# Robust H$_\infty$ Control of Discrete-Time Linear Systems with Delayed State and Frobenius Norm-Bounded Uncertainties


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**ABSTRACT:** The robust H$_\infty$ control problem is considered in this paper for discrete-time linear systems with delayed state and subject to Frobenius norm-bounded parameter uncertainties, and attention is focused on the design of memoryless state feedback controllers. A design procedure for the robust H$_\infty$ controllers is developed based on the linear matrix inequality (LMI) approach.


## 1. INTRODUCTION

Control of dynamical delay systems, a subject of great practical and theoretical importance, has received considerable attention for decades. In the past few years, the problem of designing robust controllers for uncertain time-delay systems has received increasing attention, and a number of results have been reported (Choi and Chung, 1997; Yu *et al.*, 1996). These results were developed for continuous-time systems. In practice, the use of digital hardware invariably requires that the designing of robust controller be implemented in a discrete time frame. Therefore, it is necessary to develop corresponding discrete-time results.

The analysis and synthesis problems of discrete-time systems with delay have been studied (Kapila and Haddad, 1998; Song *et al.*, 1999). They, however, did not consider the parameter uncertainty in the system model. The robust H$_\infty$ control problem for a class of systems with norm-bounded parameter uncertainty has been considered by Song and Kim (1998), but the uncertainty is limited to the state matrix $A$ only.

This paper considers the robust H$_\infty$ control problem for a class of discrete-time uncertain systems with delayed state and is also a generalization of Song and Kim (1998) to the uncertainties in the all matrices of the state equation. The uncertainties are assumed to be time-varying Frobenius norm-bounded. Attention is focused on the design of linear memoryless state feedback controllers for the above uncertain systems such that both robust stability and a prescribed H$_\infty$ disturbance attenuation is achieved despite of the uncertainties. An LMI-based robust controller design procedure is proposed.

## 2. PROBLEM FORMULATION

Consider the following linear discrete-time uncertain system with delayed state:

$$\begin{aligned}x(k+1) &= \bar{A}(k)x(k) + \bar{A}_d(k)x(k-d) + \bar{B}(k)u(k) + \bar{B}_1(k)w(k)\\ &= (A + H\Delta(k)E_1)x(k) + (A_d + H\Delta(k)E_d)x(k-d)\\ &\quad + (B + H\Delta(k)E_2)u(k) + (B_1 + H\Delta(k)E_3)w(k)\\ &= \left(A + \sum_{i=1}^{r}\sum_{j=1}^{s} H_i\Delta_{ij}(k)E_{1j}\right)x(k) + \left(A_d + \sum_{i=1}^{r}\sum_{j=1}^{s} H_i\Delta_{ij}(k)E_{dj}\right)x(k-d)\\ &\quad + \left(B + \sum_{i=1}^{r}\sum_{j=1}^{s} H_i\Delta_{ij}(k)E_{2j}\right)u(k) + \left(B_1 + \sum_{i=1}^{r}\sum_{j=1}^{s} H_i\Delta_{ij}(k)E_{3j}\right)w(k),\end{aligned} \qquad (1)$$

$$z(k) = Cx(k) + Du(k) + D_1 w(k), \qquad (2)$$

where $x(k)\in R^n$ is the state, $u(k)\in R^m$ is the control input, $w(k)\in R^p$ is the disturbance input which is from $l_2[0,\infty)$, $z(k)\in R^q$ is the controlled output, $A, A_d, B, B_1, C, D, D_1$, $E_{ij}$ and $H_i$ are known real constant matrices of appropriate dimensions that describe the nominal system,

$$H = [H_1 \quad H_2 \quad \cdots \quad H_r],$$
$$E_l = [E_{l1}' \quad E_{l2}' \quad \cdots \quad E_{lr}']', \qquad l = 1, 2, 3, d,$$

and

$$\Delta(k) = \begin{bmatrix}\Delta_{11}(k) & \Delta_{12}(k) & \dots & \Delta_{1s}(k)\\ \Delta_{21}(k) & \Delta_{22}(k) & \cdots & \Delta_{2s}(k)\\ \vdots & \vdots & \ddots & \vdots\\ \Delta_{r1}(k) & \Delta_{r2}(k) & \cdots & \Delta_{rs}(k)\end{bmatrix}$$

is an unknown real time-varying matrix satisfying

$$\sum_{i=1}^{r}\sum_{j=1}^{s}\|\Delta_{ij}(k)\|^2 \le 1, \quad \forall k. \qquad (3)$$

where $\|\cdot\|$ denotes the 2-norm of matrix. For the notational simplicity, in the sequel, we will denote $\Delta(k), \Delta_{ij}(k)$, $\bar{A}(k), \bar{A}_d(k), \bar{B}(k)$ and $\bar{B}_1(k)$ by $\Delta, \Delta_{ij}$, $\bar{A}, \bar{A}_d, \bar{B}$ and $\bar{B}_1$, respectively, whenever no confusion arises regarding the time dependence of these quantities.

Using the Frobenius norm, (3) can be denoted by $\|\bar{\Delta}\| \le 1$, where the matrix $\bar{\Delta}$ is given as

$$\bar{\Delta} = \begin{bmatrix}\|\Delta_{11}\| & \|\Delta_{12}\| & \cdots & \|\Delta_{1s}\|\\ \|\Delta_{21}\| & \|\Delta_{22}\| & \cdots & \|\Delta_{2s}\|\\ \vdots & \vdots & \ddots & \vdots\\ \|\Delta_{r1}\| & \|\Delta_{r2}\| & \cdots & \|\Delta_{rs}\|\end{bmatrix}. \qquad (4)$$

**Remark 1** The importance of the uncertainties of the form (3) has been illustrated by Lee *et al*. (1996) and Boukas and Shi (1998).

**Remark 2** The norm-bounded uncertainty is a special case of (3) when $r=1, s=1$. Furthermore, this paper is an extension of the work of Song and Kim (1998), where the uncertainties appear only in the $A$ matrix, to uncertainties in all matrices of the state equation.

A robust $H_\infty$ control problem for the system (1)-(2) will be considered as in the following definition.

**Definition 1** For a given constant $\gamma > 0$, the system (1)-(2) is said to be stabilizable with an $H_\infty$-norm bound $\gamma$ if there exists a state feedback control law $u(k) = Fx(k)$ such that for any admissible parameter uncertainty, the following conditions are satisfied:

(1) The closed-loop system is asymptotically stable;

(2) Subject to the assumption of the zero initial condition, the controlled output $z(k)$ satisfies

$$\|z(k)\|_2 \le \gamma \|w(k)\|_2 , \tag{5}$$

where $\|\cdot\|_2$ denotes the usual $l_2[0,\infty)$-norm. In this case, $u(k) = Fx(k)$ is said to be a robust $H_\infty$ controller of the system (1)-(2).

The objective of this paper is to derive the existence condition and to give a design procedure of the robust $H_\infty$ controllers. To this end, we first introduce the following lemma which will be used in the proof of our main results.

**Lemma 1** (Boukas and Shi, 1998): Given matrices $V = [V_1 \;\; V_2 \;\; \cdots \;\; V_r]$, $W = [W_1' \;\; W_2' \;\; \cdots \;\; W_s']'$ and a symmetric matrix $Q$ of appropriate dimensions, then

$$Q + V\Delta W + W'\Delta' V' = Q + \sum_{i=1}^{r}\sum_{j=1}^{s} V_i \Delta_{ij} W_j + \sum_{i=1}^{r}\sum_{j=1}^{s} W_j' \Delta_{ji}' V_i' < 0$$

for all $\Delta_{ij}, i = 1, 2, \cdots, r,\ j = 1, 2, \cdots, s$ satisfying (3), if and only if there exists some $\varepsilon > 0$ such that

$$Q + \varepsilon VV' + \varepsilon^{-1} W'W < 0 .$$

## 3. MAIN RESULTS

The closed-loop system of the system (1)-(2) with $u(k) = Fx(k)$ can be written as follows:

$$\begin{aligned} x(k+1) &= \overline{A}_c x(k) + \overline{A}_d x(k-d) + \overline{B}_1 w(k), \\ z(k) &= \overline{C}_c x(k) + \overline{D}_1 w(k), \end{aligned} \tag{6}$$

where

$$\begin{aligned} \overline{A}_c &= A_c + H_1 \Delta E_c = A + BF + H_1 \Delta (E_1 + E_2 F), \\ \overline{C}_c &= C_c + H_2 \Delta E_c = C + DF + H_2 \Delta (E_1 + E_2 F), \end{aligned}$$

The following theorem presents a sufficient condition for the existence of the robust $H_\infty$ controllers.

**Theorem 1** For a given constant $\gamma > 0$, there exist robust $H_\infty$ controllers $u(k) = Fx(k)$ for the system (1)-(2) if there exist positive definite matrices $X$, $T$ and a positive scalar $\varepsilon$ such that the following linear matrix inequality holds.

$$\begin{bmatrix} -X+T & 0 & 0 & XA_c' & XC_c' & XE_c' \\ 0 & -T & 0 & XA_d' & 0 & XE_d' \\ 0 & 0 & -\gamma^2 I & B_1' & D_1' & E_3' \\ A_c X & A_d X & B_1 & -X + \varepsilon HH' & 0 & 0 \\ C_c X & 0 & D_1 & 0 & -I & 0 \\ E_c X & E_d X & E_3 & 0 & 0 & -\varepsilon I \end{bmatrix} < 0 . \tag{7}$$

The following theorem gives the main result of this paper.

**Theorem 2** For a given constant $\gamma > 0$, the system (1)-(2) is stabilizable with an $H_\infty$-norm bound $\gamma$ if there exist symmetric positive definite matrices $X$ and $T$, a matrix $Y$ and a scalar $\varepsilon > 0$, satisfying the following LMI:

$$\begin{bmatrix} -X+T & 0 & 0 & (AX+BY)' & (CX+DY)' & (E_1X+E_2Y)' \\ 0 & -T & 0 & XA_d' & 0 & XE_d' \\ 0 & 0 & -\gamma^2 I & B_1' & D_1' & E_3' \\ AX+BY & A_d X & B_1 & -X + \varepsilon HH' & 0 & 0 \\ CX+DY & 0 & D_1 & 0 & -I & 0 \\ E_1X+E_2Y & E_d X & E_3 & 0 & 0 & -\varepsilon I \end{bmatrix} < 0 \tag{8}$$

Moreover, a robust $H_\infty$ controller is given by $u(k) = YX^{-1}x(k)$.

(8) is an LMI in the variables $\varepsilon, X, T$ and $Y$. Therefore, the robust $H_\infty$ control problem of the system (1)-(2) is reduced to an LMI feasibility problem. The latter is convex and can be effectively solved by existing LMI software. Furthermore, if the LMI (8) is feasible, then a robust $H_\infty$ controller can be constructed in terms of the feasible solution to this LMI.


## 4. ACKNOLOGEMENT

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## 5. REFERENCES


Boukas, E. K. and Shi, P.; $H_\infty$ control for discrete-time linear systems with Frobenius norm-bounded uncertainties. *Proc. of American Control Conference*, Philadelphia, Pennsylvania, 1998, pp 557-561.

Choi, H. H. and Chung, M. J.; An LMI approach to $H_\infty$ controller design for linear time-delay systems. *Automatica*, 1997, 33(4): 737-739.

Kapila, V. and Haddad, W. M.; Memoryless $H_\infty$ controllers for discrete-time systems with time delay. *Automatica*, 1998, 34(9): 1141-1149.

Lee, J. H., Kwon, W. H. and Lee, J. W.; Quadratic stability and stabilization of linear systems with Frobenius norm-bounded uncertainties. *IEEE Trans. on Automat. Control*, 1996, 41(3): 453-456.

Song, S.H., Kim, J.K., Yim, C.H., Kim, H.C.; $H_\infty$ control of discrete-time linear systems with time-varying delays in state. *Automatica*, 1999, 35: 1587-1597.

Song, S. H. and Kim, J. K.; $H_\infty$ control of discrete-time linear systems with norm-bounded uncertainties and time-delay in state. *Automatica*, 1998, 34: 137-139.

Yu, L., Chu, J. and Su, H.; Robust memoryless $H_\infty$ controller design for linear time-delay systems with norm-bounded time-varying uncertainty. *Automatica*, 1996, 32(12): 1759-1762.