On Quadratic Differential Forms for n-D systems

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1 Introduction

A study of quadratic differential forms for 1-D systems was carried out in [4]. Extention and study of this concept of quadratic differential forms to n-D systems is the main purpose of this paper. This extension opens the way to generalization of several concepts of 1-D systems, like that of conservative systems and dissipative systems and could throw new light into other control problems in the n-D case (especially in the behavioural framework). We collect together several properties of these quadratic differential forms for n-D systems and comment on their similarity and their differences from quadratic differential forms for 1-D systems.

In behavioural theory, a *system* Σ is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ where $\mathbb{T}$ is the indexing set, the set of independent variables, $\mathbb{W}$ is the signal space, the set of dependent variables, and $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$ is the behaviour. When $\mathbb{T} = \mathbb{R}$, then we have a 1-D system and when $\mathbb{T} = \mathbb{R}^n$, we have an n-D system. We will only consider behaviours $\mathfrak{B}$ that occur as kernels of systems of (partial) differential equations. Such systems of equations can be represented by matrices whose entries are polynomials in n variables [1]. Thus $\mathfrak{B} = \{w \mid R(\frac{d}{dx})w = 0\}$, where $R \in \mathbb{R}^{g \times q}[\xi]$ is a matrix in the n indeterminates $\xi = (\xi_1, \xi_2, \ldots \xi_n)$. The variables ($\xi_i$'s) represent partial derivatives ($\frac{\partial}{\partial x_i}$).

2 Quadratic Differential Forms

In [4] a theory was developed for linear (1-D) differential systems and quadratic functionals associated with these systems. It was shown that for systems described by *one-variable* polynomial matrices, the appropriate tool to express quadratic functionals are *two-variable* polynomial matrices. In the same vein, in this paper we will look at polynomial matrices in $2n$ variables to express quadratic functionals for systems defined by polynomial matrices with n variables. Let ζ denote $(\zeta_1, \ldots, \zeta_n)$ and η denote $(\eta_1, \ldots, \eta_n)$. Let $\mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ denote the set of real polynomial matrices in the (commutating) $2n$ indeterminates ζ and η. We will consider quadratic forms of the type $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$. Explicitly,

$$\Phi(\zeta, \eta) = \sum_{\mathbf{k},\mathbf{l}} \Phi_{\mathbf{k},\mathbf{l}} \zeta^{\mathbf{k}} \eta^{\mathbf{l}}$$

The sum above ranges over all non-negative multi-indices $\mathbf{k}, \mathbf{l} \in \mathbb{N}^n$ and is assumed to be finite. Moreover, $\Phi_{\mathbf{k},\mathbf{l}} \in \mathbb{R}^{q_1 \times q_2}$. Note that ζ denotes differentiation of terms to the left and η refers to differentiation of the terms to the right. Hence Φ induces a *bilinear differential form* (BLDF), defined by

$$L_\Phi(v, w)(\mathbf{x}) := \sum_{\mathbf{k},\mathbf{l}} \left(\frac{d^{\mathbf{k}} v}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x}) \right)^T \Phi_{\mathbf{k},\mathbf{l}} \left(\frac{d^{\mathbf{l}} w}{d\mathbf{x}^{\mathbf{l}}}(\mathbf{x}) \right)$$

If $q_1 = q_2 = q$ then Φ induces a *quadratic differential form* (QDF) defined by

$$Q_\Phi(w) := L_\Phi(w, w)$$

Define the * operator by $\Phi^*(\zeta, \eta) := \Phi^T(\eta, \zeta)$ where ^{T} denotes matrix transposition. Clearly $L_\Phi(v, w) = L_{\Phi^*}(w, v)$. If $\Phi \in \mathbb{R}^{q \times q}[\zeta, \eta]$ satisfies $\Phi = \Phi^*$, then Φ is called *symmetric*. For the purposes of QDF's induced by polynomial matrices, it is enough to only consider the symmetric quadratic differential forms, since $Q_\Phi = Q_{\Phi^*} = Q_{\frac{1}{2}(\Phi + \Phi^*)}$.

We also define the "div" operator that acts on a vector of BLDFs (or QDFs) $\Psi = (\Psi_1, \ldots, \Psi_n)$ by

$$\begin{aligned}(\text{div } L_\Psi)(v, w) &:= \text{div } (L_\Psi(v, w)) \\ &= \frac{\partial}{\partial x_1} L_{\Psi_1}(v, w) + \cdots + \frac{\partial}{\partial x_n} L_{\Psi_n}(v, w)\end{aligned}$$

3 Path Integrability

In addition to studying BLDFs and QDFs as maps into $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$, one is also interested in studying their integrals over all of $\mathbb{R}^n$. So we consider the integral $\int_{\mathbb{R}^n} L_\Phi(v,w)d\mathbf{x}$. In order to make sure that these integrals exist, one considers only those elements v and w, that are compactly supported. We denote by $\mathfrak{D}(\mathbb{R}^n, \mathbb{R}^q)$ the compactly supported members of $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^q)$. Consider the integral $\int L_\Phi$ defined by

$$\int L_\Phi(v,w) := \int_{\mathbb{R}^n} L_\Phi(v,w)d\mathbf{x}$$

where $v, w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^q)$. The notation $\int Q_\Phi$ follows readily from this.

Consider now the following integral $\int_\Omega L_\Phi(v,w)d\mathbf{x}$ over Ω, a closed bounded subset of $\mathbb{R}^n$ with non-empty interior. This integral is said to be independent of the "path" (or a *path integral*), if the integral only depends on the value of v and w and their derivatives on the boundary of Ω, denoted by $\partial\Omega$.

Theorem 1 *Let $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ induce a BLDF L_Φ. Then the following statements are equivalent:*

1. *$\int_\Omega L_\Phi$ is independent of path for all Ω, which are closed bounded subsets of $\mathbb{R}^n$.*
2. *$\int L_\Phi = 0$.*
3. *$\Phi(-\xi, \xi) = 0$.*
4. *There exist $\Psi_1, \ldots, \Psi_n \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$, such that*
$$\Phi(\zeta,\eta) = (\zeta_1 + \eta_1)\Psi_1(\zeta,\eta) + \cdots + (\zeta_n + \eta_n)\Psi_n(\zeta,\eta)$$
5. *There exists a $\Psi \in (\mathbb{R}^{q_1 \times q_2}[\zeta, \eta])^n$ such that*
$$\text{div} L_\Psi = L_\Phi$$
for all $v \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{q_1})$ and $w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{q_2})$.

The same equivalence holds for QDFs once the appropriate changes are made in the statements above.

At this juncture, we like to point out a principle difference in the case when $n = 1$ and $n > 1$. Although the above theorem holds for all values of n, more can be said in the case $n = 1$. In this case, Ψ in reality defines a BLDF and the last condition of the above theorem can be strengthened to state that there exists a unique $\Psi = \frac{1}{\zeta+\eta}\Phi$ such that $\frac{d}{dt}L_\Psi = L_\Phi$ On the other hand, the Ψ in the theorem above for the cases $n > 1$ are not unique.

4 Average non-negativity

Several useful functionals like supply functional, cost functional, storage, dissipation of a system etc., tend to be quadratic functionals of the system variables (QDFs). An important property of such functionals is average non-negativity, i.e., integrated over all of $\mathbb{R}^n$, these functionals yield non-negative numbers that in some sense quantifies the property defined by the functional.

Given a $\Phi \in \mathbb{R}^{q \times q}[\zeta, \eta]$, we call the QDF induced by Φ *average non-negative* if

$$\int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} \geq 0$$

for all $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^q)$. We denote an average non-negative QDF Φ by $\int Q_\Phi \geq 0$. If the inequality in the above equation is strict for $w \neq 0$, then we call the QDF *average positive* and denote it by $\int Q_\Phi > 0$.

Given a $\Delta \in \mathbb{R}^{q \times q}[\zeta, \eta]$, we call the QDF induced by it *non-negative* (denoted by $Q_\Delta \geq 0$) if $Q_\Delta(w(\mathbf{x})) \geq 0$ for all $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^q)$ evaluated at every $\mathbf{x} \in \mathbb{R}^n$.

Thus, we note that average non-negativity is a global property of the QDF whereas non-negativity is a local property (in fact, it is a property which is defined pointwise). Clearly given a non-negative QDF Q_Δ, $\int_\Omega Q_\Delta(w)d\mathbf{x} \geq 0$ for every $\Omega \subset \mathbb{R}^n$. Thus every non-negative QDF is average non-negative, but the converse is not true. Moreover, it is easy to see that every $\Delta \in \mathbb{R}^{q \times q}[\zeta, \eta]$ which can be written in the form $\Delta(\zeta,\eta) = R^T(\zeta)R(\eta)$ clearly induces a non-negative QDF. Here $R \in \mathbb{R}^{\bullet \times q}[\xi]$ is a polynomial matrix. So a natural question to ask is whether every non-negative QDF is of this particular form. This is in general not true for the N-D case.

We will now characterize the average non-negative QDFs and average positive QDFs.

Proposition 2 *Let $\Phi \in \mathbb{R}^{q\times q}[\zeta,\eta]$. Then*

- $(\int Q_\Phi \geq 0) \Leftrightarrow \Phi(-i\omega, i\omega) \geq 0 \; \forall\omega \in \mathbb{R}^n$
- $(\int Q_\Phi > 0) \Leftrightarrow \Phi(-i\omega, i\omega) \geq 0 \; \forall\omega \in \mathbb{R}^n$ *and* $\Phi(-\xi,\xi)$ *is non-singular.*

An important set of problems that crop up in systems theory deal with the question of substituting a global property by a local property. As already mentioned, average non-negativity (respectively average positivity) of a QDF is a global property. So the natural question to ask is whether this can be substituted by a local property. Since positivity of a QDF is a local property, one can ask the question - is it possible to replace an average non-negative QDF Q_Φ with a positive QDF Q_Δ such that $\int Q_\Phi = \int Q_\Delta$? We now state an important theorem which plays a crucial role in deciding the answer to the above question.

Theorem 3 *Let $\Gamma \in \mathbb{R}^{q\times q}[\xi]$ be para-Hermitian, i.e. $\Gamma^T(-\xi) = \Gamma(\xi)$ and $\Gamma(i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$. Then there exists an $F \in \mathbb{R}^{\bullet\times q}(\xi)$ such that $\Gamma(\xi) = F^T(-\xi)F(\xi)$.*

Note here that the matrix F has (in general) rational entries. In case of $\xi = (\xi_1)$, (i.e. the case of 1-D systems), we can in fact find $F(\xi)$ which have polynomial entries, i.e. $F \in \mathbb{R}^{\bullet\times q}[\xi]$, whereas finding a polynomial F is not always possible for the n-D case ($n > 1$).

Lemma 4 *Given a $\Phi \in \mathbb{R}^{q\times q}[\zeta,\eta]$, which induces an average non-negative QDF, it is possible to find a positive QDF induced by $\Delta \in \mathbb{R}^{q\times q}[\zeta,\eta]$ such that $\int Q_\Phi = \int Q_\Delta$, if there exists a polynomial matrix $F \in \mathbb{R}^{\bullet\times q}[\xi]$ such that $\Phi(-\xi,\xi) = F^T(-\xi)F(\xi)$.*

Thus we conclude that in the 1-D case, one can always find a non-negative QDF which would mimic an average non-negative QDF in the global sense. This is not always possible in the n-D case.

5 QDFs on behaviours

We shall now investigate the action of QDFs on specific behaviours. In [1, 2] (and elsewhere) differential systems have been studied in the behavioural framework where a behaviour $\mathfrak{B}$ is characterized as the kernel of (partial) differential operator. Now we consider the specific case of a behaviour $\mathfrak{B}$ given as a kernel of a system of partial differential equations. As mentioned earlier, a system of partial differential equations can be written as a polynomial matrix $R \in \mathbb{R}^{g\times q}[\xi]$ in n indeterminates. Then $\mathfrak{B} = \ker R$. There are a special class of behaviours called controllable behaviours (for definition, see [1]). For the purposes of this paper, it suffices to know that controllable behaviours are those behaviours that have an image representation, i.e., there exists an operator M, such that for every $w \in \mathfrak{B}$ there exists an ℓ in some appropriate space yielding $w = M\ell$. We denote such an image representation by $\mathfrak{B} = \text{im}\, M$.

We now ask when a QDF induced by $\Phi \in \mathbb{R}^{q\times q}[\zeta,\eta]$ is independent of path for trajectories $w \in \mathfrak{B}$, i.e. if $w_1, w_2 \in \mathfrak{B}$ and $\frac{d^{\mathbf{k}} w_1}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x}) = \frac{d^{\mathbf{k}} w_2}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$ and all $\mathbf{k} \in \mathbb{N}^n$, then

$$\int_\Omega Q_\Phi(w_1)d\mathbf{x} \;=\; \int_\Omega Q_\Phi(w_2)d\mathbf{x}$$

We first define the * operator as $X^\star(\xi) = X^T(-\xi)$. In other words, if we look at X as a partial differential operator, then $X^\star$ is the adjoint operator.

Proposition 5 *Let $\mathfrak{B}$ be a controllable behaviour, $\mathfrak{B} = \ker R = \text{im} M$, and let $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ induce a QDF on $\mathfrak{B}$. Then the following conditions are equivalent :*

1. *QDF induced by Φ is independent of path on $\mathfrak{B}$;*
2. *there exists $X \in \mathbb{R}^{\bullet\times q}[\xi]$ such that*
$$X^\star(\xi)R(\xi) + R^\star(\xi)X(\xi) = \Phi(-\xi,\xi)$$
3. *the QDF corresponding to Φ' is a path integral, where Φ' is given by $\Phi'(\zeta,\eta) := M^T(\zeta)\Phi(\zeta,\eta)M(\eta)$;*
4. *$\Phi'(-\xi,\xi) = 0$;*

5. *there exists a VQDF $Q_{\Psi'}$, with $\Psi' \in (\mathbb{R}^{m\times m}[\zeta,\eta])^n$, where m is the number of columns of M, such that*

$$\operatorname{div} Q_{\Psi'}(\ell) = Q_{\Phi'}(\ell) = Q_\Phi(w)$$

for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$ and $w = M(\frac{d}{d\mathbf{x}})\ell$.

Note that in the unconstrained case (i.e. when $R = 0$), this proposition specializes to Theorem 1. Given a behaviour $\mathfrak{B}$ that is not controllable, the condition (2) given above is sufficient but not neccessary. Let us consider an example from [3]. Consider the 1-D behaviour that arises as solutions to the equation $w - \frac{d^2w}{dt^2} = 0$. Clearly, the behaviour consists of linear combinations of e^t and e^{-t}. Furthermore, consider the QDF defined by $\Phi = 1$. Thus $Q_\Phi(w) = w^2$. Path independence holds trivially since there is at most one trajectory which satisfies the boundary conditions - in this case, the boundary $\partial\Omega$ are the points that form the limits of the integral considered. Now $\Phi(-\xi,\xi) = 1$ and so condition (2) of the above proposition yields the equation $(1-\xi^2)(X(\xi) + X(-\xi)) = 1$. This equation clearly does not have a polynomial solution solution X. Thus condition (2) in the above proposition is not necessary. We now give the general condition for path independence to hold for any behaviour.

It has been shown in [1] that given any behaviour $\mathfrak{B}$, one can find a unique sub-behaviour $\mathfrak{B}_c$ contained in this behaviour which is the largest controllable sub-behaviour (i.e. every controllable subbehaviour of $\mathfrak{B}$ is contained in this sub-behaviour $\mathfrak{B}_c$). For the purposes of our result, we have to consider the largest controllable sub-behaviour in $\mathfrak{B}$, denoted by $\mathfrak{B}_c$. Let $R_c \in \mathbb{R}^{g'\times q}[\xi]$ induce a kernel representation for $\mathfrak{B}_c$. Then we have the following

Proposition 6 *Given a $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ and any uncontrollable behaviour $\mathfrak{B}$ that has a kernel representation given by $R \in \mathbb{R}^{g\times q}[\xi]$, the QDF induced by Φ is independent of path on $\mathfrak{B}$ if and only if there exists $X \in \mathbb{R}^{\bullet\times q}[\xi]$ and some $L \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that*

$$X^\star(\xi)R_c(\xi) + R_c^\star(\xi)X(\xi) = \Phi(-\xi,\xi)$$
$$\text{and} \quad X = LR_c$$

Note that if we consider an autonomous behaviour $\mathfrak{B}$, then the largest controllable sub-behaviour is the zero behaviour. Hence $\mathfrak{B}_c = 0$ and the corresponding $R_c = I$, the identity matrix. Thus the first equation in the above proposition becomes

$$X^\star(\xi) + X(\xi) = \Phi(-\xi,\xi)$$

The second condition is trivially satisfied, since $R_c = I$. Since $\Phi = \Phi^*$, the above equation has a solution, namely $X(\xi) = \frac{1}{2}\Phi(-\xi,\xi)$ and thus every symmetric $\Phi \in \mathbb{R}^{q\times q}[\zeta,\eta]$ induces a QDF which is path independent over any given autonomous behaviour.

Summarizing, in this paper, we have gathered together several interesting properties of QDFs and BLDFs. We have obtained conditions for a BLDF or QDF to be path independent. We have obtained conditions for a QDF to be average non-negative and average positive. We show when a average positive QDF mimics a non-negative one. We also obtain conditions for a QDF to be path indepent over a given behaviour.

References

[1] H. K. Pillai and Shiva Shankar. A behavioural approach to control of distributed systems. *SIAM Journal of Control and Optimization*, 37:388–408, 1999.

[2] J. C. Willems. Paradigms and puzzles in the theory of dynamical systems. *IEEE Transactions on Automatic Control*, 36:259–294, 1991.

[3] J. C. Willems. Path integrals and stability. In J. Baillieul and J. C. Willems, editors, *Mathematical Control Theory*, Festschrift at the occasion of the 60-th birthday of R. W. Brockett. pages 1–32, Springer Verlag, 1998.

[4] J. C. Willems and H. L. Trentelman. On quadratic differential forms. *SIAM Journal of Control and Optimization*, 36(5):1703–1749, 1998.