(t)| \leq \sigma_{1,i}^{-1}(V_i(X_i(t)))$. Obviously, ρ_i' is a continuous and positive definite function. It follows from Corollary 1 that there exists a continuously increasing function $\Gamma_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with $\Gamma_i(0, \cdot) = 0$ such that $\forall t \geq 0$,

$$V_i(X_i(t)) \leq \Gamma_i(V_i(X_i(0)), t) + \int_0^t 2\sigma_{3,i}(|u_i(s)|)ds.$$

Hence,

$$\begin{aligned} \alpha_{1,i}(|X_i(t)|) &\leq V_i(X_i(t)) \leq \Gamma_i(\alpha_{2,i}(|X_i(0)|), t) \\ &\quad + \int_0^t 2\sigma_{3,i}(|u_i(s)|)ds \end{aligned}$$

which can be converted into the following form

$$\begin{aligned} |X_i(t)| &\leq \alpha_{1,i}^{-1}(\Gamma_i(\alpha_{2,i}(|X_i(0)|), t)) \\ &\quad + \alpha_{1,i}^{-1}\left(\int_0^t 2\sigma_{3,i}(|u_i(s)|)\right) \\ &\triangleq \beta_i(|X_i(0)|, t) + \gamma_{1,i}\left(\int_0^t \gamma_{2,i}(|u_i(s)|)ds\right) \end{aligned} \tag{10}$$

where $\beta_i(s, t) = \alpha_{1,i}^{-1}(\Gamma_i(\alpha_{2,i}(s), t))$, $\gamma_{1,i}(s) = \alpha_{1,i}^{-1}(s)$ and $\gamma_{2,i}(s) = 2\sigma_{3,i}(s)$. Clearly, $\beta_i(s, t)$ is a continuously increasing function with $\beta_i(0, \cdot) = 0$ since for a $\mathcal{K}_\infty$ function $\alpha_{1,i}$, its inverse function $\alpha_{1,i}^{-1}$ is also a $\mathcal{K}_\infty$ function. Also $\gamma_{2,i}(s)$ is a $\mathcal{K}$ functions.

Remark 1. *Condition (10) implies that for a pseudo-IISS system, the corresponding 0-input system is not necessarily globally asymptotically stable.*

Remark 2. *As stated in [5], condition (2) holds if and only if there exist a $\mathcal{KL}$ function β_i' and $\mathcal{K}$ functions $\gamma_{1,i}, \gamma_{2,i}$ such that*

$$|X_i(t)| \leq \beta_i'(|X_i(0)|, t) + \gamma_{1,i}\left(\int_0^t \gamma_{2,i}(|u_i(s)|)ds\right).$$

Comparing with condition (10), we can conclude that for a subsystem corresponding to mode $i \in \bar{M}$, no matter it is IISS or pseudo-IISS, its state satisfies a condition in a similar form. Concretely, if a subsystem is IISS, then the corresponding β_i in (10) is a $\mathcal{KL}$ function. Otherwise, β_i is a continuously increasing function with $\beta_i(0, \cdot) = 0$.

Remark 3. *For any β_i satisfying a condition (10), if we let $t = 0$, then we can get the following property:*

$$\beta_i(s, 0) \geq s, \ \forall s \in \mathbb{R}_+. \tag{11}$$

In particular, for a β_i corresponding to a pseudo-IISS subsystem

$$\beta_i(s,t) \geq \beta_i(s,0) \geq s, \ \forall s \in \mathbb{R}_+, t \in \mathbb{R}_+. \quad (12)$$

So far, we obtain a uniform form for the state properties of IISS and pseudo-IISS subsystems.

3 Main Results

We now introduce the concept of "cycle", which is very important for stability analysis of switched systems [6]. A logical path of length $k+1$ in system (1) is a sequence $m(\tau_j)$,$m(\tau_{j+1})$, $\cdots$, $m(\tau_{j+k})$. A finite logical path $m(\tau_j), m(\tau_{j+1}),\cdots, m(\tau_{j+k})$ is closed if $m(\tau_j) = m(\tau_{j+k})$. A closed logical path $LC = m(\tau_j), m(\tau_{j+1})$, $\cdots$, $m(\tau_{j+k})$ in which no state appears more than once except for the first and the last is a cycle. All types of cycles can be found out by graph theory [7].

In this paper, we make the following assumption for the discrete states of system (1).

Assumption 1. *For any $m(\tau)$, $m(\tau') \in \bar{M}$, there exist at least a path from $m(\tau)$ to $m(\tau')$ and at least a path from $m(\tau')$ to $m(\tau)$, respectively.*

Remark 4. *Assumption 1 implies that any pair of discrete states is reachable.*

Thus, for any $m(\tau)$, there exists at least a cycle which includes $m(\tau)$. Here, we suppose that the total number of the types of cycles is θ_0 and denote these cycles as $LC(1), LC(2)$, $\cdots$, $LC(\theta_0)$. For any cycle $LC(j)$ $(1 \leq j \leq \theta_0)$, we use j' to denote the number of elements in $LC(j)$. Figure 1 is an illustrative example of such discrete states.

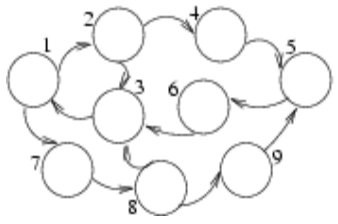

Figure 1: An illustrative example of discrete states

In general, a switched controller has two parts: local controllers for subsystems and their switching law. We now discuss some requirements on the switching law of system modes. Let $t^k_{s,i}$ and $t^k_{e,i}$ denote respectively the kth starting time and the kth ending time of mode i ($i \in \bar{M}$). The activation time of a subsystem is required to satisfy, for an IISS subsystem

$$\inf_k\{t^k_{e,i} - t^k_{s,i}\} \quad = \quad \Delta T_i > 0 \quad (13)$$

and for a pseudo-IISS subsystem

$$\sup_k\{t^k_{e,i} - t^k_{s,i}\} \quad = \quad \Delta T_i < +\infty, \quad (14)$$

and for any cycle $LC(j)$ $(1 \leq j \leq \theta_0)$ and any $s \in \mathbb{R}_+$

$$\beta_{j_1}(2\beta_{j_2}(\cdots(2\beta_{j_{j'-1}}(2\beta_{j_{j'}}(2s, \Delta T_{j_{j'}}), \Delta T_{j_{j'-1}}), \cdots), \Delta T_{j_2}), \Delta T_{j_1}) \leq \epsilon s < s, \quad (15)$$

where for all $l = 1, 2, \cdots, j'-1, j'$, $j_l \in LC(j)$ and β_{j_l} are specified as in Remark 2, $j_1 \neq j_2 \neq \cdots \neq j_{j'-1} \neq j_{j'}$, and $0 < \epsilon < 1$.

Remark 5. *Condition (15) implies that there exists at least one IISS subsystem contained in cycle $LC(j)$. This is because if there is no IISS subsystem in cycle $LC(j)$, then from (12)*

$$\begin{aligned}
&\beta_{j_1}(2\beta_{j_2}(\cdots(2\beta_{j_{j'-1}}(2\beta_{j_{j'}}(2s, \Delta T_{j_{j'}}), \Delta T_{j_{j'-1}}), \\
&\qquad \cdots), \Delta T_{j_2}), \Delta T_{j_1}) \\
&\geq \beta_{j_1}(2\beta_{j_2}(\cdots(2\beta_{j_{j'-1}}(2\beta_{j_{j'}}(2s, 0), 0), \cdots), 0), 0) \\
&\geq \quad 2^{j'}s > s,
\end{aligned}$$

which contradicts with (15).

Remark 6. *Under the above switching law of system modes, it can be easily shown that system (1) is IISS if the number of switchings is finite, since from (13) and (14), this case implies that the final mode of system (1) must be in a mode corresponding to an IISS subsystem. So, we only need consider the case that the number of switchings is infinite.*

Theorem 1. *Consider system (1) with IISS and pseudo-IISS subsystems under the control law $u(t) = u_i(t)$, $i = 1, 2, \cdots, n$. Suppose that Assumption (1) holds and the switchings of system modes satisfy condition (15). Then, the system is IISS.*

Proof. For simplicity, the following proof just considers the case that $\theta_0 = 1$. That is the total

number of types of cycles for the discrete states of system (1) is 1. The techniques being used in the following proof can be easily extended to the case that $\theta_0 > 1$.

Without loss of generality, we assume the initial mode is mode 1, the initial continuous state is $X(0)$ and the system switches its modes in the sequence $1 \to 2 \to \cdots \to n-1 \to n \to 1 \to 2 \to \cdots$. Note that $X(t) = X_i(t)$ and $u(t) = u_i(t)$ if the present mode is $i \in \bar{M}$, and the fact that for any increasing function $y : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, and any constants $a, b \in \mathbb{R}_+$,

$$\begin{aligned} y(a+b,t) &\leq \max\{y(2a,t), y(2b,t)\} \\ &\triangleq y(2a,t) \boxplus y(2b,t). \end{aligned}$$

For simplicity, let $\gamma_{i,k} = \gamma_{1,i}\left(\int_{t_{s,i}^k}^{t_{e,i}^k} \gamma_{2,i}(|u(s)|)ds\right)$. Then from (10),

$$\begin{aligned} |X(t_{e,1}^1)| &\leq \beta_1(|X(0)|, t_{e,1}^1) + \gamma_{1,1} \\ &\leq \beta_1(|X(0)|, \Delta T_1) + \gamma_{1,1}; \\ |X(t_{e,2}^1)| &\leq \beta_2(|X(t_{e,1}^1)|, t_{e,2}^1 - t_{e,1}^1) + \gamma_{2,1}; \\ &\leq \beta_2(2\beta_1(|X(0)|, \Delta T_1), \Delta T_2) \\ &\quad \boxplus \beta_2(2\gamma_{1,1}, \Delta T_2) + \gamma_{2,1}; \\ |X(t_{e,3}^1)| &\leq \beta_3(|X(t_{e,2}^1)|, t_{e,3}^1 - t_{e,2}^1) + \gamma_{3,1} \\ &\leq \beta_3(2\beta_2(2\beta_1(|X(0)|, \Delta T_1), \Delta T_2), \Delta T_3) \\ &\quad \boxplus \beta_3(2\beta_2(2\gamma_{1,1}, \Delta T_2), \Delta T_3) \boxplus \\ &\quad \beta_3(2\gamma_{2,1}, \Delta T_3) + \gamma_{3,1}; \\ &\vdots \end{aligned}$$

$$\begin{aligned} |X(t_{e,n}^1)| &\leq \beta_n(|X(t_{e,n-1}^1)|, t_{e,n}^1 - t_{e,n-1}^1) + \gamma_{n,1} \\ &\leq \beta_n(2\beta_{n-1}(\cdots(2\beta_3(2\beta_2(2\beta_1(|X(0)|, \\ &\quad \Delta T_1), \Delta T_2), \Delta T_3), \cdots), \Delta T_{n-1}), \Delta T_n) \boxplus \\ &\quad \beta_n(2\beta_{n-1}(\cdots(2\beta_3(2\beta_2(2\gamma_{1,1}, \Delta T_2), \Delta T_3), \cdots), \\ &\quad \Delta T_{n-1}), \Delta T_n) \boxplus \\ &\quad \beta_n(2\beta_{n-1}(\cdots(2\beta_3(2\gamma_{2,1}, \Delta T_3), \cdots), \\ &\quad \Delta T_{n-1}), \Delta T_n) \boxplus \cdots \boxplus \\ &\quad \beta_n(2\beta_{n-1}(2\gamma_{n-2,1}, \Delta T_{n-1}), \Delta T_n) \boxplus \\ &\quad \beta_n(2\gamma_{n-1,1}, \Delta T_n) + \gamma_{n,1} \\ &\leq \epsilon|X(0)| \boxplus \beta_n(2\beta_{n-1}(\cdots(2\beta_3(2\beta_2(2\gamma_{1,1}, \\ &\quad \Delta T_2), \Delta T_3), \cdots), \Delta T_{n-1}), \Delta T_n) \boxplus \\ &\quad \beta_n(2\beta_{n-1}(\cdots(2\beta_3(2\gamma_{2,1}, \Delta T_3), \cdots), \\ &\quad \Delta T_{n-1}), \Delta T_n) \boxplus \cdots \boxplus \\ &\quad \beta_n(2\beta_{n-1}(2\gamma_{n-2,1}, \Delta T_{n-1}), \Delta T_n) \boxplus \\ &\quad \beta_n(2\gamma_{n-1,1}, \Delta T_n) + \gamma_{n,1} \end{aligned}$$

where the last inequality holds from (15).

Utilizing the induction method, we can further get the expression of $|X(t_{e,n}^k)|$ for $k > 3$. We here omit such an expression because of space limitation.

Let

$$\begin{aligned} \gamma_1'(s) &= \sum_{i=1}^n \gamma_{1,i}(s), \\ \gamma_2'(s) &= \sum_{i=1}^n \gamma_{2,i}(s), \\ \gamma_t &= \gamma_1'\left(\int_0^t \gamma_2'(|u(s)|)ds\right). \end{aligned}$$

Note that $0 < \epsilon < 1$, then the following holds

$$\begin{aligned} &|X(t_{e,n}^k)| \leq \epsilon^k|X(0)| \boxplus \epsilon\gamma_{t_{e,n}^k} \boxplus \beta_n(2\beta_{n-1}(\cdots \\ &(2\beta_3(2\beta_2(2\gamma_{t_{e,n}^k}, \Delta T_2), \Delta T_3), \cdots), \Delta T_{n-1}), \Delta T_n) \boxplus \\ &\beta_n(2\beta_{n-1}(\cdots(2\beta_3(2\gamma_{t_{e,n}^k}, \Delta T_3), \cdots), \Delta T_{n-1}), \Delta T_n) \\ &\boxplus \cdots \boxplus \beta_n(2\beta_{n-1}(2\gamma_{t_{e,n}^k}, \Delta T_{n-1}), \Delta T_n) \\ &\boxplus \beta_n(2\gamma_{t_{e,n}^k}, \Delta T_n) + \gamma_{t_{e,n}^k} \end{aligned}$$

In what follows, we consider the general case that $t \in [t_{s,1}^{k+1}, t_{e,1}^{k+1}]$ in mode 1.

$$\begin{aligned} |X(t)| &\leq \beta_1(2\epsilon^k|X(0)|, t - t_{s,1}^k) \boxplus \beta_1(2\epsilon\gamma_t, t - t_{s,1}^k) \boxplus \\ &\quad \beta_1(2\beta_n(2\beta_{n-1}(\cdots(2\beta_3(2\beta_2(2\gamma_t, \Delta T_2), \Delta T_3), \cdots), \\ &\quad \Delta T_{n-1}), \Delta T_n), t - t_{s,1}^k) \boxplus \\ &\quad \beta_1(2\beta_n(2\beta_{n-1}(\cdots(2\beta_3(2\gamma_t, \Delta T_3), \cdots), \Delta T_{n-1}), \\ &\quad \Delta T_n), t - t_{s,1}^k) \boxplus \cdots \boxplus \\ &\quad \beta_1(2\beta_n(2\beta_{n-1}(2\gamma_t, \Delta T_{n-1}), \Delta T_n), t - t_{s,1}^k) \\ &\quad \boxplus \beta_1(2\beta_n(2\gamma_t, \Delta T_n), t - t_{s,1}^k) \boxplus \\ &\quad \beta_1(2\gamma_t, t - t_{s,1}^k) + \gamma_t \end{aligned} \tag{16}$$

where the fact that $\gamma_t \geq \gamma_{t_{e,n}^k}$ is used since $t \geq t_{e,n}^k = t_{s,1}^{k+1}$.

Without loss of generality, we assume the modes $i_1, \cdots, i_m \in \bar{M}$ correspond to IISS subsystems, the other modes $i_{m+1}, \cdots, i_n \in \bar{M}$ correspond to pseudo-IISS subsystems. For $s \geq 0$, define

$$\begin{aligned} \beta(s,t) &= \sum_{j=1}^m \beta_{i_j}(2\epsilon^k s, 0) + \sum_{j=m+1}^n \beta_{i_j}(2\epsilon^k s, \Delta T_{i_j}) \\ \Gamma(s) &= \sum_{j=1}^m \beta_{i_j}(2s, 0) + \sum_{j=m+1}^n \beta_{i_j}(2s, \Delta T_{i_j}) \end{aligned}$$

Obviously, $\beta(s,t)$ is a $\mathcal{KL}$ function since $0 < \epsilon < 1$ and $k \to +\infty$ as $t \to +\infty$, and $\Gamma(s)$ is a $\mathcal{K}$ function.

From (16) and properties (11), (12), we have

$$\begin{aligned}|X(t)| \leq & \underbrace{\Gamma(\Gamma(\cdots(\Gamma}_{n-1}(\beta(|X(0)|,t)))\cdots)) \\ & \boxplus \underbrace{\Gamma(\Gamma(\cdots(\Gamma}_{n}(\gamma_t))\cdots)) + \gamma_t \\ \leq & \underbrace{\Gamma(\Gamma(\cdots(\Gamma}_{n-1}(\beta(|X(0)|,t)))\cdots)) \\ & + \underbrace{\Gamma(\Gamma(\cdots(\Gamma}_{n}(\gamma_t))\cdots)) + \gamma_t \\ \leq & \bar{\beta}(|X(0)|,t) + \bar{\gamma}\left(\int_0^t \gamma_2'(|u(s)|)ds\right), \quad (17)\end{aligned}$$

where

$$\begin{aligned}\bar{\beta}(s,t) &= \underbrace{\Gamma(\Gamma(\cdots(\Gamma}_{n-1}(\beta(s,t)))\cdots)) \\ \bar{\gamma}(s) &= \underbrace{\Gamma(\Gamma(\cdots(\Gamma}_{n}(\gamma_1'(s)))\cdots)) + \gamma_1'(s)\end{aligned}$$

$\bar{\beta}$ and $\bar{\gamma}$ are, respectively, a $\mathcal{KL}$ function and a $\mathcal{K}$ function since β and Γ are, respectively, a $\mathcal{KL}$ function and a $\mathcal{K}$ function; and γ_2' is a $\mathcal{K}$ function from the fact that for $\mathcal{K}$ functions ϕ_i, $i = 1, \cdots, l$, $\sum_{i=1}^{l} \phi_i$ is also a $\mathcal{K}$ function. Thus, system (1) is IISS in the case of $t \in [t_{s,1}^{k+1}, t_{e,1}^{k+1}]$. In a similar way, we can consider the other cases of $t \in [t_{s,i}^{k+1}, t_{e,i}^{k+1}]$, $i = 2, \cdots, n$ and get the same result as (17).

So far, the proof is completed for the case that $\theta_0 = 1$. □

4 Conclusion

This paper investigates the issue on the integral-input-to-state stabilization of a class of switched nonlinear systems by a cycle analysis method. Any pair of discrete states of the considered system is assumed to be reachable, and the system consists of IISS subsystems and pseudo-IISS subsystems. For a pseudo-IISS subsystem, we first obtain a similar state property to that of a IISS subsystem. Then based on these state properties of IISS and pseudo-IISS subsystems, we prove that the system can be integral-input-to-state stabilized under some proper conditions including local controllers for subsystems and their switching law.

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