GLOBAL BEHAVIOR IN NONLINEAR SYSTEMS WITH DELAYED FEEDBACK [1]

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Abstract: The problem of global stability in scalar delay differential equations of the form $\dot{x}(t) = f(x(t-\tau)) - g(x(t))$ is studied. Functions f and g are continuous and such that the equation assumes a unique equilibrium. Two types of the suffiecient conditions for the global asymptotic stability of the unique equlibrium are established: (i) delay independent, and (ii) conditions involving the size τ of the delay. Delay independent stability conditions make use of the global stability in the limiting (as $\tau \to \infty$) difference equation $g(x_{n+1}) = f(x_n)$: the latter always implying the global stability in the differential equation for all values of the delay $\tau \geq 0$. The delay dependent conditions involve the global attractivity in specially constructed one-dimensional maps (difference equations) that include the nonlinearities f and g, and the delay τ.

1. Object and Assumptions

Nonlinear delay differential equation

$$\dot{x}(t) = f(x(t-\tau)) - g(x(t)) \qquad (1)$$

is considered, where functions f and g are defined and continuous on the positive semiaxis $\mathbf{R}_+ := \{x : x \geq 0\}$ with the values in $\mathbf{R}_+$.

The following basic hypotheses are assumed throughout the paper without further mentioning: **(H1)** $g(x)$ is strictly increasing, $g(0) = 0$, and $\lim_{x\to+\infty} g(x) = +\infty$; **(H2)** there is exactly one point $\bar{x} > 0$ such that $f(\bar{x}) = g(\bar{x})$; moreover, $f(x) > g(x)$ in $(0, \bar{x})$ and $f(x) < g(x)$ in $(\bar{x}, \infty)$.

The above hypotheses are motivated by specific applications, see e.g. [1-3] and further references therein.

2. Main Results

2.1 Invariance. Equation (1) is equivalent, via the change of variables $t = \tau \cdot s$, to the equation

$$\mu\dot{x}(t) = f(x(t-1)) - g(x(t)) \qquad (2)$$

where $\mu = 1/\tau$. The limiting case $\mu = 0 (\tau = +\infty)$ in (2) results in the difference equation

$$f(x_n) = g(x_{n+1}), \quad n \in \mathbf{Z}_+ \qquad (3)$$

which can be solved explicitly for x_{n+1}:

$$x_{n+1} = g^{-1}(f(x_n)) := F(x_n), \quad n \in \mathbf{Z}_+. \qquad (4)$$

Some dynamical properties of the one-dimensional map F can be translated to those of equation (2), for arbitrary positive values of the parameter μ. This is true, in particular, with regard to the *invariance* property and the *global stability* property.

Let $I \subset \mathbf{R}_+$ be a closed invariant under F interval. Set $X := C([-1, 0], \mathbf{R}_+)$, and $X_I := \{\varphi \in X : \varphi(s) \in I \ \forall s \in [-1, 0]\}$.

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Theorem 1. *The set X_I is invariant under equation (2). That is, with $\varphi \in X_I$ the corresponding solution $x(t) = x(t;\varphi)$ satisfies $x(t) \in I$ for all $t \geq 0$ and every $\mu \geq 0$. The interval I can be determined as $I := [m, M]$, where $M := g^{-1}(\max_{[0,\bar{x}]} f(x))$ and $m := g^{-1}(\min_{[0,M]} f(x))$. Moreover, for every $\varphi \in X$ there exists time $T = T(\varphi, \mu)$ such that $x(t) \in I$ for all $t \geq T$.*

The above interval $[m, M]$ provides an upper estimate for the largest possible invariant interval of the map $F = g^{-1} \circ f$. The actual maximal interval with the same properties as described by the Theorem is given by $I_0 = \cap_{n \geq 0} F^n(I)$ where the upper index n stands for the $n-$th iteration of the map F.

2.2 Global Stability. In the case when the interval I_0 degenerates into a single point, which is necessarily the fixed point $x = \bar{x}$, one has the globally attracting fixed point for the map F. The implication for the delay differential equation (2) is that it is globally asymptotically stable.

Theorem 2. *Assume that $x = \bar{x}$ is the globally attracting fixed point of the map F, that is* $\lim_{n\to\infty} F^n(x_0) = \bar{x}$ *for every $x_0 \geq 0$. Then for any initial function $\varphi \in X$ and every $\mu > 0$ the corresponding solution $x(t)$ of equation (2) satisfies* $\lim_{t\to\infty} x(t) = \bar{x}$.

Theorem 2 provides delay independent sufficient conditions for the global asymptotic stability in equation (1). Its proof is based on a detailed comparison of solutions of equation (2) and the dynamics of the map F (i.e., difference equation (3)). Informally stated, the dynamics of the latter dominates the dynamics of equation (1).

The delay dependent conditions for the global asymptotic stability in equation (1) make use of the following one-dimensional map

$$G(x) := \frac{1 - e^{-\alpha\tau}}{\alpha} \cdot f(x) + [\bar{x} - \frac{1 - e^{-\alpha\tau}}{\alpha} \cdot f(\bar{x})], \quad (5)$$

where $\alpha := \inf\{\frac{g(x)-g(\bar{x})}{x-\bar{x}}, x \in I\}$ is assumed to be positive. When $g(x)$ is differentiable the quantity $\beta := \inf\{g'(x), x \in I\}$ can be used as an approximation to α.

Theorem 3. *Suppose that $x = \bar{x}$ is the globally attracting fixed point of the map G. Then the constant solution $x(t) = \bar{x}$ of equation (1) is globally asymptotically stable.*

3. Conclusion

Sufficient conditions for the global asymptotic stability in the infinite-dimensional nonlinear dynamical system defined by equation (1) are given in terms of global attractivity in specially constructed one-dimensional maps. Theorem 2 provides delay independent conditions while Theorem 3 gives conditions explicitly dependent on the delay τ as well as the nonlinearities f and g involved. The conditions of Theorem 3 are generally more strong than those of Theorem 2; this can be easily seen for the particular case of linear function $g(x), g(x) = g(\bar{x}) + \alpha(x - \bar{x})$. Because of the comparison with one-dimensional maps both sets of conditions are easily verifiable in practice, e.g. numerically. Theorem 2 also includes the results on global stability of equation (1) from [1] as partial cases. It also indicates that they all are basically a consequence of the global attractivity of the one-dimensional map (4).

References

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