Smooth Exponential Stabilization of Nonholonomic Systems via time-varying feedback

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The problem of global exponential stabilization of a class of nonholonomic systems via smooth time-varying control is addressed. It is shown that smooth aperiodic time-varying controls for globally exponentially stabilizing a nonholonomic system can be easily synthesized if the system is augmented with some auxiliary state(s). The proposed method proves to be effective for a wide class of nonholonomic systems including the chained form system, multiple chained form system, power form system, Brockett system, etc. Simulation examples are introduced to demonstrate the effectiveness of the method.

1 Introduction

Controlling nonholonomic mechanical systems has attracted much attention recently and has become a very active research area. As pointed in an early paper of Brockett [6], a nonholonomic system with more degrees of freedom than controls can not be asymptotically stabilized by any continuous pure-state feedback. In order to circumvent this obstruction, three main research directions have been followed. The first solution consists of using discontinuous feedback ,the second one consists of using time-varying continuous feedback, and the third one consists of middle strategies, i.e., discontinuous and time-varying strategies.

The first possibility was first proposed by [4] and then further discussed by [5] and [7] to make a kinematic model of three-dimensional robot globally converge to a given configuration. This system is equivalent to a three-dimensional chained form. In [1] an elegant discontinuous coordinate transformation approach is proposed for solving the exponential stabilization problem for a class of nonholonomic systems. The designed control strategy is discontinuous only on a superplane of the state space.

The second possibility was first proposed and investigated in [13] for the stabilization of a unicycle-type vehicle whose equations can be converted into a three dimensional chained system. Since then this kind of control designs has drawn much attention of many authors [11, 12, 14, 16]. In [8], Coron established some general existence theorems which basically state that any driftless controllable system (such as the chained system) can be asymptotically stabilized, eventually in finite time, by means of time-varying continuous feedback. However, these time varying control laws produce very slow convergence of system states. As so far, there is no results on exponential stabilization of nonholonomic systems by using smooth time varying feedback.

In this paper, we restrict our attention to smooth time-varying control of nonholonomic systems. We show for the first time that smooth aperiodic time-varying controls for exponentially stabilizing a nonholonomic system can be easily synthesized if the system is augmented with some auxiliary state(s). By comparison with other existing (continuous and discontinuous) time-varying control results, besides the simplicity of the design procedure, the trajectory of system states designed by our control strategy avoids the zig-zag path and is thus more reasonable, and the convergence exponents can be evaluated easily and assigned arbitrarily. The method also overcomes the drawback of discontinuity of the existing pure-state feedback approaches. The proposed method proves to be effective for a wide class of nonholonomic systems including the chained form system, multiple chained form system, power form system, Brockett system, etc. Simulation examples are introduced to demonstrate the effectiveness of the method.

2 Stabilization of the chained form system

The modelling equations of a number of nonholonomic systems, e.g., unicycle-type carts, car-like vehicles with trailer, etc., can be converted into canonical chained form control equations (see, e.g., [11]).

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ &\vdots \\ \dot{x}_i &= x_{i-1}u_1, \qquad i = 3,\cdots,n \end{aligned} \tag{1}$$

Before proceeding to our main result of this section we in-

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troduce some preliminary lemmas.

Lemma 1:[15] Consider the linear time-varying system

$$\dot{x} = (A_0 + A_1(t))x \tag{2}$$

where $x \in R^n$ is the state vector of the system. Suppose $A_0 \in R^{n\times n}$ is a Hurwitz matrix, $A_1(t) \in R^{n\times n}$ is smooth in t and satisfies

1) $\int_0^\infty ||A_1(t)||\mathrm{d}t < \infty$,

2) $\lim_{t\to\infty} A_1(t) = 0$.

Then the state $x(t)$ of system (2) is uniformly exponentially stable.

Similarly we can establish the following lemma.

Lemma 2: Consider the linear time-varying control system

$$\dot{x} = (A_0 + A_1(t))x + (B_0 + B_1(t))u \tag{3}$$

with $x \in R^n$, $u \in R^m$. Suppose (A_0, B_0) is a stabilizable pair, $A_1(t)$ and $B_1(t)$ satisfy

1) $\int_0^\infty ||A_1(t)||\mathrm{d}t < \infty, \quad \int_0^\infty ||B_1(t)||\mathrm{d}t < \infty$

2) $\lim_{t\to\infty} A_1(t) = 0, \quad \lim_{t\to\infty} B_1(t) = 0$

then there exists a state feedback $u = -Kx$ which make the closed system be uniformly exponentially stable.

Theorem 1: The chained form system (1) can be globally exponentially stabilized by the following smooth time-varying state feedback control:

$$u_1(t) = e^{-\beta t} f(t) \tag{4}$$
$$u_2(t) = -k_2x_2 - \cdots - k_nx_ne^{(n-2)\beta t} \tag{5}$$

where $k_i \in R, i = 2,\cdots,n$ are some constants,

$$f(t) = f_0 + f_1(t) = f_0 + Ce^{-(\alpha-\beta)t} \tag{6}$$

where f_0, C, α, β are some non-zero real constants, and $\alpha > \beta > 0$.

Proof: We prove the theorem by a constructive method. Let us first consider how to design the control law for u_1. A key idea of this paper is to introduce an auxiliary state $x_0(t)$ into the system such that

$$\dot{x}_0 = x_1.$$

Then the states x_0 and x_1 compose a linear subsystem as follows

$$\begin{aligned} \dot{x}_0 &= x_1, \\ \dot{x}_1 &= u_1. \end{aligned} \tag{7}$$

A simple stabilizer for this subsystem can be designed as

$$u_1 = -k_0x_0 - k_1x_1 \tag{8}$$

where $k_0, k_1 > 0$. If we choose such k_0, k_1 that $k_1^2 - 4k_0 > 0$, then the system (7) has two negative real eigenvalues $-\alpha < 0, -\beta < 0$. Suppose that, without loss of generality, $\alpha > \beta$. The solution of the system can be then written as

$$x_0(t) = c_0e^{-\alpha t} + c_1e^{-\beta t} \tag{9}$$
$$x_1(t) = -\alpha c_0e^{-\alpha t} - \beta c_1e^{-\beta t} \tag{10}$$

where c_0, c_1 are some constants determined by initial values of the system states:

$$c_0 = \frac{-\beta x_0(0) - x_1(0)}{\alpha - \beta}, \tag{11}$$
$$c_1 = \frac{\alpha x_0(0) + x_1(0)}{\alpha - \beta}. \tag{12}$$

Let z denote the slower mode of the solution of the subsystem (7), i.e.,

$$z(t) = e^{-\beta t}. \tag{13}$$

With this notation the equations (9) and (10) can be rewritten as

$$x_0(t) = z(t)(c_0e^{-(\alpha-\beta)t} + c_1), \tag{14}$$
$$x_1(t) = -z(t)(\alpha c_0e^{-(\alpha-\beta)t} + \beta c_1). \tag{15}$$

The control law for u_1 can be also rewritten as

$$u_1 = z(t) f(t) \tag{16}$$

with

$$f(t) = f_0 + f_1(t) \tag{17}$$

where

$$f_0 = (k_1\beta - k_0)c_1, \tag{18}$$
$$f_1(t) = (k_1\alpha - k_0)c_0e^{-(\alpha-\beta)t}. \tag{19}$$

Note that $k_0 = \alpha\beta$, $k_1 = \alpha + \beta$. So we have

$$f_0 = ((\alpha+\beta)\beta - \alpha\beta)\frac{\alpha x_0(0) + x_1(0)}{\alpha - \beta} = \beta^2\frac{\alpha x_0(0) + x_1(0)}{\alpha - \beta}.$$

We can always choose an initial value of the auxiliary state $x_0(0)$ such that $f_0 \neq 0$.

Now we consider the design procedure for u_2. Introduce the following time-varying coordinate transformation:

$$\begin{aligned} y_2 &= x_2 \\ y_3 &= \tfrac{x_3}{z} \\ y_4 &= \tfrac{x_4}{z^2} \\ &\vdots \\ y_n &= \tfrac{x_n}{z^{n-2}} \end{aligned} \tag{20}$$

Note that $\dot{z}(t) = -\beta z(t)$. So we obtain

$$\begin{aligned} \dot{y}_2 &= u_2 \\ \dot{y}_3 &= f(t)y_2 + \beta y_3 \\ \dot{y}_4 &= f(t)y_3 + 2\beta y_4 \\ &\vdots \\ \dot{y}_n &= f(t)y_{n-1} + (n-2)\beta y_n \end{aligned} \tag{21}$$

Define a new state vector $y = [y_2, \cdots, y_n]^T$. The system (21) can be rewritten in the matrix form:

$$\dot{y} = (A_0 + A_1(t))y + Bu_2 \tag{22}$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ f_0 & \beta & 0 & \cdots & 0 & 0 \\ 0 & f_0 & 2\beta & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & f_0 & (n-2)\beta \end{bmatrix}, \tag{23}$$

$$A_1(t) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ f_1(t) & 0 & 0 & \cdots & 0 & 0 \\ 0 & f_1(t) & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & f_1(t) & 0 \end{bmatrix}, \tag{24}$$

$$B = [1\ 0\ \cdots\ 0]^T. \tag{25}$$

It is easy to see $A_1(t)$ satisfies the condition

$$\int_0^\infty \|A_1(t)\| \mathrm{d}t < \infty, \qquad \text{and} \qquad \lim_{t\to\infty} A_1(t) = 0.$$

So according to Lemma 2 the system (22) can be stabilized by a state feedback

$$u_2 = -Ky$$

if (A_0, B) is stabilizable, where

$$K = [k_2,\ \cdots,\ k_n].$$

Straightforward calculation shows that the rank of matrix $[B\ A_0B\ \cdots\ A_0^{n-2}B]$ is equal to $n-1$ if and only if $f_0 \neq 0$. Since $f_0 \neq 0$ can be always guaranteed as we have noted before, (A_0, B) is controllable and the spectrum of matrix $A_0 - BK$ can be assigned arbitrarily through the state feedback gain matrix K. By Lemma 1, the state $y(t)$ is uniformly exponentially stable if $A_0 - BK$ is arranged as a Hurwitz matrix. Clearly, when $y(t)$ is uniformly exponentially stable, by (20), the states $x_i(t), i = 2, \cdots, n$ of the original system are also uniformly exponentially stable. The theorem is thus proved. Q.E.D

Remark 1:With the help of the auxiliary x_0 we have avoid the problem of uncontrollability of the second subsystem on the superplane $x_1(0) = 0$. Indeed, if u_1 were designed simply as [1]

$$u_1 = -\beta x_1,$$

then we would have

$$x_1(t) = x_1(0)e^{-\beta t}.$$

Under the coordinate transformation

$$\begin{aligned} y_2 &= x_2 \\ y_3 &= \tfrac{x_3}{z} \\ y_4 &= \tfrac{x_4}{z^2} \\ &\vdots \\ y_n &= \tfrac{x_n}{z^{n-2}} \end{aligned}$$

where $z = e^{-\beta t}$ (note that this transformation is different from that used by [1], where instead of $e^{-\beta t}$ the author of [1] used the first state x_1 to "dilate" the coordinates), we would obtain a LTI subsystem

$$\dot{y} = Ay + Bu_2$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ x_1(0) & \beta & 0 & \cdots & 0 & 0 \\ 0 & x_1(0) & 2\beta & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & x_1(0) & (n-2)\beta \end{bmatrix},$$
$$B = [1\ 0\ 0\ \cdots\ 0]^T$$

It is obvious that the system is uncontrollable on the superplane $x_1(0) = 0$.

We can extend the result of Theorem 1 to the following multiple chained form system with single generator [2]

$$\begin{array}{ccc} & \dot{x}_1 = u_1, & \\ \dot{x}_2^{(1)} = u_2^{(1)} & \cdots & \dot{x}_2^{(m)} = u_2^{(m)}, \\ \dot{x}_3^{(1)} = x_2^{(1)}u_1 & \cdots & \dot{x}_3^{(m)} = x_2^{(m)}u_1 \\ \cdots & \cdots & \cdots \\ \dot{x}_{n_1-1}^{(1)} = x_{n_1-2}^{(1)}u_1 & \cdots & \dot{x}_{n_m-1}^{(m)} = x_{n_m-2}^{(m)}u_1, \\ \dot{x}_{n_1}^{(1)} = x_{n_1-1}^{(1)}u_1 & \cdots & \dot{x}_{n_m}^{(m)} = x_{n_m-1}^{(m)}u_1. \end{array} \tag{26}$$

Theorem 2: The multiple chained form system can be globally exponentially stabilized by the following smooth time-varying state feedback control:

$$u_1(t) = e^{-\beta t} f(t) \tag{27}$$

$$\begin{aligned} u_2^{(i)}(t) &= -k_2^{(i)}x_2^{(i)} - k_3^{(i)}x_3^{(i)}e^{\beta t} - \cdots \\ &\quad -k_{n_i}^{(i)}x_{n_i}^{(i)}e^{(n_i-2)\beta t}, i = 1, \cdots, m \end{aligned} \tag{28}$$

where

$$f(t) = f_0 + f_1(t) = f_0 + Ce^{-(\alpha-\beta)t} \tag{29}$$

and f_0, C, α, β are some non-zero real constants, $\alpha > \beta > 0$.

Simulation with 4-dimensional chained form system has been conducted. The initial condition is chosen as

$$(x_1(0), x_2(0), x_3(0), x_4(0)) = (0, -1, 0.5, 0.5).$$

Note that the state feedback control law of [1] can not be directly applied because the control law for u_2 given in [1] is undefined (discontinuous) on the superplane $x_1(0) = 0$. Now apply our method. Select $k_0 = 3, k_1 = 4, x_0(0) = 2.0$. Then $\alpha = 3, \beta = 1, f_0 = 3.0$. The gain vector is synthesized by using LQ optimal control design method with the optimality criterion

$$J = \int_0^\infty [y^T(t)Qy(t) + u_2^T R u_2]dt \tag{30}$$

where the weight matrices Q, R are chosen as $R = 16$ and

$$Q = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Then we obtain the gain vector

$$K = [k_2\ k_3\ k_4] = [7.51\ 9.37\ 5.91].$$

This gain vector assigns the eigenvalues of the matrix $A_0 - BK$ at $\lambda_1 = -2.23, \lambda_{2,3} = -1.14 \pm 1.08j$. Simulation results are shown in Fig. 1 to 3. Note that all the state and input variables are bounded and (exponentially) converge to zero. By comparison with other existing (continuous and discontinuous) time-varying control results, the trajectory of the system states designed by our control strategy avoids the zig-zag path and is thus more reasonable.

3 Stabilization of the Brockett system

Brockett system (or Brockett intergrator) described by

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= x_2 u_1 - x_1 u_2 \end{aligned} \tag{31}$$

is a well known nonholonomic system given by Brockett in his famous work [6]. Since the Brockett's necessary condition for stabilization by pure state feedback was proposed, controlling this system has become a benchmark problem in the field of nonlinear system control. The problem has been considered by many authors and, in particular, the exponential stabilization problem has been solved by using discontinuous state feedback law [3]. In this section we will show that the exponential stabilization problem for Brockett system can be alternatively solved by using a simple smooth time-varying state feedback approach.

Using Lemma 2 and the design procedure given in the previous section we can prove the following result.

Theorem 3: The Brockett system (31) can be globally exponentially stabilized by the following smooth time-varying state feedback control:

$$u_1(t) = e^{-\beta t} f(t), \tag{32}$$

$$u_2(t) = -k_2 x_2 - k_3 x_3 e^{\beta t} \tag{33}$$

with

$$f(t) = f_0 + f_1(t) = f_0 + Ce^{-(\alpha-\beta)t} \tag{34}$$

where f_0, C, α, β are some non-zero real constants, and $\alpha > \beta > 0$.

Simulation for controlling the Brockett system has been conducted. The initial states was chosen as

$$(x_1(0), x_2(0), x_3(0)) = (0, 1, 1),$$

and the initial value of the auxiliary state was chosen as $x_0(0) = 2$. For designing the control law u_1 we took $k_0 = 3, k_1 = 4$. Then we have two eigenvalues of the subsystem composed of (x_0, x_1) as $\alpha = 3, \beta = 1$. Using (18) we have $f_0 = c_1 = 3$. The gain vector K for the control u_2 is synthesized by using the optimality index (26) with weight matrices as

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \quad R = 16.$$

Then we obtain the gain vector $K = [k_2\ k_3] = [1.00\ 1.17]$. Simulation results are shown in Fig. 4 and 5. Note that all the state and input variables are bounded and (exponentially) converge to zero.

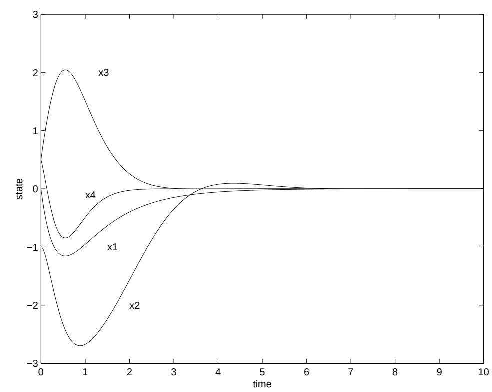

Figure 1: Time responses of state variables of the single chain system.

4 Generalization and discussion

Let us consider nonholonomic systems which are subject to the following generalized form

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= (\sum_{k=1}^{n-1} A^{(k)} x_1^{k-1}) x_2 u_1 + B f(x_1) u_2 \end{aligned} \tag{35}$$

where $u_1, u_2 \in R$ are control inputs, $x_1 \in R$, $x_2 = [x_{2,1}\ \cdots\ x_{2,n-1}]^T \in R^{n-1}$ are system states, $f(x_1) : R \mapsto R^{n-1}$ is a vector function of the form:

$$f(x_1) = [1\ x_1\ x_1^2\ \cdots\ x_1^{n-2}]^T, \tag{36}$$

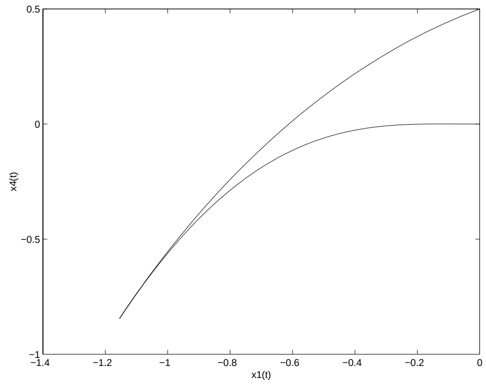

Figure 2: Path of $(x_1(t), x_4(t))$ of the single chain system.

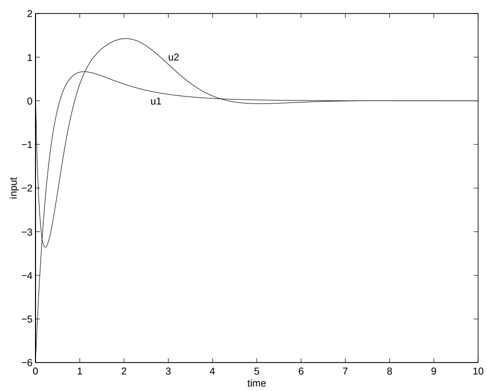

Figure 3: Control inputs for the single chain system.

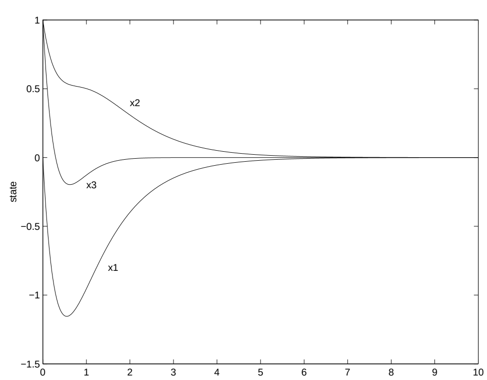

Figure 4: Time responses of state variables of the Brockett system.

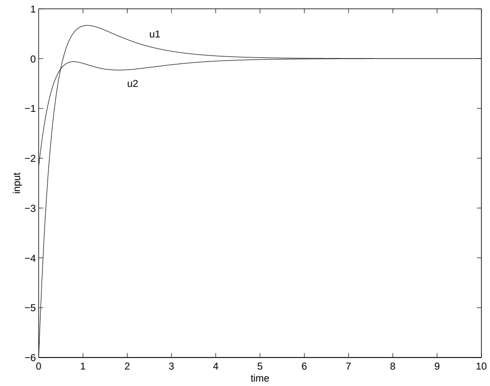

Figure 5: Control inputs for the Brockett system.

matrices $A^{(k)}, k = 1, \cdots, n-1$ and B are assumed to be subject to the following structure:

$$A^{(k)} = \begin{bmatrix} a^{(k)}_{1,1} & a^{(k)}_{1,2} & \cdots & \cdots & \cdots & \cdots & a^{(k)}_{1,n-1} \\ \vdots & & & & & & \vdots \\ a^{(k)}_{k+1,1} & a^{(k)}_{k+1,2} & \cdots & \cdots & \cdots & \cdots & a^{(k)}_{k+1,n-1} \\ 0 & a^{(k)}_{k+2,2} & \cdots & \cdots & \cdots & \cdots & a^{(k)}_{k+2,n-1} \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & a^{(k)}_{n-1,n-1-k} & \cdots & a^{(k)}_{n-1,n-1} \end{bmatrix},$$

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,n-1} \\ 0 & b_{2,2} & b_{2,3} & \cdots & b_{2,n-1} \\ 0 & 0 & b_{3,3} & \cdots & b_{3,n-1} \\ & & & \cdots & \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & b_{n-1,n-1} \end{bmatrix}.$$

It is clear that the chained form system, the Brockett system and the power form system [10] are special cases of system (35). As shown in preceding sections, by augmenting the system with some auxiliary state x_0 we can design the control law for u_1 as

$$u_1(t) = z(t) f(t)$$

where $z(t)$ and $f(t)$ are defined by (13) and (17) respectively. The time response of $x_1(t)$ is

$$\begin{aligned} x_1(t) &= -z(t)(\alpha c_0 e^{-(\alpha-\beta)t} + \beta c_1) \\ &= z(t) g(t). \end{aligned}$$

Then the time-varying transformation (20) can be used, i.e.,

$$\begin{aligned} y_2 &= x_{2,1}, \\ y_3 &= x_{2,2}/z, \\ &\vdots \\ y_n &= x_{2,n-1}/z^{n-2}. \end{aligned} \tag{37}$$

Denote $y = [y_2 \ \cdots \ y_n]^T$, we have

$$\dot{y} = (A_0 + A_1(t))y + (B_0 + B_1(t))u_2. \tag{38}$$

It can be verify that

$$\int_0^\infty \|A_1(t)\| \mathrm{d}t < \infty, \ \int_0^\infty \|B_1(t)\| \mathrm{d}t < \infty$$

and

$$\lim_{t\to\infty} A_1(t) = 0, \;\; \lim_{t\to\infty} B_1(t) = 0.$$

So according to Lemma 2 the system (38) can be stabilized by a state feedback

$$u_2 = -Ky$$

if (A_0, B_0) is controllable. As we have shown in preceding sections, the controllability of (A_0, B_0) can be usually guaranteed by choosing some appreciate initial value for the auxiliary state x_0.

Remark 2: The method developed in this paper can be applied to systems which belong to a more generalized class beyond (35). First, one can add some drift terms both in the generator subsystem with input u_1 and in the second subsystem with input u_2, an example of such systems is the extended power form system discussed in [9]. Second, each scalar state variable x_1, $x_{2,i}, i = 1, \cdots, n-1$ in (35) and (37) can be replaced with a sub-vector of states. Note that in the second case the structure of $A^{(k)}$ and B should be modified.

5 Conclusion

In this paper we have considered the problem of exponential stabilization of a class of nonholonomic systems via smooth time-varying control. We have shown that smooth aperiodic time-varying controls for exponentially stabilizing a nonholonomic system can be easily synthesized if the system is augmented with some auxiliary state(s). By comparison with other existing (continuous and discontinuous)time-varying control results, the main advantages of the proposed method are its simplicity and the exponential converging rate. Moreover, trajectory of system states designed by our control strategy avoids the zig-zag path (oscillatory behavior) which is usually associated with time-varying control. The method also overcomes the discontinuity of the existing pure-state feedback approaches.

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