STABILITY ANALYSIS OF PULSE-WIDTH-MODULATED FEEDBACK SYSTEMS

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Abstract

We present new Lyapunov and Lagrange stability results for pulse-width-modulated (PWM) feedback systems with linear plants. We consider the *non-critical* case, where the poles of the transfer function of the plant are all in the left-half of the complex plane and the *critical* case, where one pole is at the origin while the remaining poles are all in the left-half of the complex plane. For these systems we apply the Direct Method of Lyapunov to establish new and improved stability results. As in most existing results for PWM feedback systems obtained by the Lyapunov method, we employ *quadratic* Lyapunov functions in our analysis. However, in the proofs we make use of different majorizations, requiring hypotheses that differ significantly from those used in the existing results. Additionally, and perhaps more importantly, we incorporate into our results optimization procedures that improve our results significantly. We demonstrate the applicability and quality of our results by means of two specific examples that are identical to examples presented in the literature.

1 Introduction

Pulse-width-modulation has extensively been used in electronic and electrical systems including attitude control systems, adaptive control systems, signal processing, power control systems, modeling of neuron behavior, and the like. The classical example of pulse-width-modulated (PWM) control is the constant temperature oven suggested by Gouy in 1897 [1] while the most well-known modern application is the attitude control of satellites and space vehicles (see, e.g., [2]). In the latter it is usually required that power be modulated in an on-off fashion and that the control computer be time-shared, thus almost always necessitating the use of pulse-width modulation if anything more than simple relay control is desired. Indeed, PWM systems include some of the most important specific classes of practical nonlinear control systems (For additional applications of PWM feedback systems, see, e.g., [10], [13]). One advantage of PWM control is the simplicity of its realization: the control variable assumes only two or three values, say $+M, -M$ and 0, and hence, the control action is realized through the operation of a switch. One of the reasons for their wide applicability is that pulse-width modulators make it possible to process large signals with high efficiency and low sensitivity to noise. On the other hand, the rather unique and inherent *nonlinear* and *discontinuous* characteristics of pulse-width modulators give rise to challenges in the stability analysis of PWM feedback systems.

Stability results for PWM feedback systems have been established by a variety of methods (see, e.g., [3]–[9], [11], [12], and the survey by Tzafestas [14]). Most of these results constitute sufficient conditions and have inherent limitations (e.g., they may have limited applicability or they may be overly conservative).

In this paper we apply Lyapunov's Direct Method to establish new and improved stability results for PWM feedback systems with *linear* plants. As is the case in most of the existing results for PWM feedback systems obtained by the Lyapunov method, we also employ *quadratic* Lyapunov functions in our analysis. However, in the proofs we make use of different kinds of majorizations, resulting in hypotheses that differ significantly from those used in the existing results. Additionally, and perhaps more importantly, our results incorporate optimization procedures to determine maximal stability bounds for the parameters of the pulse-width modulator.

2 Mathematical Description of the Pulse-Width Modulator

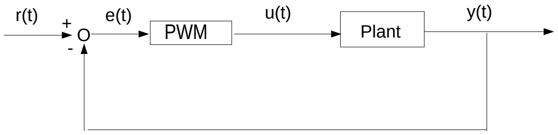

Figure 1: PWM feedback system

The PWM feedback system to be considered is shown in Fig. 1. The output of the pulse-width modulator is given by

$$u(t) = m(e(t)) = \begin{cases} M \cdot sgn(e(kT)), & t \in [kT, kT+T_k], \\ 0, & \text{otherwise} \end{cases}, \tag{2.1}$$

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for $k = 0, 1, 2, \cdots$, where the pulse width T_k and the signum function are given by

$$T_k = \begin{cases} \beta|e(kT)|, & |e(kT)| \le \frac{T}{\beta} \\ T, & |e(kT)| > \frac{T}{\beta} \end{cases}, \quad sgn(\sigma) = \begin{cases} 1, & \sigma > 0 \\ 0, & \sigma = 0 \\ -1, & \sigma < 0 \end{cases}.$$

The sampling period T, the amplitude of the pulse M, and β are all assumed to be constant.

The pulse-width modulator yields piecewise continuous outputs, as is illustrated in Fig. 2. The amplitude of the pulses is fixed while their duration varies, depending on the magnitude of the sample signal ($e(kT)$).

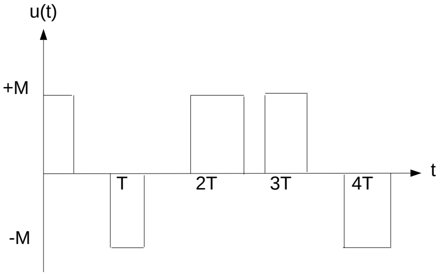

Figure 2: PWM feedback system

3 PWM Systems with Hurwitz Stable Plant

We assume that the plant is linear and has a state-space representation of the form

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}, u \in \mathbb{R}$, and A, B and C are real matrices of appropriate dimensions. Also, in the present section, we assume that A is *Hurwitz stable.*

Combining (2.1) and (3.1), the PWM feedback system of Fig. 1 assumes the form (with $r(t) \equiv 0$)

$$\dot{x}(t) = \begin{cases} Ax(t) - BMsgn(Cx(kT)), & t \in [kT, kT+T_k] \\ Ax(t), & t \in (kT+T_k,\ kT+T) \end{cases}. \tag{3.2}$$

We note that the trivial solution $x_e = 0$ is an equilibrium of the PWM feedback system (3.2).

Remark 3.1 The above is a typical *discontinuous dynamical system* which contains discontinuous dynamics (the output of the pulse-width modulator $u(t)$), even though the state of system (3.2), $x(t)$, is continuous with respect to time t. The discontinuity in $u(t)$ gives rise to discontinuities in $\dot{x}(t)$. □

Over the time intervals $[kT, kT + T_k]$ and $[kT + T_k,\ kT + T], k = 0, 1, 2, \cdots$, equation (3.2) can be solved to yield the exact solution

$$x(t) = \begin{cases} e^{A(t-kT)}x(kT) - \int_{kT}^{t} e^{A(t-\tau)} d\tau BMsgn(Cx(kT)), \\ \qquad\qquad\qquad\qquad t \in [kT, kT+T_k] \\ e^{A(t-kT-T_k)}x(kT + T_k), \quad t \in [kT+T_k, kT+T] \end{cases}. \tag{3.3}$$

In particular, at $t = kT + T$, we have

$$\begin{aligned} x(kT + T) &= e^{A(T-T_k)}x(kT + T_k) \\ &= e^{AT}\Big(x(kT) - \int_0^{T_k} e^{-A\tau} d\tau BMsgn(Cx(kT))\Big). \end{aligned} \tag{3.4}$$

The above equation describes a nonlinear discrete-time system that specifies the states of the PWM feedback system (3.2) at discrete instants (kT, $k = 0, 1, 2, \cdots$). To simplify equation (3.4), we let

$$\begin{aligned} \tilde{x}(kT) &\triangleq -\int_0^{T_k} e^{-A\tau} d\tau BMsgn(Cx(kT)) \\ &= -M\beta W_{\tau_k} x(kT), \end{aligned} \tag{3.5}$$

where

$$\tau_k \triangleq \beta|Cx(kT)| \begin{cases} = T_k, & T_k < T \\ \ge T, & T_k = T \end{cases}, \tag{3.6}$$

and

$$W_{\tau_k} \triangleq \begin{cases} 0, & \tau_k = 0 \\ \frac{I - e^{-A\tau_k}}{\tau_k} A^{-1}BC, & \tau_k < T \\ \frac{I - e^{-AT}}{\tau_k} A^{-1}BC = \frac{T}{\tau_k} W_T, & \tau_k \ge T \end{cases}. \tag{3.7}$$

Equation (3.4) is then reduced to

$$x(kT + T) = e^{AT}\big(x(kT) + \tilde{x}(kT)\big) = e^{AT}(I - M\beta W_{\tau_k})x(kT),$$

$k = 0, 1, 2, \cdots$, where I is the identity matrix.

We recall that if A is Hurwitz stable, then e^{AT} is Schur stable. In this case, there exists a positive definite matrix $P = P^T$ such that

$$(e^{AT})^T P(e^{AT}) - P = -I. \tag{3.8}$$

Choosing the quadratic form Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}^+$, $V(x) = x^T Px$, we obtain for the first forward difference of V along the solutions of the discrete-time system (3.4), the expression

$$\begin{aligned} &\Delta V(x(kT)) \triangleq V(x(kT + T)) - V(x(kT)) \\ &= -x(kT)^T\Big(I + M\beta W_{\tau_k}^T(P - I) + M\beta(P - I)W_{\tau_k} \\ &\quad - M^2\beta^2 W_{\tau_k}^T(P - I)W_{\tau_k}\Big)x(kT). \end{aligned} \tag{3.9}$$

The following result follows now readily.

Theorem 3.1 Assume that the matrix A in (3.1) is Hurwitz stable. Then the trivial solution of the PWM feedback system (3.2) is *uniformly asymptotically stable in the large* whenever

$$M\beta < \inf_{\tau_k \in (0,T]} \frac{\lambda_m(G_1(\tau_k)) + \sqrt{\lambda_m(G_1(\tau_k))^2 + 4\lambda_M(G_2(\tau_k))}}{2\lambda_M(G_2(\tau_k))}, \tag{3.10}$$

where $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote the minimum and maximum eigenvalues of a real symmetric matrix, respectively, $G_1(\tau_k) \triangleq W_{\tau_k}^T(P - I) + (P - I)W_{\tau_k}$, $G_2(\tau_k) \triangleq W_{\tau_k}^T(P - I)W_{\tau_k}$, W_{τ_k} is given by (3.7), $0 < \tau_k < +\infty$, and P is given in (3.8).

Proof: Notice that when $\tau_k > T$, it is true that

$$\begin{aligned} & I+M\beta G_1(\tau_k)-M^2\beta^2 G_2(\tau_k) \\ = \; & I+M\beta\frac{T}{\tau_k}G_1(T)-M^2\beta^2\frac{T^2}{\tau_k^2}G_2(T). \qquad (3.11) \end{aligned}$$

Therefore, if we can show that the matrix $I+M\beta G_1(T)-M^2\beta^2 G_2(T)$ is positive definite for all $M\beta$ less than a certain value, say $\alpha > 0$, then in view of (3.11), the matrix $I + M\beta G_1(\tau_k) - M^2\beta^2 G_2(\tau_k)$ is positive definite for all $\tau_k > T$ and all $M\beta < \alpha$.

It can easily be verified that $M\beta$ satisfying (3.10) yields $\sup\limits_{\tau_k\in(0,T]} \lambda_M\big(- I - M\beta W_{\tau_k}^T(P - I) - M\beta(P - I)W_{\tau_k} + M^2\beta^2 W_{\tau_k}^T(P - I)W_{\tau_k}\big) < 0$. In particular, the matrix $I + M\beta G_1(T) - M^2\beta^2 G_2(T)$ is positive definite for all $M\beta$ under the present assumption. Since the matrix $I+M\beta\frac{T}{\tau_k}G_1(T)-M^2\beta^2\frac{T^2}{\tau_k^2}G_2(T)$ is continuous with respect to $\frac{T}{\tau_k}$, and tends to the identity matrix I as τ_k goes to ∞, we conclude that

$$\begin{aligned} \Theta_{M\beta} \overset{\Delta}{=} & \sup_{\tau_k\geq 0}\lambda_M\big(-I-M\beta W_{\tau_k}^T(P-I) \\ & -M\beta(P-I)W_{\tau_k}+M^2\beta^2 W_{\tau_k}^T(P-I)W_{\tau_k}\big) \qquad (3.12) \end{aligned}$$

is negative, where $\Theta_{M\beta}$ depends on the choice of $M\beta$.

It follows from (3.9) that $\Delta V(x(kT)) \leq -|\Theta_{M\beta}|\|x(kT)\|^2$. Therefore, $\Delta V(x(kT))$ is negative definite when $M\beta$ satisfies (3.10). It follows from the usual Lyapunov stability results (see, e.g., [6]) that the trivial solution of the discrete-time system (3.4) is uniformly asymptotically stable in the large.

Next, we obtain an estimate for $x(t)$ when $t\in[kT, kT{+}T)$. It is easily seen from the definition of T_k that $T_k \leq \beta|e(kT)| \leq \beta\|C\|\|x(kT)\|$. Since $\|e^{A\tau}\| \leq e^{\|A\|T}$ for all $\tau \in [0,T]$, it follows from (3.3) that

$$\|x(t)\| \leq (1 + M\beta\|C\|\|B\|)e^{\|A\|T}\|x(kT)\| \qquad (3.13)$$

when $t \in [kT, kT{+}T_k]$, and that

$$\|x(t)\| \leq (1 + M\beta\|C\|\|B\|)e^{2\|A\|T}\|x(kT)\| \qquad (3.14)$$

when $t \in (kT{+}T_k, kT{+}T)$. This implies that each state $x(t)$ in the interval $(kT, kT{+}T)$ is bounded by $c\|x(kT)\|$ where $c{>}0$ is a constant.

Therefore, $x(t)$ converges to the origin simultaneously with $x(kT)$. We conclude that the trivial solution of (3.2) is uniformly asymptotically stable in the large. □

Note that the trivial solution of (3.2) is uniformly asymptotically stable in the large for all $M\beta$ such that $\Theta_{M\beta}$ given in inequality (3.12) is negative. The above theorem gives an upper bound for such $M\beta$ (by (3.10)). To obtain the least conservative stability results by Theorem 3.1, we need to determine the largest upper bound of $M\beta$ such that $\Theta_{M\beta} < 0$ is satisfied for *all state representations* of (3.1). We will denote this value by $(M\beta)_{opt}$. In Remark 3.2 given below, we outline a procedure for computing an estimate of the optimal value of $M\beta$ such that $\Theta_{M\beta} < 0$ for a *given state representation.* We call this $(M\beta)^*_{opt}$. The values of $(M\beta)^*_{opt}$ for different but equivalent state representations will in general vary. In Remark 3.3 given below, we outline a procedure for determining an *estimate* of $(M\beta)_{opt}$ using the different values of $(M\beta)^*_{opt}$ obtained by employing different state representations of (3.1). We will denote the estimate of $(M\beta)_{opt}$ by $\overline{(M\beta)}_{opt}$.

Remark 3.2 To obtain $(M\beta)^*_{opt}$ for a given state representation, we proceed as follows. Let $m_0 > 0$ be such that $\Theta_{M\beta} < 0$ is true for all $M\beta < m_0$ (m_0 can be initialized by choosing, for example, the right side of (3.10)), and let

$$\begin{aligned} & \widetilde{G}_0(\tau_k) = I + m_0 G_1(\tau_k) - {m_0}^2 G_2(\tau_k), \\ & \widetilde{G}_1(\tau_k) = G_1(\tau_k) - 2m_0 G_2(\tau_k), \qquad (3.15) \\ & \widetilde{m}_0 = m_0. \end{aligned}$$

In order that $\Theta_{M\beta}$ given in (3.12) be negative, it is necessary that $I+M\beta G_1(\tau_k)-M^2\beta^2 G_2(\tau_k) = \widetilde{G}_0+(M\beta-m_0)\widetilde{G}_1(\tau_k)-(M\beta-m_0)^2 G_2(\tau_k)$ be positive definite. For this to be true, we obtain, using the same arguments as in the proof of Theorem 3.1 (see (3.11)-(3.12)), that $\Theta_{M\beta} < 0$ is true for all $M\beta$ such that

$$\begin{aligned} M\beta < \widetilde{m}_0 + \inf_{\tau_k\in(0,T]} & \Bigg(\frac{\lambda_m(\widetilde{G}_1(\tau_k))}{2\lambda_M(G_2(\tau_k))} \\ & + \frac{\sqrt{\lambda_m(\widetilde{G}_1(\tau_k))^2+4\lambda_m(\widetilde{G}_0(\tau_k))\lambda_M(G_2(\tau_k))}}{2\lambda_M(G_2(\tau_k))}\Bigg). \qquad (3.16) \end{aligned}$$

We repeat the above computation, replacing in (3.15) m_0 by the right hand side of (3.16) until the increment of m_0 is negligible. Set $(M\beta)^*_{opt}$ equal to the final value of m_0. □

In *implementing* the above procedure, we evaluate G_1 and G_2 at the points of a sufficiently fine partition, say $\{t_0 = 0 < t_1 < \cdots < t_N = T\}$, $0 < t_{j+1} - t_j < \delta$, $j = 0, 1, \cdots, N - 1$. The procedure outlined in Remark 3.2 is repeated for different partitions (involving decreasing values of δ), until no further improvements are realized.

Remark 3.3 To determine $\overline{(M\beta)}_{opt}$, we compute $(M\beta)^*_{opt}$ for different state representations, $\widetilde{A} = SAS^{-1}, \widetilde{B} = SB, \widetilde{C} = CS^{-1}$, where S is a nonsingular matrix. In doing so, we choose a set of nonsingular matrices S, say Ω, using a random generator (e.g., the rand command in MATLAB). An estimate of $(M\beta)_{opt}$, $\overline{(M\beta)}_{opt}$, can be determined by setting $\overline{(M\beta)}_{opt} = \max\limits_{S\in\Omega}(M\beta)^*_{opt}$. The above procedure is repeated, increasing the size of Ω, until no further improvements are realized. □

In Section 5, we present an algorithm to determine $\overline{(M\beta)}_{opt}$ based on Remarks 3.2 and 3.3 above.

Remark 3.4 If M is allowed to assume *negative values* (corresponding to positive feedback in Fig. 1), then similarly as above, we can obtain a *lower bound* for $M\beta$ given by

$$M\beta > \sup_{\tau_k \in (0,T]} \frac{\lambda_M(G_1(\tau_k)) - \sqrt{\lambda_M(G_1(\tau_k))^2 + 4\lambda_M(G_2(\tau_k))}}{2\lambda_M(G_2(\tau_k))},$$

where $G_1(\tau_k)$ and $G_2(\tau_k)$ are given in Theorem 3.1. $\square$

The next result is concerned with the Lagrange stability of the PWM feedback system (3.2).

Theorem 3.2 Assume that A in (3.1) is Hurwitz stable. Then the solutions of system (3.2) are uniformly ultimately bounded for *any choice* of M and β.

Proof: To show that the solutions of system (3.2) are uniformly ultimately bounded, we first verify that the solutions at discrete instants kT, $k = 0, 1, 2, \cdots$, are uniformly ultimately bounded.

The solutions of system (3.2) at $t = kT + T$ are given by

$$x(kT+T) = e^{AT}\Big(x(kT) + \tilde{x}(kT)\Big), \tag{3.17}$$

where $\tilde{x}(kT)$ given in (3.5) is bounded, since

$$\|\tilde{x}(kT)\| = \left\| \int_0^{T_k} e^{-A\tau} d\tau BM \right\| \leq TMe^{\|A\|T}\|B\|. \tag{3.18}$$

Choose P as in (3.8) and $V: \mathbb{R}^n \to \mathbb{R}^+$ as $V(x) = x^T Px$. Along the solutions of the discrete-time system (3.17) we have

$$\begin{aligned} & \Delta V(x(kT)) \triangleq V(x(kT+T)) - V(x(kT)) \\ \leq \; & -\|x(kT)\|^2 + 2TMe^{\|A\|T}\|B\|\|P-I\|\|x(kT)\| \\ & +(TMe^{\|A\|T}\|B\|)^2\|P-I\|. \end{aligned}$$

It is readily verified that $\Delta V(x(kT))$ is negative whenever $\|x(kT)\| > \Omega \triangleq TMe^{\|A\|T}\|B\|\Big(\|P-I\| + \sqrt{\|P-I\|^2 + \|P-I\|}\Big)$. If $\|x(kT)\| < \Omega$, we have that

$$\begin{aligned} & \|x(kT+T)\|^2 \leq \frac{V(x(kT)) + \Delta V(x(kT))}{\lambda_m(P)} \\ \leq \; & \frac{1}{\lambda_m(P)}\Big(\|P-I\|\Omega^2 + 2TMe^{\|A\|T}\|B\|\|P-I\|\Omega \\ & +(TMe^{\|A\|T}\|B\|)^2\|P-I\|\Big). \end{aligned}$$

From the usual Lagrange stability results it follows that $x(kT)$ is uniformly ultimately bounded.

For $t \in [kT, kT+T)$, the estimates given by (3.13) and (3.14) are still valid. Thus, $x(t)$ is uniformly bounded by $x(kT)$. Therefore, the solutions of system (3.2) are uniformly ultimately bounded. . $\square$

Remark 3.5 Datta [4] establishes ultimate boundedness results for PWM feedback systems which are based on the method of Murphy and Wu [8]. He shows that the solutions are ultimately bounded only when $M\beta$ is in a certain range. Our result stated in Theorem 3.2 is obviously more general. $\square$

4 PWM Systems with Linear Plants That Have One Pole at the Origin

In this section, we assume that the plant described by (3.1) has one and only one pole at the origin . Under this condition, there exists a nonsingular matrix $Q \in \mathbb{R}^{n\times n}$ such that $QAQ^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, QB = \begin{bmatrix} B_1 \\ b \end{bmatrix}, CQ^{-1} = [C_1 \; 1]$, where A_1 is Hurwitz stable. Letting $(x_1^T, x_2^T)^T = Qx$, where $x_1 \in \mathbb{R}^{n-1}$ and $x_2 \in \mathbb{R}$, system (3.1) can be rewritten as

$$\begin{cases} \dot{x}_1 = A_1x_1 + B_1u, \\ \dot{x}_2 = bu, \\ u = -m(C_1x_1 + x_2) \end{cases}. \tag{4.1}$$

We note that the trivial solution $(x_1^T, x_2^T)^T = (0^T, 0)^T$ is an equilibrium of system (4.1).

Following along similar lines as for the results established in Section 3, we can prove the following results for the PWM feedback system (4.1).

Theorem 4.1 Assume that in (4.1) A_1 is Hurwitz stable, b is positive, and $C_1 \neq 0$. Then, the trivial solution of the PWM feedback system (4.1) is *uniformly asymptotically stable in the large* whenever $M\beta$ satisfies

$$M\beta < \inf_{\tau_k \in (0,T]} \frac{\lambda_m(L_1(\tau_k)) + \sqrt{\lambda_m(L_1(\tau_k))^2 + 4\lambda_M(L_2(\tau_k))}}{2\lambda_M(L_2(\tau_k))},$$

and

$$\begin{aligned} \frac{1}{M\beta} \geq \sup_{\tau_k \in (0,T]} \Bigg(& \frac{\xi b^2 + Z_{\tau_k}^T(P_1 - I)Z_{\tau_k}}{2\xi b} \\ & + \frac{\Big(Z_{\tau_k}^T(P_1 - I) - \xi bC_1\Big)\Big((P_1 - I)Z_{\tau_k} - \xi bC_1^T\Big)}{2\xi b}\Bigg), \end{aligned}$$

for some positive constant ξ, where $L_1 = C_1^T Z_{\tau_k}^T(P_1 - I) + (P_1 - I)Z_{\tau_k}C_1$, $L_2 = \Big(Z_{\tau_k}^T(P_1 - I)Z_{\tau_k} + \xi b^2\rho_{\tau_k}^2\Big)C_1^T C_1$, and

$$Z_{\tau_k} \triangleq \begin{cases} 0, & \tau_k = 0 \\ \frac{I - e^{-A_1\tau_k}}{\tau_k}A_1^{-1}B_1, & \tau_k < T \\ \frac{I - e^{-A_1T}}{\tau_k}A_1^{-1}B_1 = \frac{T}{\tau_k}Z_T, & \tau_k \geq T \end{cases}. \tag{4.2}$$

Theorem 4.2 Assume that in (4.1) A_1 is Hurwitz stable, $C_1 = 0$, and $b > 0$. Then the trivial solution of the PWM feedback system (4.1) is uniformly asymptotically stable in the large whenever $M\beta < \frac{2}{b}$. $\square$

Remark 4.1 When $b < 0$, the system (4.1) cannot be stabilized by a pulse-width modulator with $M > 0$. In fact, we will show in the next result that the solutions of (4.1) are unbounded in this case. Therefore, the condition that $b > 0$ is necessary for the uniform asymptotic stability in the large of the trivial solution. However, the system can be stabilized by letting $u(t) = +m(C_1x_1 + x_2)$ instead of $u(t) = -m(C_1x_1 + x_2)$. This is equivalent to letting M assume negative values. $\square$

Theorem 4.3 Assume that A_1 in (4.1) is Hurwitz stable. Then the solutions of system (4.1) are uniformly ultimately bounded provided that $b > 0$. When $b \leq 0$, the solutions are unbounded. □

Refer to [15] for the proofs of Theorems 4.1, 4.2, and 4.3.

Remark 4.2 Theorem 4.3 presents *necessary and sufficient* conditions for the ultimate boundendess of the solutions of the PWM feedback system (4.1). □

5 Examples

To demonstrate the applicability of the results established in Sections 3 and 4, and to illustrate how to compute estimates of upper stability bounds $\overline{(M\beta)}_{opt}$, we consider in the present section two examples. In order to be able to make comparisons with existing results, we chose the *identical* examples that were considered by Skoog [11] and Gelig and Churilov [5], and Murphy and Wu [8]. However, before doing so, we outline the following simple procedure for computing estimates for the optimal stability bounds for $M\beta$, based on Theorems 3.1 and Remarks 3.2 and 3.3.

Procedure I The upper bound given in (3.10) can be computed and optimized in the following manner:

(1) Determine the matrix P by solving $(e^{AT})^T P e^{AT} - P = -I$.

(2) Choose a precision level $\delta > 0$ and a correspondingly dense partition of the interval $[0, T]$, say the set $\{t_0 = 0, t_1, \cdots, t_N = T\}$, where $0 < t_{j+1} - t_j < \delta$, $j = 0, 1, \cdots, N-1$.

(3) For each j, $j = 0, 1, \cdots, N$, compute $W_{t_j} = \frac{I - e^{-At_j}}{t_j} A^{-1}BC, G_1(t_j) = W_{t_j}^T(P-I) + (P-I)W_{t_j}, G_2(t_j) = W_{t_j}^T(P-I)W_{t_j}$.

(4) Initialize m_0 by setting (see (3.10))

$$m_0 = \min_{0 \leq j \leq N} \frac{\lambda_m(G_1(t_j)) + \sqrt{\lambda_m(G_1(t_j))^2 + 4\lambda_M(G_2(t_j))}}{2\lambda_M(G_2(t_j))}.$$

(5) Let (see (3.15)) $\widetilde{G}_0(t_j) = I + m_0 G_1(t_j) - m_0^2 G_2(t_j), \widetilde{G}_1(t_j) = G_1(t_j) - 2m_0 G_2(t_j), \widetilde{m}_0 = m_0$. Replace m_0 in Step 4 by (see (3.16)) $m_0 = \widetilde{m}_0 + \min\limits_{0 \leq j \leq N} \frac{\lambda_m(\widetilde{G}_1(t_j)) + \sqrt{\lambda_m(\widetilde{G}_1(t_j))^2 + 4\lambda_m(\widetilde{G}_0(t_j))\lambda_M(G_2(t_j))}}{2\lambda_M(G_2(t_j))}$.

(6) Repeat the above computation in Step 5 until the increment of m_0 is negligible, say, $m_0 - \widetilde{m}_0 < \epsilon$, where $\epsilon > 0$ is a chosen precision level. Set $\overline{(M\beta)}_{opt}^* = m_0$, where $\overline{(M\beta)}_{opt}^*$ is an estimate of $(M\beta)_{opt}^*$.

(7) Repeat Steps 1 – 6 for different but equivalent matrices $\widetilde{A}, \widetilde{B}$, and $\widetilde{C}$. This can be done, for example, by generating a set Ω of random (nonsingular) matrices, and for each $S \in \Omega$ letting $\widetilde{A} = SAS^{-1}$, $\widetilde{B} = SB$, and $\widetilde{C} = CS^{-1}$. Determine an optimal upper bound for $M\beta$ by setting $\overline{(M\beta)}_{opt} = \max\limits_{S \in \Omega} \overline{(M\beta)}_{opt}^*$. In general, the larger the size of Ω, the closer the computed value $\overline{(M\beta)}_{opt}$ to the *actual* upper bound of $M\beta$.

(8) Repeat Steps 1 – 7, using finer partitions of the interval $[0, T]$ (i.e, smaller δ), until there is no further significant improvement for $\overline{(M\beta)}_{opt}$.

Remark 5.1 The values of $\overline{(M\beta)}_{opt}$ obtained by Procedure I are approximations to the upper bounds given in Theorems 3.1 and 4.1, respectively. For sufficiently large sets Ω and sufficiently dense partitions, these approximations will be as close as desired. □

We can similarly as above derive a procedure for computing estimates for the optimal stability bounds for $M\beta$, based on Theorems 4.1.

We are now in a position to consider two specific examples. Additional examples, including examples of systems with plants that have one pole at the origin, can be found in [15].

Example 5.1 (First-Order Systems) In the present case, the plant is characterized by a transfer function of the form

$$G(s) = \frac{c}{s+a}, \quad a > 0,$$

or by the state-space representation (3.1) with $A = -a$, $B = 1$, $C = c$. P, W_{τ_k}, $G_1(\tau_k)$ and $G_2(\tau_k)$, required in Theorem 3.1, are computed in this case as $P = \frac{1}{1 - e^{-2aT}}$, $W_{\tau_k} = \frac{e^{a\tau_k} - 1}{a\tau_k} c$, $G_1(\tau_k) = 2W_{\tau_k}(P - 1)$, $G_2(\tau_k) = W_{\tau_k}^2 (P - 1)$. Thus,

$$\inf_{\tau_k \in (0,T]} \frac{G_1(\tau_k) + \sqrt{G_1(\tau_k)^2 + 4G_2(\tau_k)}}{2G_2(\tau_k)} \geq \begin{cases} \frac{aT(1+e^{-aT})}{c(1-e^{-aT})} & c > 0 \\ \frac{aT}{|c|}, & c < 0 \end{cases}. \tag{5.1}$$

It follows from Theorem 3.1 that the trivial solution of system (3.2) is uniformly asymptotically stable in the large if

$$M\beta < \begin{cases} \frac{aT(1+e^{-aT})}{c(1-e^{-aT})} & c > 0 \\ \frac{aT}{|c|}, & c < 0 \end{cases}.$$

Note that the optimal bound obtained for $M\beta$ above is the exact value, since in the present case it was not necessary to invoke Procedure I to apply Theorem 3.1.

Using a method that employs averaging of the pulse-width modulator output, and assuming $M = 1$ and $c > 0$, the stability condition

$$\frac{1}{\beta} > \frac{2}{\pi} c + \frac{2}{\pi\sqrt{3}} acT \tag{5.2}$$

is obtained in [5]. In Fig. 3 we depict the plots of $c\beta$ *vs.* aT obtained by using (5.1) (with $M = 1$ and $c > 0$) and by using (5.2). For the particular example on hand, the results of the present paper are clearly less conservative than those obtained in [5].

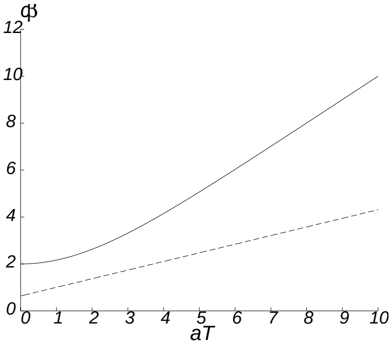

Figure 3: Upper bounds for $c\beta$ for the first-order PWM feedback system of Example 5.1 (solid plot obtained by (5.1), dashed plot obtained by (5.2))

It is interesting to note that an analysis *tailored specifically to the present example* [12] shows that for $c > 0$, relation (5.1) is also a necessary condition. □

Example 5.2 (Second-Order Systems with Hurwitz Stable Plant) In this case the plant is characterized by the transfer function

$$G(s) = \frac{K}{(s+1)(s+2)}.$$

The state-space representation is given by $A = S\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}S^{-1}$, $B = S\begin{bmatrix}K\\ K\end{bmatrix}$, $C = [1 \quad -1]S^{-1}$, where S is a non-singular matrix. In applying Procedure I we let $\delta = 0.001$ and $\epsilon = 0.0001$ (the improvements of the computed results were negligible for smaller δ and ϵ), and we generated 200 random matrices S to form the set Ω. In particular, when $S = \begin{bmatrix}-2.0894 & -2.1267\\ -2.2581 & 5.3609\end{bmatrix}$ the upper and lower bounds for MK are computed to be 6.3278 and -0.1242, respectively; when $S = \begin{bmatrix}0.9660 & 0.9010\\ 0.1530 & 2.6370\end{bmatrix}$, the upper and lower bounds are computed to be 1.0962 and -1.9789, respectively. It follows from Theorem 3.1 that the trivial solution of (3.2) is uniformly asymptotically stable in the large if $-1.9789 < MK < 6.3278$.

To determine the quality of the estimates of the bounds for MK obtained above, we note that if $MK = -2$, then $x(kT) = (2,1)^T$ is an equilibrium of the discrete-time system (3.4) with $S = I$. This means that there exists a periodic solution $x(t)$ of (3.2) with $x(kT) = (2,1)^T$ for all k. Also, when $MK = 6.62$, the discrete-time system (3.4) has a periodic solution $x(2kT) = (-0.4491, -0.2241)$, $x(2kT+T) = (0.4491, 0.2241)$ (approximately). Therefore, the trivial solution of the PWM feedback system (3.4) cannot be uniformly asymptotically stable in the large for the above two cases. This shows that our result, $-1.9789 < MK < 6.3278$, obtained by Theorem 3.1, is very close to the *actual* lower and upper bounds for MK that ensure stability. We would like to point out that the above result is less restrictive than the result $-0.9 < MK < 5.57$ obtained by Murphy and Wu [8] for this particular example. □

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