Algorithms for Quantum Control Based on Decompositions of Lie Groups

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Abstract

Algorithms for the control of quantum mechanical systems are presented which are based on decompositions of Lie groups. The derived control laws drive the state of two and four level systems to any desired final configuration. The algorithms for the two-level case allow the use of an arbitrarily bounded control.

Keywords: Control of Quantum Mechanical Systems, Systems on Lie Groups, Quantum Computing.

1 Introduction

The state of a general multilevel quantum system is described by a vector $|\psi(t)>$, in a finite dimensional Hilbert space. At every time t, we have

$$|\psi(t)>= X(t)|\psi(0)> . \tag{1}$$

The operator $X(t)$ is the evolution operator solution of Schrödinger equation

$$i\hbar\dot{X} = H(t)X, \tag{2}$$

with initial condition equal to the Identity. In many experimental situations $H(t)$ has the form $H(t) := H_0 + \sum_{i=1}^{m} H_i u_i(t)$, where the operators (matrices) $H_0, H_1, ..., H_m$ are time invariant and $u_i(t)$, $i = 1, ..., m$ are typically time varying components of electromagnetic fields which play the role of controls. In this situation, system (2) can be written as [4] [8]

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{m} B_i X(t) u_i(t), \quad X(0) = I_{n\times n}. \tag{3}$$

In (3), A, $B_1, ..., B_m$ are matrices in $su(n)$, where n is the number of levels of the system under consideration. The solution of (3), $X(t)$, varies in the Lie group of special unitary matrices, of dimension n, $SU(n)$. In this paper we are interested in the control of X in (2) (3). This is a way to obtain the control of the state $|\psi>$ in (1) and also to control the *operation* performed on the state of the quantum system. This is of interest in scenarios like *quantum computing*, where the state of the system encodes information and the evolution of the system has to be driven in order to perform a prescribed logic operation.

2 Control of two-level systems

We consider the model (3) in the two level case and with single input,

$$\dot{X} = AX + BXu. \tag{4}$$

The matrices A and B are in $su(2)$ and we assume the system to be controllable which is equivalent to assuming that A and B are linearly independent.

F. Lowenthal [7] has shown that every two linearly independent matrices in $su(2)$, say Z_1 and Z_2, generate the Lie group $SU(2)$. This means that every element X_f in $SU(2)$ can be written as the finite product of alternate elements of the one dimensional subgroups corresponding to Z_1 and Z_2, namely

$$X_f = e^{Z_1 t_1} e^{Z_2 t_2} e^{Z_1 t_3} e^{Z_2 t_4} \cdots e^{Z_1 t_m}, \tag{5}$$

for some parameters $t_1, t_2, ..., t_m$. Notice that, since the one dimensional subgroups for every Z_1, $Z_2 \in su(2)$ are closed, we can always assume that $t_1, t_2, ..., t_m$ are positive, which is important in our context since they will have meaning of time. The number of factors in (5) depends on the value of the inner product $< Z_1, Z_2 >:= Trace(Z_1 Z_2^*)$. In particular, it is minimal and equal to three if Z_1 and Z_2 are orthogonal. F. Lowenthal has derived a formula which relates $< Z_1, Z_2 >$ to the number of factors m in (5). In particular we have $m = 3$, if $< Z_1, Z_2 >= 0$ and $m = f + 2$ if

$$cos(\frac{\pi}{f}) < \frac{|< Z_1, Z_2 >|}{< Z_1, Z_1 >^{\frac{1}{2}} < Z_2, Z_2 >^{\frac{1}{2}}} \leq cos(\frac{\pi}{f+1}), \tag{6}$$

with $f \geq 2$. Now, if the control u is bounded by $|u| \leq M$, we choose a bang bang control $u = \pm M$ in order to obtain a final condition of the form (5) with $Z_1 = A + MB$ and $Z_2 = A - MB$. The number of factors or, which is the same, the (maximum) number of switches in the control can be obtained from formula (6) by substituting $Z_1 = A + MB$, $Z_2 = A - MB$. After some manipulations, we obtain for the second term in the inequality (6) the following function of M, that we denote by $L(M)$

$$L(M) := \frac{|k^2 - M^2|}{\sqrt{(k^2 + M^2)^2 - 4M^2(\frac{<A,B>}{<B,B>})^2}}. \tag{7}$$

The parameter k is defined as

$$k := \sqrt{\frac{< A, A >}{< B, B >}}. \tag{8}$$

Notice that the number of factors in (5) increases with the parameter $L(M)$. A study of this quantity as a function of M shows that $M = k$ achieves the minimum value of number of switches. The function is decreasing in the interval $[0, k)$ and increasing in $[k, +\infty)$. The number of switches and the number of factors in (5) tends to infinity as M goes to zero as well as M goes to $+\infty$. In any case, we have an algorithm to steer the state of system (4) via a bang bang control. Once the coefficients $t_1, ..., t_m > 0$, in (5) with $Z_1 = A + MB$, $Z_2 = A - MB$, are known the control is kept constant for intervals of length equal to these parameters.

3 Control of two interacting spin $\frac{1}{2}$ particles

The control algorithm presented in the previous section is based on the decomposition of the elements of $SU(2)$ in (5). Algorithms for problems of higher dimension can be derived in a similar fashion if decompositions of the type (5) are available. In fact, decompositions exist for higher dimensional Lie groups (see e.g. [2]) but, in general, we cannot insist that the matrices appearing as factors are exponentials of the matrices appearing in the equations of the system (3). This problem as been dealt with in [3] for the important system of two interacting spin $\frac{1}{2}$ particles in a magnetic field. This system is a four level one and it has been used to implement the logic operation XOR in quantum computing [5]. For this system and the corresponding matrices in (3) (with $m = 3$) an analysis of the Lie algebra $su(4)$ and an application of Cartan decomposition [6] lead to the following result.

Theorem *Consider the dynamics of two interacting spin $\frac{1}{2}$ particles [1] [3] and the corresponding equation (3) that we rewrite as*

$$\dot{X} = \tilde{A} + B_x X u_x + B_y X u_y + B_z X u_z. \tag{9}$$

Every element $X_f \in SU(4)$ can be written as

$$X_f = K_1 A_1 K_2 A_2 K_3 A_3 K_4. \tag{10}$$

In (10) $A_{1,2,3}$ are elements of the one dimensional closed subgroup $\{X_f \in SU(4) : X_f = e^{\tilde{A}s}, \quad s \in \mathbf{R}\}$ and K_j, $j = 1, 2, 3, 4$, can be written as

$$K_j = e^{a_{1j}B_z} e^{\tilde{A}\frac{\pi}{4}} e^{a_{2j}B_z} e^{\tilde{A}\frac{7\pi}{4}} e^{a_{3j}B_y} e^{\tilde{A}\frac{\pi}{4}} e^{a_{4j}B_y} e^{\tilde{A}\frac{7\pi}{4}} \times e^{a_{5j}B_z} e^{\tilde{A}\frac{\pi}{4}} e^{a_{6j}B_z} e^{\tilde{A}\frac{7\pi}{4}}. \tag{11}$$

The decomposition of the previous theorem is given in terms of the matrices that appear in the equation of the system. It is clear that a control which is identically zero will steer the Identity matrix to a matrix of the form $e^{\tilde{A}s}$ while a control with all the components equal to zero except $u_{y,z} = \frac{a}{\Delta}$ in an interval $\Delta > 0$ will steer the state very close to $e^{B_{z,y}a}$ if Δ is sufficiently small. Concatenation of these controls will give (with arbitrary accuracy) a final condition as in X_f in (10) and therefore every final condition in $SU(4)$.

Proofs of the above results along with a discussion on the computation of the parameters in (11), once a desired final condition X_f is given, are presented in [3].

References

[1] R. Brockett and N. Khaneja, On the stochastic control of quantum ensambles, *System Theory: Modeling, Analysis and Control*, 1999, pg. 75-96.

[2] F. Silva Leite, Uniform controllable sets of left-invariant vector fields on compact Lie groups, *Systems & Control Letters,* **6**, 1986, 329-335.

[3] D. D'Alessandro, Control of spin systems based on decomposition of Lie groups; the one and two spin case, Technical Report, Department of Mathematics, Iowa State University, 2000.

[4] D. D'Alessandro and M. Dahleh, Optimal control of two-level quantum systems, *Proceedings of 2000 American Control Conference* Chicago, IL, June 2000.

[5] D. P. DiVincenzo, Quantum Computation, *Science*, Vol. 270, 13 October 1995.

[6] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.

[7] F. Lowenthal, Uniform finite generation of $SU(2)$ and $SL(2, \mathbf{R})$, *Can. J. Math.*, Vol. XXIV, No. 4, 1972,pp. 713-727.

[8] V. Ramakrishna, M. V. Salapaka, M. Dahleh, H. Rabitz and A. Peirce, Controllability of molecular systems, *Physical Review A*, 51, no. 2, 1995.