MOMENT STABILITY OF PULSE-WIDTH-MODULATED FEEDBACK SYSTEMS SUBJECTED TO RANDOM DISTURBANCES

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Abstract

We study the stability properties of pulse-width-modulated (PWM) feedback systems with stable plants, subjected to multiplicative and additive random disturbances (modeled by the derivative of a Wiener process). We show that when the parameters of the pulse-width modulator are within a computable range and the random disturbances are sufficiently small, then the PWM feedback system is globally asymptotically stable in the pth mean. We also show that in the presence of additive disturbances, such PWM feedback systems are bounded in the pth mean for *arbitrarily* large disturbances.

1 Introduction

Pulse-width modulation has extensively been used in attitude control systems, adaptive control systems, signal processing, and the like. Indeed, such systems include one of the most important specific classes of practical nonlinear control systems (see, e.g., [11]–[13] for applications of pulse-width-modulated (PWM) feedback systems). Stability results for PWM feedback systems have been established by a variety of methods (see, e.g., [5] and [10]). There are, however, only a few results concerning the qualitative properties of PWM feedback systems subjected to random disturbances. Gupta and Jury [8] developed a method of determining the mean square value of the output of a PWM system with Gaussian random input while Heinen [9] studied the existence of limit cycles in PWM feedback systems subjected to random disturbances at the system inputs using a modified describing function method. Recently, Gelig *et al* [6], [7] used frequency domain techniques to study the mean square stability of PWM feedback systems with random disturbances at the plants. The stability conditions in [6], [7] are analogous to similar conditions given in [5] for deterministic PWM feedback systems. The latter have proved to be rather conservative (see, [15] for comparisons). The conditions in [6] and [7] appear extremely complicated and no specific examples are given in [6], [7] to demonstrate the applicability and the quality of these results.

In this paper we investigate moment stability properties of PWM feedback systems with the plants subjected to random disturbances. In [14], we initiated a systematic study of stochastic dynamical systems for which the motions may be discontinuous with respect to time. For these systems, we established new Lyapunov and Lagrange stability results in the pth mean. We will employ the method developed in [14] in the analysis of the class of problems considered herein. We would like to point out that although the general theory developed in [14] can be applied directly in the analysis of the class of problems considered in the present paper, due to the special nature of these systems, we will find it more convenient to use a direct approach.

2 Notation and Some Definitions

We let $(\Omega, \mathcal{F}, P)$ denote the underlying probability space for all the systems that will be considered, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space, and P is the probability measure. An $\mathbb{R}^n$-valued random variable x with domain X is a measurable function from Ω to $X \subset \mathbb{R}^n$. A family $\{x(t), t \in I\}$ of $\mathbb{R}^n$-valued random variables with domain X defined on a probability space $(\Omega, \mathcal{F}, P)$ is called a *stochastic process* with index set I and state space $(X, \mathcal{F}^n)$.

Definition 2.1 Let (X, d) be a metric space, let $X \subset \mathbb{R}^n$, let $A \subset X$, and let $T \subset \mathbb{R}^+$. For any fixed $a \in A$ (a is called the initial state), $t_0 \in T$, a stochastic process $\{x(t, \omega, a, t_0), t \in T_{a,t_0}\}$ with domain X is called a **stochastic motion** if $x(t_0, \omega, a, t_0) = a$ for all $\omega \in \Omega$, where $T_{a,t_0} = [t_0, t_1) \cap T$, $t_1 > t_0$, and t_1 is finite or infinite. □

Definition 2.2 Let S be a family of stochastic motions with domain X given by $S \subset \{x(\cdot,\cdot,a,t_0) : x(t_0, \omega, a, t_0) = a, \omega \in \Omega, a \in A, t_0 \in T\}$. We call the four-tuple $\{T, X, A, S\}$ a **stochastic dynamical system**. □

Definition 2.3 Let $\{T, X, A, S\}$ be a stochastic dynamical system. A set $M \subset A$ is said to be **invariant** with respect to system S (or short, (S, M) is invariant) if $a \in M$ implies that $P\{\omega : x(t, \omega, a, t_0) \in M \text{ for all } t \in T_{a,t_0}\} = 1$ for all $t_0 \in T$ and for all $x(\cdot,\cdot,a,t_0) \in S$. □

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Definition 2.4 $x_0 \in A$ is called an **equilibrium** of a stochastic dynamical system $\{T, X, A, S\}$ if the set $\{x_0\}$ is invariant with respect to S. □

Definition 2.5 Let $\{T, X, A, S\}$ be a stochastic dynamical system and let d be the metric on X. A set $M \subset A$ is said to be **stable in the pth mean** (i.e., (S, M) is said to be stable in the pth mean) if for any $\epsilon > 0$, $t_0 \in T$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that $Ed(x(t, \omega, a, t_0), M)^p < \epsilon$ for any process $x(\cdot, \cdot, a, t_0) \in S$, whenever $d(a, M) < \delta$, where E denotes the expectation of a random process. If δ is independent of t_0, (S, M) is said to be **uniformly stable in the pth mean**. (S, M) is said to be **uniformly asymptotically stable in the pth mean** if it is uniformly stable in the pth mean and if for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ and a $\tau = \tau(\epsilon)$ such that for any process $x(\cdot, \cdot, a, t_0) \in S$, $d(a, M) < \delta$ implies that $Ed(x(t, \omega, a, t_0), M)^p < \epsilon$ for all $t > t_0 + \tau$. □

Definition 2.6 S is said to be **uniformly bounded in the pth mean** if for every $\alpha > 0$ and for every $t_0 \in T$ there exists a $\beta = \beta(\alpha) > 0$ such that if $d(a, x_0) < \alpha$, then for all $x(\cdot, \cdot, a, t_0) \in S$, $Ed(x(t, \omega, a, t_0), x_0)^p < \beta$ for all $t \in T_{a,t_0}$, where x_0 is a fixed point in X. S is said to be **uniformly ultimately bounded in the pth mean** if there exists $B > 0$ and if for every $\alpha > 0$ and for every $t_0 \in T$ there exists a $\tau = \tau(\alpha)$ such that for all $x(\cdot, \cdot, a, t_0) \in S$, $Ed(x(t, \omega, a, t_0), x_0)^p < \beta$, for all $t > t_0 + \tau$, whenever $d(a, x_0) < \alpha$, where x_0 is a fixed point in X. □

We can similarly define *asymptotic stability in the pth mean, asymptotic stability in the large in the pth mean, uniform asymptotic stability in the large in the pth mean, exponential stability in the pth mean*, and *exponential stability in the large in the pth mean*.

When $p = 2$, we speak of various stability and boundedness concepts in the *mean square sense*.

3 Main Results

The PWM feedback control system to be considered is shown in Fig. 1.

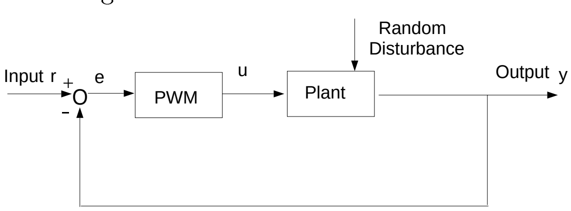

Figure 1: PWM feedback systems subjected to disturbances

The pulse-width modulator is described by the expression

$$u(t) = m(e(t)) = \begin{cases} M \cdot sgn(e(kT)), & t \in [kT, kT + T_k], \\ 0, & \text{otherwise} \end{cases}, \quad (1)$$

for $k = 0, 1, 2, \cdots$, where the pulse width T_k and the signum function are given by

$$T_k = \begin{cases} \beta |e(kT)|, & |e(kT)| \le \frac{T}{\beta} \\ T, & |e(kT)| > \frac{T}{\beta} \end{cases}, \quad sgn(\sigma) = \begin{cases} 1, & \sigma > 0 \\ 0, & \sigma = 0 \\ -1, & \sigma < 0 \end{cases}.$$

The sampling period T, the amplitude of the pulse M, and β are all assumed to be constant.

When the random disturbances are in the form of multiplicative white noise, the overall system with $r(t) \equiv 0$ can be described by equations of the form

$$\begin{cases} dx(t) = (Ax(t) + Bu(t))dt + Gx(t)d\eta(t), \\ y(t) = Cx(t) \end{cases} \quad (2)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}$ is the output of the pulse-width modulator, A, B, C, and G are matrices of appropriate dimensions, and $\eta(t)$ is a scalar Wiener process. A block diagram of system (2) is shown in Fig. 2.

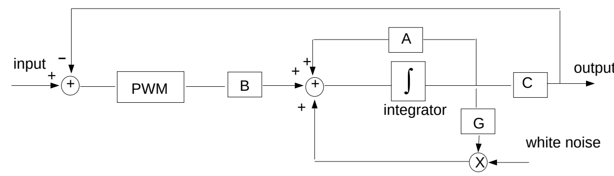

Figure 2: Block diagram of PWM feedback systems subjected to multiplicative disturbances

Let $S_{(2)}$ denote the set of solutions of (2) with initial conditions in $\mathbb{R}^n$. Then $\{\mathbb{R}^+, \mathbb{R}^n, \mathbb{R}^n, S_{(2)}\}$ is a stochastic dynamical system and we will say that this system is a stochastic dynamical system determined by equation (2), or short, system (2). We note that $x_e = 0$ is an equilibrium of the PWM feedback system (2). We are now in a position to present the first stability result.

Theorem 3.1 The equilibrium $x_e = 0$ of the PWM feedback system (2) is *uniformly asymptotically stable in the pth mean* when the matrix A is Hurwitz stable, $M\beta$ is sufficiently small (an upper bound of $M\beta$ will be given in the proof), and $\|G\|$ is sufficiently small.

Proof: Since $(E\|x\|^p)^{1/p}$ is nondecreasing in p, the equilibrium $x_e = 0$ is stable (respectively, asymptotically stable, and so forth) in the pth mean implies that it is stable (respectively, asymptotically stable, and so forth) in the qth mean for all $q < p$. Therefore, it suffices to prove the theorem for even integers.

By the variation of constants formula, the solution of the first equation in (2) is given by $x(t) = e^{A(t-kT)}x(kT) + \int_{kT}^{t} e^{A(t-s)}Bu(t)ds + \int_{kT}^{t} e^{A(t-s)}Gx(s)d\eta(s)$ for all $t \in [kT, kT + T]$ and $k = 0, 1, 2, \cdots$. Specifically, at $t = kT + T$, we have (assuming $r(t) \equiv 0$)

$$x(kT+T) = e^{AT}\Big(x(kT) - \int_0^{T_k} e^{-As}ds BM sgn(Cx(kT))\Big) + \int_{kT}^{kT+T} e^{A(kT+T-s)}Gx(s)d\eta(s). \quad (3)$$

To simplify our notation, we let

$$z(kT)=\int_{kT}^{kT+T} e^{A(kT+T-s)}Gx(s)d\eta(s),$$

$$\widetilde{z}(kT)\triangleq-\int_0^{T_k} e^{-As}dsBMsgn(Cx(kT))=-M\beta W_{s_k}x(kT), \quad (4)$$

where $s_k \triangleq \beta|Cx(kT)| \begin{cases} =T_k, & T_k<T \\ \geq T, & T_k=T \end{cases}$, (5)

and

$$W_{s_k} \triangleq \begin{cases} 0, & s_k=0 \\ \frac{I-e^{-As_k}}{s_k}A^{-1}BC, & s_k<T \\ \frac{I-e^{-AT}}{s_k}A^{-1}BC=\frac{T}{s_k}W_T, & s_k\geq T \end{cases}. \quad (6)$$

Equation (3) is then reduced to

$$x(kT+T)=e^{AT}(x(kT)+\widetilde{z}(kT))+z(kT).$$

We recall that if A is Hurwitz stable, then e^{AT} is Schur stable. Therefore, there exists a positive definite matrix $P=P^{\mathcal{T}}$ such that

$$(e^{AT})^{\mathcal{T}}P(e^{AT})-P=-I. \quad (7)$$

Choosing the quadratic form Lyapunov function $V:\mathbb{R}^n\to\mathbb{R}^+$, $V(x)=x^{\mathcal{T}}Px$, we obtain for the first forward difference of EV along the solutions of the discrete-time system (3)

$$\begin{aligned} &EV(x(kT+T))-EV(x(kT))= \\ &E(x(kT)+\widetilde{z}(kT))^{\mathcal{T}}(P-I)(x(kT)+\widetilde{z}(kT))-Ex(kT)^{\mathcal{T}}Px(kT) \\ &+2Ez(kT)^{\mathcal{T}}Pe^{AT}(x(kT)+\widetilde{z}(kT))+Ez(kT)^{\mathcal{T}}Pz(kT). \quad (8)\end{aligned}$$

We will show in the following that when $M\beta$ is sufficiently small, it is true that

$$\begin{aligned} &(x(kT)+\widetilde{z}(kT))^{\mathcal{T}}(P-I)(x(kT)+\widetilde{z}(kT))-x(kT)^{\mathcal{T}}Px(kT) \\ &=-x(kT)^{\mathcal{T}}(I+M\beta W_{s_k}^T(P-I)+M\beta(P-I)W_{s_k} \\ &-M^2\beta^2W_{s_k}^T(P-I)W_{s_k})x(kT)\leq-|\Theta_{M\beta}|\|x(kT)\|^2, \quad (9)\end{aligned}$$

where $\Theta_{M\beta}<0$ is a constant depending on the choice of $M\beta$.

Let $G_1(s_k)\triangleq W_{s_k}^T(P-I)+(P-I)W_{s_k}$, and $G_2(s_k)\triangleq W_{s_k}^T(P-I)W_{s_k}$. Notice that when $s_k>T$, it is true that $I+M\beta G_1(s_k)-M^2\beta^2G_2(s_k)=I+M\beta\frac{T}{s_k}G_1(T)-M^2\beta^2\frac{T^2}{s_k^2}G_2(T)$. Therefore, if we can show that the matrix $I+M\beta G_1(T)-M^2\beta^2G_2(T)$ is negative definite for all $M\beta$ less than a certain value, say $\alpha>0$, then the matrix $I+M\beta G_1(s_k)-M^2\beta^2G_2(s_k)$ is negative definite for all $s_k>T$ and all $M\beta<\alpha$.

Let $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote the minimum and maximum eigenvalues of a matrix, respectively. It can easily be verified that when $M\beta$ satisfies

$$M\beta<\inf_{s_k\in(0,T]}\left(\frac{\lambda_m(G_1(s_k))}{2\lambda_M(G_2(s_k))}+\frac{\sqrt{\lambda_m(G_1(s_k))^2+4\lambda_M(G_2(s_k))}}{2\lambda_M(G_2(s_k))}\right), \quad (10)$$

it is true that $\sup_{s_k\in(0,T]}\lambda_M\big(-I-M\beta W_{s_k}^T(P-I)-M\beta(P-I)W_{s_k}+M^2\beta^2W_{s_k}^T(P-I)W_{s_k}\big)<0$. In particular, the matrix $I+M\beta G_1(T)-M^2\beta^2G_2(T)$ is negative definite for all $M\beta$ under the present assumption. Since the matrix $I+M\beta\frac{T}{s_k}G_1(T)-M^2\beta^2\frac{T^2}{s_k^2}G_2(T)$ is continuous with respect to $\frac{T}{s_k}$, and tends to the identity matrix I as s_k goes to ∞, we conclude that

$$\begin{aligned}\Theta_{M\beta}\triangleq\sup_{s_k\geq0}&\lambda_M\big(-I-M\beta W_{s_k}^T(P-I) \\ &-M\beta(P-I)W_{s_k}+M^2\beta^2W_{s_k}^T(P-I)W_{s_k}\big)<0, \quad (11)\end{aligned}$$

when (10) is satisfied.

For $\widetilde{z}(kT)$, we have the estimate, $\|\widetilde{z}(kT)\|=\big\|\int_0^{T_k}e^{-As}dsBM\big\|\leq M\beta e^{\|A\|T}\|B\|\|C\|\|x(kT)\|$.

We next establish an estimate for $z(kT)$. Let $\mu>0$ be arbitrary. We will show that there exists a $\delta>0$ such that whenever $\|G\|<\delta$, then $E\|z(kT)\|^2<\mu E\|x(kT)\|^2$ is true for all k.

For $t\in[kT,kT+T)$, we have $x(t)=x(kT)+\int_{kT}^t(Ax(s)+Bu(s))ds+\int_{kT}^t Gx(s)d\eta(s)$, and therefore,

$$\begin{aligned}\frac{E\|x(t)\|^2}{4}&\leq E\|x(kT)\|^2+\|A\|^2\int_{kT}^t E\|x(s)\|^2ds \\ &+E(\int_{kT}^t\|B\||u(s)|ds)^2+E\int_{kT}^t x(s)^{\mathcal{T}}G^{\mathcal{T}}Gx(s)ds \\ &\leq K_0E\|x(kT)\|^2+(\|A\|^2+\|G\|^2)\int_{kT}^t E\|x(s)\|^2ds,\end{aligned}$$

where $K_0=1+M^2\beta^2\|B\|^2\|C\|^2$. In the above inequality we have used the fact that $E(\int_{kT}^t\|B\||u(s)|ds)^2\leq M^2\beta^2\|B\|^2\|C\|^2E\|x(kT)\|^2$ and the fact that $E(\int_{kT}^tGx(s)d\eta(s))^{\mathcal{T}}(\int_{kT}^tGx(s)d\eta(s))=E\int_{kT}^tx(s)^{\mathcal{T}}G^{\mathcal{T}}Gx(s)ds$.

By the Gronwall inequality, we have

$$\begin{aligned}E\|x(t)\|^2&\leq 4K_0E\|x(kT)\|^2e^{(\|A\|^2+\|G\|^2)(t-kT)} \\ &\leq 4K_0e^{T(\|A\|^2+\|G\|^2)}E\|x(kT)\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}E\|z(kT)\|^2&=E\int_{kT}^{kT+T}x(s)^{\mathcal{T}}G^{\mathcal{T}}e^{A^{\mathcal{T}}(kT+T-s)}e^{A(kT+T-s)}Gx(s)ds \\ &\leq\|G\|^2K_1\int_{kT}^{kT+T}E\|x(s)\|^2ds \\ &\leq K\|G\|^2e^{T\|G\|^2}E\|x(kT)\|^2, \quad (12)\end{aligned}$$

where $K_1=\max_{s\in[0,T]}\|e^{A^{\mathcal{T}}s}e^{As}\|$ and $K=4K_1K_0e^{T\|A\|^2}$ are constants. Thus, there exists a $\delta>0$ such that $K\|G\|^2e^{T\|G\|^2}<\mu$ whenever $\|G\|<\delta$.

Choosing $\mu>0$ so that $c(\mu)=1-2\|Pe^{AT}\|(1+M\beta e^{\|A\|T}\|B\|\|C\|)\sqrt{\mu}-\|P\|\mu>0$, we obtain

$$EV(x(kT+T)) - EV(x(kT)) \leq -\frac{c(\mu)}{\lambda_M(P)} EV(x(kT)). \tag{13}$$

Therefore $EV(x(kT+T)) - EV(x(kT))$ is negative definite.

If we define $\widetilde{M} = \{0\}$ and $\widetilde{S} = \{\widetilde{x} : \widetilde{x}(t,a,t_0) = EV(x(t,a,t_0))\}$, then we have determined in an unambiguous way a deterministic dynamical system $\{\mathbb{R}^+, \mathbb{R}^+, \mathbb{R}^+, \widetilde{S}\}$, corresponding to the stochastic dynamical system $\{\mathbb{R}^+, \mathbb{R}^n, \mathbb{R}^n, S_{(2)}\}$. In this case $\widetilde{M}$ is an invariant set for $\{\mathbb{R}^+, \mathbb{R}^+, \mathbb{R}^+, \widetilde{S}\}$, by construction.

Now recall that $(\widetilde{S}, \widetilde{M})$ is said to be uniformly stable if for every $\epsilon > 0$ and every $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\widetilde{x}(t, \widetilde{a}, t_0) < \epsilon$ for all $t \in \mathbb{R}^+_{\widetilde{a},t_0} = \mathbb{R}^+_{a,t_0}$ and for all $\widetilde{x}(\cdot, \widetilde{a}, t_0) \in \widetilde{S}$, whenever $\widetilde{a} < \delta$. Since $\widetilde{x}(t,\widetilde{a},t_0) = EV(x(t))$, $(\widetilde{S}, \widetilde{M})$ is uniformly stable means that for every $\epsilon > 0$ and every $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(\epsilon) > 0$ such that $EV(x(t)) < \epsilon$ for all $t \in \mathbb{R}^+_{a,t_0}$ and for all $x(\cdot,\cdot,a,t_0) \in S$, whenever $EV(a) < \delta$, which yields $E\|x(t)\|^2 \leq \frac{\epsilon}{\lambda_m(P)}$ for all $t \in \mathbb{R}^+_{a,t_0}$ and for all $x(\cdot,\cdot,a,t_0) \in S$, whenever $\|a\| < \frac{\delta}{\lambda_M(P)}$ since $\lambda_m(P)\|x(t)\|^2 \leq V(x(t)) \leq \lambda_M(P)\|x(t)\|^2$. This is precisely the definition of uniform stability in the mean square of the trivial solution $x_e = 0$ of system $S_{(2)}$. Therefore, the uniform stability of $(\widetilde{S}, \widetilde{M})$ is equivalent to the uniform stability in the mean square of $(S_{(2)}, \{x_e\})$. Similarly, the uniform asymptotic stability and the exponential stability of $(\widetilde{S}, \widetilde{M})$, the uniform boundedness of $\widetilde{S}$, and so forth, are equivalent to the corresponding stability in the pth mean of (S, M) or boundedness in the pth mean of S.

It follows from the usual Lyapunov stability results for discrete-time dynamical systems (see, e.g., [17]) that the trivial solution of the discrete-time system $\{EV(x(kT))\}$ is uniformly asymptotically stable.

For $t \in (kT, kT+T)$, we have

$$\begin{aligned} EV(x(t)) &\leq \lambda_M(P)K\|G\|^2 e^{\|G\|^2} E\|x(kT)\|^2 \\ &\leq \frac{\lambda_M(P)K\|G\|^2 e^{\|G\|^2}}{\lambda_m(P)} EV(x(kT)). \end{aligned} \tag{14}$$

This implies that $EV(x(t))$ in the interval $(kT, kT+T)$ is bounded by $\gamma EV(x(kT))$ where $\gamma > 0$ is a constant.

Therefore, $EV(x(t))$ converges to the origin simultaneously with $EV(x(kT))$. We conclude that the trivial solution of the deterministic system $\widetilde{S}$ is asymptotically stable. We can now conclude that the equilibrium $x_e = 0$ of the PWM feedback system (2) is *uniformly asymptotically stable in the mean square.*

For $p = 2q$, $q \geq 1$, we have

$$\begin{aligned} &EV(x(kT+T))^q - EV(x(kT))^q \\ &\leq E\Big(V(x(kT+T)) - V(x(kT))\Big)\Big(V(x(kT+T))^{q-1} \\ &\quad + \cdots + V(x(kT))^{q-1}\Big) \\ &\leq \frac{-1}{\lambda_M(P)} EV(x(kT))^q + 2E\big(z(kT)^T P e^{AT}(x(kT) + \widetilde{z}(kT)) \\ &\quad + z(kT)^T P z(kT)\big) \cdot \big(V(x(kT+T))^{q-1} + \cdots + V(x(kT))^{q-1}\big). \end{aligned}$$

It can easily be verified that the second expectation in the above inequality can always be chosen to be less than $\mu EV(x(kT))^q$ for arbitrary μ when $\|G\|$ is sufficiently small. Similarly as in (14), we can show that $EV(x(t))^q$ is bounded by $\alpha EV(x(kT))^q$ when $t \in [kT, kT+T)$, where $\alpha > 0$ is a constant. The rest of the proof of the uniform asymptotic stability in the pth mean of the trivial solution of (2) with $p = 2q$ proceeds similarly as the proof of uniform asymptotic stability in the mean square of the trivial solution of (2). We omit the details in the interests of brevity.

Therefore, we have shown that the trivial solution of system (2) is uniformly asymptotically stable in the pth mean for even integers, and hence, for all $p > 0$. □

Remark 3.1 The upper bound of $M\beta$ given by (10) can easily be computed and optimized. A simple procedure is presented in [15] to accomplish this. We will employ this procedure in a specific example in Section 4. □

Remark 3.2 For PWM feedback system (2), Gelig, *et al* established in [6] and [7] frequency conditions for mean square stability of system (2) with Hurwitz stable A matrix and with A having one pole at the origin. Their results appear very involved and seem, in general, very difficult to verify. No specific examples are included in [6] and [7] to demonstrate the applicability and quality of these results. □

Next, we consider *PWM feedback systems with additive noise in the plant*, described by equations of the form

$$\begin{cases} dx(t) = (Ax(t) + Bu(t))dt + \epsilon d\eta(t), \\ y(t) = Cx(t) \end{cases} \tag{15}$$

where $\epsilon \in \mathbb{R}^n$ has positive components and denotes the magnitude of the random noise. A block diagram of system (15) is shown in Fig. 3.

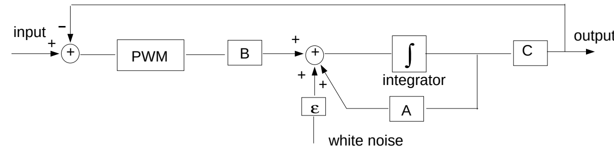

Figure 3: Block diagram of PWM feedback systems subjected to additive disturbances

For this system, we have the following boundedness result.

Theorem 3.3 The solution processes of the PWM feedback system given by (15) are *uniformly ultimately bounded in the pth mean* if the matrix A is Hurwitz stable and $M\beta$ is sufficiently small (for arbitrary ϵ).

Proof: We will present the proof for $p = 2$.

Solving the first equation in (15), we obtain

$$x(t)=e^{A(t-kT)}x(kT)+\int_{kT}^{t}e^{A(t-s)}Bu(k)ds+\int_{kT}^{t}e^{A(t-s)}\epsilon d\eta(s) \tag{16}$$

for all $t \in [kT, kT+T]$. At $t = kT+T$, we have

$$x(kT+T) = e^{AT}\Big(x(kT)-\int_0^{T_k}e^{-As}dsBMsgn(Cx(kT))\Big) + \int_{kT}^{kT+T}e^{A(kT+T-s)}\epsilon d\eta(s). \tag{17}$$

Let $\tilde{z}(kT) = -\int_0^{T_k}e^{-As}dsBMsgn(Cx(kT))$, and $z(kT)=\int_{kT}^{kT+T}e^{A(kT+T-s)}\epsilon d\eta(s)$. Choose P such that $(e^{AT})^{\mathcal{T}}P(e^{AT}) - P = -I$ and a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}^+$ as $V(x) = x^{\mathcal{T}}Px$. We note that $\int_{kT}^{kT+T}e^{A(kT+T-s)}\epsilon d\eta(s)$ is independent of $x(kT)$, and that $E\int_{kT}^{kT+T}e^{A(kT+T-s)}\epsilon d\eta(s) = 0$. The first forward difference of EV evaluated along the solutions of the discrete-time system (17) yields

$$\begin{aligned} & EV(x(kT+T)) - EV(x(kT)) \\ \le\ & -|\Theta_{M\beta}|E\|x(kT)\|^2 + \|P\|E\|z(kT)\|^2, \end{aligned}$$

where $\Theta_{M\beta} < 0$ is a constant depending on the choice of $M\beta$ (refer to the proof of Theorem 3.1) and $E\|z(kT)\|^2$ satisfies the relation $E\|z(kT)\|^2 = E\int_{kT}^{kT+T}\epsilon^{\mathcal{T}}e^{A^{\mathcal{T}}(kT+T-s)}e^{A(kT+T-s)}\epsilon ds \le \|\epsilon\|^2e^{2\|A\|T}$. Thus, whenever $EV(x(kT)) > \Lambda \triangleq \frac{\lambda_M(P)}{\Theta_{M\beta}}e^{2\|A\|T}\|\epsilon\|^2$, the first forward difference of $EV(x(kT))$ is negative definite, which in turn implies that $EV(x(kT))$ will reach a value less than Λ uniformly in a finite number of steps. We next show that $EV(x(kT))$ does not exceed $\Gamma \triangleq \Lambda + e^{2\|A\|T}\|\epsilon\|^2$ thereafter. If $EV(x(k_0T)) \le \Lambda$ for some $k_0 \in \mathbb{N}$, then

$$\begin{aligned} & EV(x(k_0T+T)) \\ \le\ & EV(x(k_0T))-|\Theta_{M\beta}|E\|x(k_0T)\|^2+\|P\|E\|z(k_0T)\|^2 \\ \le\ & \Lambda+e^{2\|A\|T}\|\epsilon\|^2 = \Gamma. \end{aligned} \tag{18}$$

If $EV(x(k_0T+T)) > \Lambda$, then $EV(x(k_0T+2T)) < EV(x(k_0T+T)) \le \Gamma$. By induction, it follows that $EV(x(kT+T)) \le \Gamma$ is true for all $k > k_0$. Therefore, $E\|x(kT)\|^2 < \frac{\Gamma}{\lambda_m(P)}$ is true for all $k \ge k_0$.

For $t \in (kT, kT+T)$, $k \ge k_0$, it follows from (16) that

$$\begin{aligned} E\|x(t)\|^2 &\le 3e^{2\|A\|T}(E\|x(kT)\|^2+TM\|B\|^2+\|\epsilon\|^2) \\ &\le 3e^{2\|A\|T}(\frac{\Gamma}{\lambda_m(P)} + TM\|B\|^2 + \|\epsilon\|^2) \triangleq \Delta. \end{aligned} \tag{19}$$

Therefore, we have shown that the solutions of system (15) are uniformly ultimately bounded in the mean square. □

4 An Example

We consider PWM feedback system (2) with second-order Hurwitz stable plant described by the transfer function $G(s) = \frac{1}{(s+1)(s+2)}$. The state-space representation of this system is given by $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = [1 \quad -1]$.

An estimation of the right side of (10) can be obtained and optimized as follows (for further details and the rationale of this procedure, see [15]):

(1) Determine the matrix P by solving $(e^{AT})^{T}Pe^{AT} - P = -I$. In the present case we have $P = \begin{bmatrix} 1.1565 & 0 \\ 0 & 1.0187 \end{bmatrix}$.

(2) Choose a precision level $\delta > 0$ and a correspondingly dense partition of the interval $[0, T]$, say the set $\{t_0 = 0, t_1, \cdots, t_N = T\}$, where $0 < t_{j+1} - t_j < \delta$, $j = 0, 1, \cdots, N-1$.

(3) For each j, $j = 0, 1, \cdots, N$, compute $W_{t_j} = \frac{I-e^{-At_j}}{t_j}A^{-1}BC, G_1(t_j) = W_{t_j}^T(P-I) + (P-I)W_{t_j}, G_2(t_j) = W_{t_j}^T(P-I)W_{t_j}$.

(4) Initialize m_0 by setting (see (10)) $m_0 = \min_{0\le j\le N} \frac{\lambda_m(G_1(t_j))+\sqrt{\lambda_m(G_1(t_j))^2+4\lambda_M(G_2(t_j))}}{2\lambda_M(G_2(t_j))}$.

(5) m_0 can be optimized as follows (refer to [15]). Let $\widetilde{G}_0(t_j) = I + m_0G_1(t_j) - m_0^2G_2(t_j), \widetilde{G}_1(t_j) = G_1(t_j) - 2m_0G_2(t_j), \widetilde{m}_0 = m_0$. Replace m_0 in Step 4 by $m_0 = \widetilde{m}_0 + \min_{0\le j\le N} \frac{\lambda_m(\widetilde{G}_1(t_j))+\sqrt{\lambda_m(\widetilde{G}_1(t_j))^2+4\lambda_m(\widetilde{G}_0(t_j))\lambda_M(G_2(t_j))}}{2\lambda_M(G_2(t_j))}$.

(6) Repeat the above computation in Step 5 until the increment of m_0 is negligible, say, $m_0 - \widetilde{m}_0 < \epsilon$, where $\epsilon > 0$ is a chosen precision level.

(7) Repeat Steps 1 – 6, using finer partitions of the interval $[0, T]$ (i.e, smaller δ), until there is no further significant improvement for m_0.

For the present example, we let $\delta = 0.001$ and $\epsilon = 0.0001$ (the improvements of the computed results were negligible for smaller δ and ϵ). We obtained the estimate 1.0022 for the upper bound of $M\beta$, assuming $T = 1$.

For $M\beta \in (0, 1.0022)$, we compute μ_{max} such that $c(\mu) = 1-2\|Pe^{AT}\|(1+M\beta e^{\|A\|T}\|B\|\|C\|)\sqrt{\mu}-\|P\|\mu > 0$ is true for all $\mu < \mu_{max}$. Next, we compute δ_{max} such that $K\delta^2e^{T\delta^2} < \mu_{max}$ is true for all $\delta < \delta_{max}$, where $K = K_1(1 + M^2\beta^2\|B\|^2\|C\|^2)e^{T\|A\|^2}$ and $K_1 = \max_{s\in[0,T]}\|e^{A^{\mathcal{T}}s}e^{As}\|$. In Fig. 4, we depict estimates of the upper bound δ_{max} of $\|G\|$ *vs.* $M\beta$. We observe that δ_{max} decreases as $M\beta$ increases. This can be interpreted as follows. When the states are sufficiently far away from the origin so that $T_k = T$, the output of the pulse-width modulator is either $+M$ or $-M$. This may be viewed as another disturbance to the Hurwitz stable plant, in addition to the noise term. Therefore, as M increases (for fixed β), the maximum $\|G\|$ allowable to ensure uniform asymptotic stability in the pth mean will decrease, as shown in Fig. 4. In Fig. 5 we plot the sample response of $x = (x_1, x_2)^{\mathcal{T}}$ *vs.* time t with $M = 0.2$, $T = \beta = 1$, $G = (0.01, 0.006)^{\mathcal{T}}$, and $x(0) = (2,1)^{\mathcal{T}}$. We generated 100 sample responses of $x(t)$ and computed the average $\overline{\|x(t)\|}$ of $\|x(t)\|$. Fig. 6 shows the average $\overline{\|x(t)\|}$ tending to zero as time t increases. However, $\overline{\|x(t)\|}$ does not diminish entirely to zero, since it is an approximation to the mean $E\|x\|$.

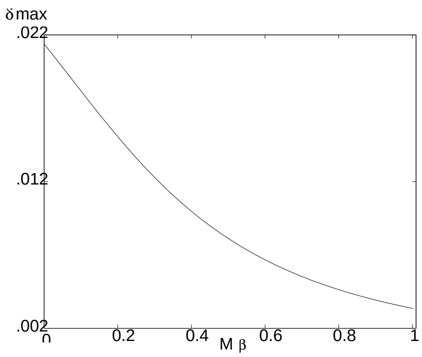

Figure 4: Upper bounds for $\|G\|$ when $M\beta \in (0, 1.0022)$

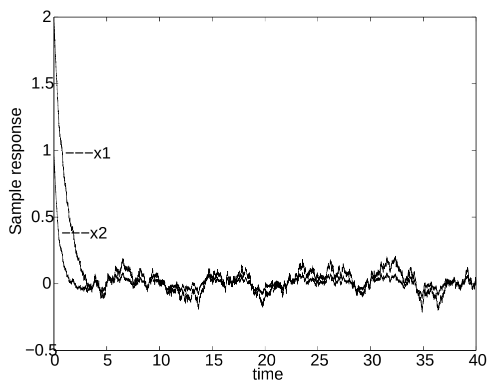

Figure 5: Sample response of PWM feedback system (2)

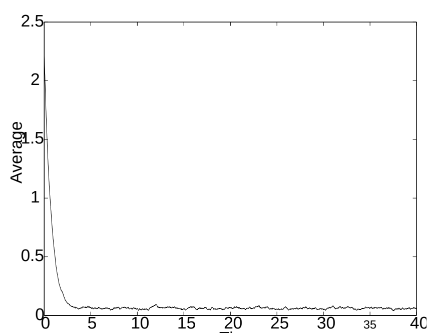

Figure 6: $\overline{\|x(t)\|}$ of PWM feedback system (2)

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